QUENCHING AND PROPAGATION OF COMBUSTION WITHOUT IGNITION TEMPERATURE CUTOFF

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Abstract. We study a reaction-diffusion equation in the cylinder $\Omega = \mathbb{R} \times \mathbb{T}^m$, with combustion-type reaction term without ignition temperature cutoff, and in the presence of a periodic flow. We show that if the reaction function decays as a power of T larger than three as $T \to 0$ and the initial datum is small, then the flame is extinguished — the solution *quenches*. If, on the other hand, the power of decay is smaller than three or initial datum is large, then quenching does not happen, and the burning region spreads linearly in time. This extends results of Aronson-Weinberger for the no-flow case. We also consider shear flows with large amplitude and show that if the reaction power-law decay is larger than three and the flow has only small plateaux (connected domains where it is constant), then any compactly supported initial datum is quenched when the flow amplitude is large enough (which is not true if the power is smaller than three or in the presence of a large plateau). This extends results of Constantin-Kiselev-Ryzhik for combustion with ignition temperature cutoff. Our work carries over to the case $\Omega = \mathbb{R}^n \times \mathbb{T}^m$, when the critical power is $1 + \frac{2}{n}$, as well as to certain non-periodic flows.

1. Introduction and main results

We study the reaction-diffusion-advection equation

$$\begin{aligned} T_t + u \cdot \nabla T &= \Delta T + Mf(T) \\ T(0,x) &= T_0(x) \geq 0 \end{aligned} \tag{1.1}$$

in $\Omega \subseteq \mathbb{R}^n$, which models flame propagation in a premixed combustible fluid [4] advected by a prescribed flow $u(x)$. Here T is the normalized temperature that takes values in $[0,1]$ and $f : [0,1] \to \mathbb{R}_0^+$ with $f(0) = f(1) = 0$ is the non-linear reaction term, with coupling $M > 0$. There is a vast mathematical and physical literature on the subject and we refer to recent reviews [3, 20] for an extensive bibliography. In the present paper we will mainly focus on the question of *quenching* (extinction) of the flame

$$\lim_{t\to\infty} \|T(t,\cdot)\|_\infty = 0 \tag{1.2}$$

(in which case we say that T *quenches*), or its absence. This means that we will assume the spatial domain to be unbounded and the initial datum compactly supported (fast enough decay at infinity would be sufficient). That is, the fluid will be initially "hot" in a finite (but possibly large) central region and "cold" at infinity. Our main interest is in the study of situations when quenching depends on the size of (the support of) the initial datum (Theorems 1.1 and 3.1, Corollary 2.4), or when it results from strong fluid motion (Theorem 1.3).

We mainly want to consider *combustion-type* reaction terms with $f'(0) = 0$. However, unlike most previous works studying quenching in reaction-diffusion models, we will not assume f to have an *ignition temperature cutoff*, that is, we will not require the existence of $\theta_0 > 0$ such that $f(T) = 0$ for $T \in [0, \theta_0]$. Such an assumption simplifies the proof of quenching to showing the existence of a time t_0 at which T is below the ignition temperature θ_0, uniformly in space. Then the maximum principle shows that this will remain the case for all later times and we are left with a linear equation after t_0. Quenching is now provided by the diffusion term Δ.

Dispensing with this assumption allows us to treat the physically important case of *Arrhenius reaction term* $f(T) \equiv e^{-c/T}(1-T)$. More generally, our quenching results will hold when $f(T) \le cT^p$ for certain $p > 1$. Without the ignition temperature cutoff the equation will never become linear but can be close to it when T is small. The idea is that if at low temperatures the reaction is weak (i.e., if p is large enough), then the decay of temperature caused by diffusion may still be sufficient to ensure quenching. Hence we will consider the non-linear equation as a perturbation of its linear counterpart

$$\Phi_t + u \cdot \nabla \Phi = \Delta \Phi. \tag{1.3}$$

Then we will apply a lemma of Meier (Lemma 2.1 below) to show that the solution of the latter can be used to estimate that of the former. (There is the obvious estimate $T(t,x,y) \le e^{ct}\Phi(t,x,y)$ with $c \equiv M\|f(T)/T\|_\infty$, following from the maximum principle, but it is insufficient for our purposes.) To do all this we will need good estimates on the decay of the solutions of (1.3), which enter into Lemma 2.1.

The first work studying the extinction and propagation of flames in the case of combustion non-linearity with ignition temperature cutoff was the paper [10] by Kanel', who considered equation (1.1) in one spatial dimension and with no advection. He showed that if the initial condition is $T_0(x) \equiv \chi_{[-L,L]}(x)$, then there are two length scales L_0, L_1 such that flame extinction/propagation happens when L is smaller than L_0/larger than L_1. That is,

$$T(t,x) \to 0 \text{ as } t \to \infty \text{ uniformly in } x \in \mathbb{R} \text{ if } L < L_0,$$
$$T(t,x) \to 1 \text{ as } t \to \infty \text{ for all } x \in \mathbb{R} \text{ if } L > L_1.$$

Both length scales are of the order of the *laminar front width* $\ell_c \equiv M^{-1/2}$. But quenching often operates on larger scales, especially in the presence of strong fluid motion (see Theorem 1.3).

Kanel's result was generalized by Roquejoffre [16] to the case of *shear flows* $u(x,y) = (u(y), 0)$ in a cylindrical domain $\mathbb{R} \times D$ with $D \subset \mathbb{R}^m$ bounded and Neumann boundary conditions at $\mathbb{R} \times \partial D$. The length scales L_0, L_1 then also depend on u. Xin [19] extended the propagation part of Kanel's result to smooth periodic flows on $\mathbb{R} \times [0,h]^m$ with periodic boundary conditions. The following theorem is an extension of these results to the case of combustion without ignition temperature cutoff, when u is a periodic flow on $\mathbb{R} \times [0,h]^m$. It identifies the critical exponent $p^* \equiv 3$ such that the above extinction–propagation dichotomy picture is valid when $p > p^*$ and $f(T) \le cT^p$ close to $T = 0$, whereas if $p < p^*$ and $f(T) \ge cT^p$ close to $T = 0$, then no non-trivial non-negative solution of (1.1) satisfies (1.2).

Theorem 1.1. *Consider* (1.1) *in* $\Omega \equiv \mathbb{R} \times [0,h]^m$ *with periodic boundary conditions. Let* $u(x,y)$ *be a* C^1*, periodic, divergence-free flow on* Ω*, and let* f *be Lipschitz with* $f(0) = f(1) = 0$ *and* $f(T) > 0$ *for* $T \in (0,1)$*. Let* $c, \eta, \theta > 0$ *and assume* $0 \le T_0 \le 1$*.*

(i) *There are* $0 < \gamma_1 < \gamma_2 < \infty$*, independent of* η*, and* $L_1(\eta) < \infty$ *such that if* $T_0(x,y) \ge \eta\chi_{[-L_1,L_1]}(x)$ *is compactly supported, the solution of* (1.1) *satisfies*

$$\lim_{t\to\infty} \inf_{|x|\le\gamma_1 t} T(t, x+bt, y) = 1, \tag{1.4}$$

$$\lim_{t\to\infty} \sup_{|x|\ge\gamma_2 t} T(t, x+bt, y) = 0, \tag{1.5}$$

where b *is the first coordinate of the mean of* u*.*

(ii) *If* $p > 3$ *and* $f(T) \le cT^p$ *for* $0 \le T \le \theta$*, then there is* $\varepsilon > 0$ *such that if* $\|T_0\|_1 \le \varepsilon$*, then the solution of* (1.1) *quenches.*

(iii) *If* $p < 3$ *and* $f(T) \ge cT^p$ *for* $0 \le T \le \theta$*, then the solution of* (1.1) *quenches only if* $T_0 \equiv 0$*. Moreover, there are* $0 < \gamma_1 < \gamma_2 < \infty$ *such that any solution* T *with compactly supported* $T_0 \not\equiv 0$ *satisfies* (1.4)*,* (1.5)*.*

Remarks. 1. Part (i) is essentially a result of Xin [19] and we only include it for the sake of completeness. Part (iii) for $p = 1$ and u a shear flow was proved by Roquejoffre [16].

2. We assume u to be C^1 when Ω is viewed as $\mathbb{R} \times (h\mathbb{T})^m$ (similarly in Theorem 1.3). We do not need to assume u divergence-free in (ii) and (iii), but then b in (1.4), (1.5) need not be the first coordinate of the mean of u (see Definition 2.2).

3. Parts (ii) and (iii) extend to the case $\Omega \equiv \mathbb{R}^n \times [0, h]^m$, with the critical exponent being $p^* \equiv 1 + \frac{2}{n}$, as follows from Theorem 3.1.

In Theorem 1.1 quenching results from smallness of the initial datum, thanks to which T quickly becomes small enough so that the effects of reaction are weak. On the other hand, large initial flames can be extinguished by a strong wind. Constantin-Kiselev-Ryzhik [5] studied quenching by large amplitude shear flows, and considered the problem

$$\begin{aligned} T_t + Au(y)T_x &= \Delta T + Mf(T) \\ T(0, x, y) &= T_0(x, y) \geq 0 \end{aligned} \tag{1.6}$$

on the strip $\mathbb{R} \times [0, h]$ with periodic boundary conditions and flow amplitude A. Their interest was in identifying flow profiles u such that quenching happens for any compactly supported T_0 when A is large enough. They made the following definition.

Definition 1.2. We say that the profile u is *quenching* if for any compactly supported $T_0(x, y)$, there exists A_0 such that for all $|A| \geq A_0$ the solution of (1.6) quenches.

Of course, whether u is quenching depends on f and M. Under the ignition temperature cutoff assumption on f it is proved in [5] that if a C^∞ profile u has no *plateaux* (intervals on which u is constant) or has only one small plateau, then it is quenching. On the other hand if u has a large enough plateau, then it is not quenching. Both these plateau sizes depend on f and M.

Kiselev-Zlatoš [11] later obtained a sharp result in this direction by showing that there is a critical length $\ell_0(f, M)$ such that $u \in C^1(h\mathbb{T})$ is quenching when all its plateaux are shorter than ℓ_0 and it is not quenching when at least one plateau is longer than ℓ_0. They also provided estimates on the minimal *quenching amplitude* A_0 as a function of the size of the support of T_0 and studied the dependence of this relation on the (large and small period) scaling of the flow profile in y. All their results agree with previously obtained numerical experiments (see, e.g., [18]). Finally, quenching by large amplitude cellular flows was recently studied by Fannjiang-Kiselev-Ryzhik [7].

The following theorem is an extension of the results in [5, 11] to the case of combustion without ignition temperature cutoff, when u

is a shear flow on $\mathbb{R} \times [0,h]^m$ (in which case plateaux of u are connected sets in $(h\mathbb{T})^m$ on which u is constant). It again identifies the critical exponent $p^* \equiv 3$ such that the above quenching–non-quenching dichotomy picture is valid when $p > p^*$ and $f(T) \le cT^p$ close to $T = 0$, whereas if $p < p^*$ and $f(T) \ge cT^p$ close to $T = 0$, then quenching never happens.

Theorem 1.3. *Consider* (1.6) *on* $\Omega \equiv \mathbb{R} \times [0,h]^m$ *with periodic boundary conditions. Let* $u(x,y) = (u(y),0)$ *be a* C^1 *shear flow profile on* Ω *and let* $0 \le f \not\equiv 0$ *be Lipschitz with* $f(0) = f(1) = 0$*. Let* $c, \theta > 0$*.*

(i) *If* u *has at least one large enough (depending on* f, M*) plateau, then* u *is not quenching.*
(ii) *If* $p > 3$ *and* $f(T) \le cT^p$ *for* $0 \le T \le \theta$*, and if* u *has none or only small enough (depending on* f, M*) plateaux, then* u *is quenching.*
(iii) *If* $p < 3$ *and* $f(T) \ge cT^p$ *for* $0 \le T \le \theta$*, then* u *is not quenching.*

Remarks. 1. Part (i) for $m = 1$ is a result of Constantin-Kiselev-Ryzhik [5].

2. Large/small enough plateau in (i)/(ii) means one containing/contained in a large/small enough ball in $(h\mathbb{T})^m$. The change of variables $\tilde{T}(t,x,y) \equiv T(M^{-1}t, M^{-1/2}x, M^{-1/2}y)$ shows that bounds on the sizes of both balls (upper on the large one and lower on the small one) are of the order of the laminar front width $\ell_c \equiv M^{-1/2}$ for any fixed f.

3. This result holds with Neumann boundary conditions as well. It also generalizes to shear flows $u(x,y) = (u(y),0)$ on $\mathbb{R}^n \times [0,h]^m$. The critical exponent is then $p^* \equiv 1 + \frac{2}{n}$.

4. (1.4),(1.5) hold in (i) and (iii) (in (i) by extension of an argument from [5], in (iii) by Theorem 3.1).

The second group of papers addressing problems related to ours study the semi-linear heat equation

$$T_t + u \cdot \nabla T = \Delta T + T^p \tag{1.7}$$

with $p > 1$ on $\mathbb{R}^n$, and the first of them was the work of Fujita [8]. In the case $u \equiv 0$ he showed that if $p > p^* \equiv 1 + \frac{2}{n}$, then there are global positive solutions to (1.7), whereas if $1 < p < p^*$, then all non-trivial non-negative solutions blow up in finite time. The critical case $p = p^*$ was shown to belong to the blowup regime by Hayakawa [9].

Bandle-Levine [2] extended Fujita's result to divergence free flows with x^{-1} decay at infinity, and the existence of a critical exponent p^* for any flow was proved by Meier [13]. In both of these works the Hayakawa case $p = p^*$ is left open. Several authors have studied the problem on conical or general sectorial domains, or with additional potential or non-linear terms in (1.7). We refer to the reviews by Levine [12] and Deng-Levine [6] for more details and bibliography. In this direction we prove Corollary 2.4 which extends Fujita's theorem to more general classes of flows, periodic in particular, and is a direct application of lemmas by Meier [13] and Norris [14]. It shows that in $\mathbb{R}^n$, the critical exponent for these flows is again $p^* \equiv 1 + \frac{2}{n}$.

The rest of the paper is organized as follows. In Section 2 we state the abovementioned lemmas of Meier and Norris, and their consequence, Corollary 2.4. In Section 3 we prove a general extinction–propagation result (Theorem 3.1), as well as Theorems 1.1 and 1.3.

For the sake of simplicity of notation, in what follows we will be studying the equation

$$T_t = \Delta T + u \cdot \nabla T + f(T)$$

in $\mathbb{R}^n \times \mathbb{T}^m$ instead of (1.1) in $\mathbb{R}^n \times [0, h]^m$ with periodic boundary conditions. This is no loss as one can be obtained from the other by a change of variables. Indeed — if T satisfies (1.1) in $\mathbb{R}^n \times [0, h]^m$, then $\tilde{T}(t, x, y) \equiv T(h^2 t, hx, hy)$ satisfies

$$\tilde{T}_t = \Delta \tilde{T} + v \cdot \nabla \tilde{T} + g(\tilde{T})$$

in $\mathbb{R}^n \times \mathbb{T}^m$, with $v(x, y) \equiv -hu(hx, hy)$ and $g(T) \equiv h^2 M f(T)$.

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2. Lemmas of Meier and Norris

We now state a lemma of Meier [13] which enables one to treat certain reaction-diffusion non-linear PDE's as perturbations of associated linear equations when one is interested in qualitative phenomena like extinction and blowup. We state it in the form we will need here and provide the proof for later reference.

We let $\Omega \subseteq \mathbb{R}^n$ be a domain with a piecewise smooth (possibly empty) boundary $\partial\Omega$. We assume that $u : \Omega \to \mathbb{R}^n$ and $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $f(0) = 0$ are bounded and f is Lipschitz. We let $T(t, x)$, $\Phi(t, x)$ be the solutions of

$$T_t = \Delta T + u \cdot \nabla T + f(T) \tag{2.1}$$

$$\Phi_t = \Delta \Phi + u \cdot \nabla \Phi \tag{2.2}$$

on Ω with Dirichlet, Neumann, or periodic boundary conditions at $\partial\Omega$, and initial conditions $T_0(x), \Phi_0(x) \geq 0$ (hence, by the maximum principle, $T, \Phi \geq 0$). In this section $\| \cdot \|$ stands for $\| \cdot \|_\infty$.

Lemma 2.1 (Meier)**.** *Consider T and Φ as above and let $c, \alpha > 0$.*

(i) *If $f(T) \leq cT^{1+\alpha}$ and $I \equiv \int_0^\infty \|\Phi(t,\cdot)\|^\alpha \, dt$, then for $0 \leq \delta_0 < (c\alpha I)^{-1/\alpha}$ and $T_0(x) \equiv \delta_0 \Phi_0(x)$ the solution T quenches.*

(ii) *If $f(T) \geq cT^{1+\alpha}$ and $J \equiv \sup_t t\|\Phi(t,\cdot)\|^\alpha$, then for $\delta_0 > (c\alpha J)^{-1/\alpha}$ and $T_0(x) \equiv \delta_0\Phi_0(x)$ the solution T blows up in finite time.*

Remarks. 1. A more general form is valid with $f(T)$ replaced by $h(t)f(T)$ where h is non-negative and continuous (see [13]). In this case $I \equiv \int_0^\infty h(t)\|\Phi(t,\cdot)\|^\alpha dt$ and $J \equiv \sup_{t,x} \Phi(t,x)^\alpha \int_0^t h(s)ds$. If $h \in L^1(\mathbb{R}^+)$, we also need $\|\Phi(t,\cdot)\| \to 0$ in (i). Meier only considers the non-linear term $h(t)T^{1+\alpha}$ but the general case is identical.

2. In our applications Ω is unbounded and decay of Φ will be provided by the diffusion term in (2.1).

3. We note that one can replace $u \cdot \nabla T$ by a C^1 function $g(t,x,\nabla T)$ as long as $g(t,s,0) = 0$ and $g(t,x,sv) \geq sg(t,x,v)$ for any $v \in \mathbb{R}^n$ and $s \geq s_0$, in which case we also need $\delta_0 \leq (s_0^\alpha + c\alpha I)^{-1/\alpha}$ in (i) and $\delta_0 \geq s_0$ in (ii). Interestingly enough, if instead $g(t,s,0) = 0$ and $g(t,x,sv) \leq sg(t,x,v)$ for any $v \in \mathbb{R}^n$ and $s \geq s_0$, and $(c\alpha I)^{-1/\alpha} > s_0$, then the conclusion of (i) is still valid — by first obtaining it as below for $s_0 \leq \delta_0 < (c\alpha I)^{-1/\alpha}$ and then for all smaller δ_0 by comparison theorems (see, e.g., [17, Chapter 10]).

Proof. (i) We can assume $I < \infty$, otherwise there is nothing to prove. Let $\delta(t)$ with $\delta(0) \equiv \delta_0$ solve

$$\delta'(t) = c\|\Phi(t,\cdot)\|^\alpha \delta(t)^{1+\alpha}$$

so that

$$\delta(t) = \left(\delta_0^{-\alpha} - c\alpha \int_0^t \|\Phi(s,\cdot)\|^\alpha \, ds\right)^{-1/\alpha}.$$

If $\delta_0^{-\alpha} > c\alpha I$, then $\delta(t)$ exists and is bounded for all $t \in \mathbb{R}_0^+$. Now define $\tilde{T}(t,x) \equiv \delta(t)\Phi(t,x)$. Then

$$\tilde{T}_t = \Delta\tilde{T} + u \cdot \nabla\tilde{T} + c\delta^{1+\alpha}\Phi\|\Phi\|^\alpha,$$

so $\tilde{T}$ is a supersolution of (2.1) with $\tilde{T}_0 = T_0$, and we have $\tilde{T} \geq T$. Since $\|\Phi(t,\cdot)\|$ is non-increasing by the maximum principle, $I < \infty$ gives $\|\Phi(t,\cdot)\| \to 0$. Hence $\|\tilde{T}(t,\cdot)\| \to 0$ and the same is true for T.

(ii) Let $w(t,\phi)$ solve

$$\frac{\partial w}{\partial t} = cw^{1+\alpha}$$

with $w(0,\phi) \equiv \phi \geq 0$ and define $\tilde{T}(t,x) \equiv w(t,\delta_0\Phi(t,x))$ so that

$$\tilde{T}_t = \Delta\tilde{T} - \frac{\partial^2 w}{\partial\phi^2}\delta_0^2|\nabla\Phi|^2 + u \cdot \nabla\tilde{T} + c\tilde{T}^{1+\alpha}.$$

Now

$$w(t,\phi) = \big(\phi^{-\alpha} - c\alpha t\big)^{-1/\alpha},$$

so $\frac{\partial^2 w}{\partial\phi^2} \geq 0$ and hence $\tilde{T}$ is a subsolution of (2.1). Since $\tilde{T}_0 = T_0$, we have $\tilde{T} \leq T$. Finally, blow-up of $\tilde{T}$ (and of T) is guaranteed by the existence of t, x such that $(\delta_0\Phi(t,x))^{-\alpha} \leq c\alpha t$, which follows from $\delta_0 > (c\alpha J)^{-1/\alpha}$. □

To apply Lemma 2.1 we need to obtain good large-time asymptotic estimates of heat kernels corresponding to certain linear equations. One such result is the following lemma of Norris [14] (for a proof see Theorem 1.1 in [14]). We start with

Definition 2.2. A function $u : \mathbb{R}^n \to \mathbb{R}^n$ is of *type (N)* if $u(x) \equiv \operatorname{div}\beta(x) + (\mathrm{Id} - \beta(x))\nabla \log \mu(x) + \bar{b}/\mu(x)$ with β a bounded, differentiable, and antisymmetric $n \times n$ matrix, μ positive, differentiable, and bounded away from 0 and ∞, and $\bar{b} \in \mathbb{R}^n$ a constant vector. If $\bar{b} \neq 0$, we also require the existence of a bounded, differentiable vector field ξ such that $\operatorname{div}(\mu\xi) + \mu \equiv 1$. By the discussion on p. 168 of [14], this includes all u periodic with bounded $\operatorname{div} u$.

Remark. Theorems 1.1 and 1.3 involve periodic divergence-free u. Such functions of type (N) can be written as $u(x) \equiv \operatorname{div}\beta(x) + \bar{b}$ (see [14, p. 168]), and so the *effective drift* $\bar{b}$ is just the mean of u.

Lemma 2.3 (Norris). *If $u : \mathbb{R}^n \to \mathbb{R}^n$ is of type (N), then there is $C < \infty$ such that for any $x, y \in \mathbb{R}^n$ and $t > 0$, the heat kernel $k(t,x,y)$ of* (2.2) *in $\mathbb{R}^n$ satisfies*

$$C^{-1}t^{-n/2}e^{-C|x-y|^2/t} \leq k(t, x - \bar{b}t, y) \leq Ct^{-n/2}e^{-|x-y|^2/Ct}. \tag{2.3}$$

Remark. Of course, k is such that

$$\Phi(t,x) = \int_{\mathbb{R}^n} k(t,x,y)\Phi_0(y)dy.$$

As an immediate application of Lemmas 2.1 and 2.3 we obtain a generalization of a result of Fujita [8].

Corollary 2.4. *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and of type (N), and consider*

$$T_t = \Delta T + u \cdot \nabla T + T^{1+\alpha} \tag{2.4}$$

in $\mathbb{R}^n$.

(i) *If $\alpha > \frac{2}{n}$, then there are global positive solutions of* (2.4) *that quench.*

(ii) *If $0 < \alpha < \frac{2}{n}$, then all non-trivial non-negative solutions of* (2.4) *blow up in finite time.*

Remarks. 1. Fujita proved this for $u \equiv 0$ (in that case the conclusion of (ii) also holds when $\alpha = \frac{2}{n}$ [9]). Even in this setting our result (i) is slightly stronger in that quenching is provided by small enough L^1 and L^∞ norms of T_0, with no additional conditions on its decay.

2. Since both Lemmas 2.1 and 2.3 hold when Δ is replaced by a uniformly elliptic operator $\sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}$ with bounded differentiable $a_{i,j}(x)$, so does this corollary. The same is true for Theorem 3.1 below.

Proof. (i) Let Φ be the solution of

$$\Phi_t = \Delta\Phi + u \cdot \nabla\Phi \tag{2.5}$$

in $\mathbb{R}^n$ with initial condition $0 < \Phi_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then by (2.3) and the maximum principle,

$$\|\Phi(t,\cdot)\| \le \min\{\|\Phi_0\|, \|\Phi_0\|_1 \|k(t,\cdot,\cdot)\|\} \le \min\{\|\Phi_0\|, Ct^{-n/2}\|\Phi_0\|_1\}.$$

Hence $I \equiv \int_0^\infty \|\Phi(t,\cdot)\|^\alpha \, dt < \infty$ and Lemma 2.1(i) gives the result.

(ii) Assume $T_0(x_0) > 0$ and $\Phi_0 \equiv T_0 \ge 0$. Then by (2.3),

$$\Phi(t, x_0 - \bar{b}t) = \int_{\mathbb{R}^n} k(t, x_0 - \bar{b}t, y)\Phi_0(y)\, dy \ge C^{-1}t^{-n/2}e^{-C}D$$

for $t \ge 1$ and $D \equiv \int_{B(x_0,1)} \Phi_0(y)\, dy > 0$ (here $B(x_0, 1)$ is the ball in $\mathbb{R}^n$ with center x_0 and radius 1). But then $J \equiv \infty$ in Lemma 2.1(ii), so T blows up in finite time. □

3. Proofs of the main results

We now proceed to prove Theorems 1.1 and 1.3. We will start with a general result in the domain $\mathbb{R}^n \times \mathbb{T}^m$, which is related to Corollary 2.4. We will assume u to be C^1 and $f : [0, 1] \to \mathbb{R}_0^+$ to be Lipschitz with $f(0) = f(1) = 0$.

Theorem 3.1. *Consider* (2.1) *on* $\mathbb{R}^n \times \mathbb{T}^m$ *with* $n \ge 1$ *and* $m \ge 0$ *and let* $u : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R}^{n+m}$ *be of type (N). Let* $c, \theta > 0$ *and* $0 \le T_0 \le 1$.

(i) *If* $\alpha > \frac{2}{n}$ *and* $f(T) \le cT^{1+\alpha}$ *for* $0 \le T \le \theta$, *then there is* $\varepsilon > 0$ *such that if* $\|T_0\|_1 \le \varepsilon$, *then the solution of* (2.1) *quenches.*

(ii) *If* $\alpha < \frac{2}{n}$ *and* $f(T) \ge cT^{1+\alpha}$ *for* $0 \le T \le \theta$, *then the solution of* (2.1) *quenches only if* $T_0 \equiv 0$. *If also* $f(T) > 0$ *for* $T \in (0, 1)$, *then there are* $0 < \gamma_1 < \gamma_2 < \infty$ *such that any solution* T *with compactly supported* $T_0 \not\equiv 0$ *satisfies*

$$\lim_{t\to\infty} \inf_{|x|\le\gamma_1 t} T(t, x - \bar{b}t) = 1, \tag{3.1}$$

$$\lim_{t\to\infty} \sup_{|x|\ge\gamma_2 t} T(t, x - \bar{b}t) = 0, \tag{3.2}$$

with $\bar{b}$ *from Definition 2.2.*

Remarks. 1. If f is as in (ii), the theorem says that no flame can be extinguished, even in the presence of strong advection of type (N).

2. For $u \equiv 0$ in $\mathbb{R}^n$ this was proved by Aronson-Weinberger [1], using results from [8].

Proof. Let $q(t,x,y)$ be the heat kernel for $\Delta + u \cdot \nabla$ in $\mathbb{R}^n \times \mathbb{T}^m$ and $k(t,x,y)$ the one in $\mathbb{R}^{n+m}$ (with u periodically continued in the last m coordinates). Then

$$q(t,x,y) = \sum_{j \in \mathbb{Z}^m} k(t,x,y+(0,j))$$

where $(0,j) \in \mathbb{R}^{n+m}$. Hence from (2.3) with $n+m$ in place of n we obtain for all $x,y \in \mathbb{R}^n \times \mathbb{T}^m$ and $t \geq t_0 > 0$

$$C^{-1} t^{-n/2} e^{-C|x-y|_*^2/t} \leq q(t, x - \bar{b}t, y) \leq C t^{-n/2} e^{-|x-y|_*^2/Ct} \tag{3.3}$$

with some new $C = C(u,t_0) < \infty$. Here $\bar{b} \in \mathbb{R}^{n+m}$ and $|x|_*$ denotes the norm of the $\mathbb{R}^n$ component of x. In particular,

$$C^{-1} t^{-n/2} \leq \|q(t,\cdot,\cdot)\|_\infty \leq C t^{-n/2}$$

for $t \geq t_0$.

(i) By changing c we can assume $f(T) \leq cT^{1+\alpha}$ for all T. Let Φ satisfy (2.2) in $\mathbb{R}^n \times \mathbb{T}^m$ with $\Phi_0 \equiv T_0$. Then we have $\|\Phi(t,\cdot)\|_\infty \leq 1$, and for $t \geq t_0$

$$\|\Phi(t,\cdot)\|_\infty \leq \|\Phi_0\|_1 \|q(t,\cdot,\cdot)\|_\infty \leq C\varepsilon t^{-n/2}$$

Hence we get

$$I \equiv \int_0^\infty \|\Phi(t,\cdot)\|_\infty^\alpha \, dt \leq t_0 + \int_{t_0}^\infty (C\varepsilon t^{-n/2})^\alpha \, dt < \frac{1}{c\alpha}$$

if t_0 and then ε are chosen small enough. Lemma 2.1(i) with $\delta_0 = 1$ then gives the result.

(ii) It is obviously sufficient to consider $\alpha > 0$. Assume $\Phi_0 \equiv T_0 \not\equiv 0$ so that $0 < \Phi(t,x) \leq T(t,x)$ for all x and $t > 0$. If (1.2) were true, we would have $T(t,x) \leq \theta$ and thus $f(T(t,x)) \geq cT(t,x)^{1+\alpha}$ for all x and $t \geq t_1 > 0$. Let $\tilde{\Phi}(t,x) \equiv \Phi(t+t_1,x)$ and let $\tilde{T}$ be the solution of (2.1) with $\tilde{T}(0,x) \equiv \tilde{\Phi}(0,x)$, so that $\tilde{T}(t-t_1,x) \leq T(t,x) \leq \theta$ for $t \geq t_1$. Then by (3.3),

$$\tilde{\Phi}(t,-\bar{b}t) = \int_{\mathbb{R}^n \times \mathbb{T}^m} q(t,-\bar{b}t,y)\tilde{\Phi}_0(y)\,dy \geq C^{-1} t^{-n/2} e^{-C} D$$

for $t \geq 1$ and $D \equiv \int_{B(0,1)\times\mathbb{T}^m} \tilde{\Phi}_0(y)\,dy > 0$. But then $J \equiv \infty$ in Lemma 2.1(ii), so $\tilde{T}$ blows up in finite time, a contradiction.

Let us now prove (3.1). For the sake of transparency we will assume $m = 0$. The proof in the general case is identical, with all domains D replaced by $D \times \mathbb{T}^m$. Let us also assume $\bar{b} = 0$ in order to simplify some of the notation below. In the case of a general $\bar{b}$ the argument is identical, but the obtained bounds will hold on balls that travel with speed $-\bar{b}$, due to (3.3). For instance, (3.5) below will read

$$T(2\tau^2, x) \ge \frac{1}{4}\chi_{B(x_0 - 2\bar{b}\tau^2, \tau)}(x).$$

Since $f(T) > 0$ for $T \in (0,1)$, we can change $c > 0$ so that we can take $\theta \equiv \frac{1}{2}$. If $0 \le T \not\equiv 0$, we have $\|T_0\chi_{B(x_0,1)}\|_1 = Ce^{9C}\varepsilon$ for some x_0 and $\varepsilon > 0$. Note that all the following estimates will be uniform in x_0 and will depend on ε. If we let Φ satisfy (2.2) with $\Phi_0 \equiv T_0$, we have for $\tau^2 \ge 1$

$$\Phi(\tau^2, x) \ge \varepsilon\tau^{-n}\chi_{B(x_0, 2\tau)}(x) \equiv \bar{\Phi}_0(x) \tag{3.4}$$

by (3.3) for $t_0 \equiv 1$. Obviously if we let $\bar{T}, \bar{\Phi}$ satisfy (2.1),(2.2) with initial data $\bar{T}_0 \equiv \bar{\Phi}_0$, we have $T(\tau^2 + t, x) \ge \bar{T}(t, x)$ by comparison theorems, so it is sufficient to prove the claim for $\bar{T}$.

Next let

$$\tilde{T}(t, x) \equiv w(t, \bar{\Phi}(t, x)) = \big(\bar{\Phi}(t, x)^{-\alpha} - c\alpha t\big)^{-1/\alpha}.$$

We obviously have $\bar{\Phi} \le \varepsilon\tau^{-n}$ and so $\tilde{T}(t, x) \le (\varepsilon^{-\alpha}\tau^{\alpha n} - c\alpha t)^{-1/\alpha}$. Hence up to time $t^2 \equiv (\varepsilon^{-\alpha}\tau^{\alpha n} - 2^\alpha)/c\alpha$ we have $\tilde{T} \le \frac{1}{2}$ (if $\tau > (2\varepsilon)^{1/n}$). So by the argument in Lemma 2.1, $\tilde{T}(t^2, x) \le \bar{T}(t^2, x)$. Moreover, if τ is large, we have $t \le \tau^\omega$ for some $\omega < 1$. Thus we can take τ large enough so that

$$\int_{\mathbb{R}^n \setminus B(0,\tau)} Ct^{-n}e^{-|y|^2/Ct^2}dy < \frac{(4^\alpha - 2^\alpha)\varepsilon^\alpha}{2\alpha}\tau^{-\alpha n}.$$

Then we have for any x

$$\int_{B(x,\tau)} q(t^2, x, y)dy > 1 - \frac{(4^\alpha - 2^\alpha)\varepsilon^\alpha}{2\alpha}\tau^{-\alpha n}$$

by (3.3) with $\bar{b} = 0$ because the integral over $\mathbb{R}^n$ is 1.

Now for $x \in B(x_0, \tau)$

$$\bar{\Phi}(t^2, x) \ge \int_{B(x,\tau)} q(t^2, x, y)\varepsilon\tau^{-n}dy \ge \left(1 - \frac{(4^\alpha - 2^\alpha)\varepsilon^\alpha}{2\alpha}\tau^{-\alpha n}\right)\varepsilon\tau^{-n}$$

and so

$$\tilde{T}(t^2, x) = \big(\bar{\Phi}(t^2, x)^{-\alpha} - c\alpha t^2\big)^{-1/\alpha} \ge \frac{1}{4},$$

using

$$\left(1 - \frac{(4^\alpha - 2^\alpha)\varepsilon^\alpha}{2\alpha}\tau^{-\alpha n}\right)^{-\alpha} \le 1 + (4^\alpha - 2^\alpha)\varepsilon^\alpha \tau^{-\alpha n}$$

for large τ, and the definition of t. Hence for large enough τ and $t(\tau)$ as above we have

$$T(\tau^2 + t^2, x) \ge \bar{T}(t^2, x) \ge \tilde{T}(t^2, x) \ge \frac{1}{4}$$

for $x \in B(x_0, \tau)$. And since $\tau^2 + t(\tau)^2$ is continuous in τ and belongs to $(\tau^2, 2\tau^2)$ when τ is large, this implies

$$T(2\tau^2, x) \ge \frac{1}{4}\chi_{B(x_0,\tau)}(x) \tag{3.5}$$

for all $\tau \ge \tau_0$. All the above estimates are uniform in x_0, and so (3.5) holds for any x_0 and $\tau \ge \tau_0$, with $\tau_0 = \tau_0(\delta)$ depending only on $\delta \equiv \|T_0\chi_{B(x_0,1)}\|_1$ (and u, f, of course).

Let T_0 be given and assume $T_0(x) \ge \frac{1}{4}\chi_{B(x_0,1)}(x)$ for some x_0 (otherwise first pick t so that $T(t,x) \ge \frac{1}{4}\chi_{B(x_0,1)}(x)$ and then reset $T_0(x)$ to $T(t,x)$). If $\tau \ge \tau_1 \equiv \max\{2, \tau_0(\frac{1}{4}|B(0,1)|)\}$, then by the above,

$$T(2\tau^2, x) \ge \frac{1}{4}\chi_{B(x_0,2)}(x).$$

Applying this argument again, with initial datum $T(2\tau^2, x)$ instead of $T_0(x)$, we obtain

$$T(4\tau^2, x) \ge \frac{1}{4}\chi_{B(x_0,3)}(x)$$

because $T(2\tau^2, x) \ge \frac{1}{4}\chi_{B(x_1,1)}(x)$ for any $x_1 \in B(x_0, 1)$. Iteration of this gives us

$$T(2\tau^2 j, x) \ge \frac{1}{4}\chi_{B(x_0,j)}(x).$$

This holds for any $\tau \in [\tau_1, 2\tau_1]$ and it follows that

$$T(\gamma t, x) \ge \frac{1}{4}\chi_{B(0,t)}(x). \tag{3.6}$$

for $\gamma \equiv 16\tau_1^2$ and $t \ge |x_0|$.

The proof of (3.1) will be finished by yet another application of the above argument. Let $\varepsilon > 0$ be arbitrary and let $c_\varepsilon > 0$ be such that $f(T) \ge c_\varepsilon T^{1+\alpha}$ for $0 \le T \le 1 - \varepsilon$. We will show

$$\liminf_{t\to\infty} \inf_{|x|\le t} T(2\gamma t + s_\varepsilon, x) \ge 1 - 2\varepsilon \tag{3.7}$$

for some $s_\varepsilon < \infty$, which will imply (3.1) with, for instance, $\gamma_1 \equiv (3\gamma)^{-1}$.

Let Φ solve (2.1) with initial condition $\Phi_0 \equiv \frac{1}{4}\chi_{B(0,2t)}$ for some $t \geq |x_0|$, and let

$$\tilde{T}(s,x) \equiv (\Phi(s,x)^{-\alpha} - c_\varepsilon \alpha s)^{-1/\alpha}. \tag{3.8}$$

Since $\Phi \leq \frac{1}{4}$, up to time $s_\varepsilon \equiv (4^\alpha - (1-\varepsilon)^{-\alpha})/c_\varepsilon\alpha$ we have $\tilde{T} \leq 1-\varepsilon$, and so by the proof of Lemma 2.1, (3.6), and comparison theorems, $\tilde{T}(s_\varepsilon, x) \leq T(2\gamma t + s_\varepsilon, x)$. For all $x \in B(0,t)$ we have by (3.3),

$$\Phi(s_\varepsilon, x) \geq \frac{1}{4}\left(1 - \int_{\mathbb{R}^n \setminus B(0,t)} C s_\varepsilon^{-n/2} e^{-|y|^2/Cs_\varepsilon} dy\right) > \left(4^\alpha + (1-2\varepsilon)^{-\alpha} - (1-\varepsilon)^{-\alpha}\right)^{-1/\alpha}$$

if t is large enough. Plugging this into (3.8), we obtain $\inf_{|x|\leq t} \tilde{T}(s_\varepsilon, x) \geq 1-2\varepsilon$ (for any large t). This gives (3.7), and (3.1) is proved.

We are left with (3.2). Since f is Lipschitz, there is d such that $f(T) \leq dT$. Then by the maximum principle,

$$T(t,x) \leq e^{dt}\Phi(t,x),$$

with Φ solving (2.2) and $\Phi_0 \equiv T_0$ compactly supported. By (3.3),

$$\Phi(t, x - \bar{b}t) \leq \int_{\mathbb{R}^n \times \mathbb{T}^m} C t^{-n/2} e^{-|x-y|_*^2/Ct} \Phi_0(y) dy,$$

which is less than $t^{-n/2}e^{-dt}$ whenever $|x|_* \geq \sqrt{2Cd}\, t$ and t is large. The proof is finished. □

Proof of Theorem 1.1. (i) Eq. (1.4) follows from the same result for combustion with ignition temperature cutoff $\theta_0 < \eta$ [19] and comparison theorems. Eq. (1.5) is just (3.2) (which holds for any Lipschitz f with $f(0) = 0$) because by the remark after Definition 2.2, $\bar{b}$ is the mean of u. Note that the difference in sign results from u having opposite signs in (1.1) and (2.1).

(ii), (iii) Follow directly from Theorem 3.1(i),(ii) with $\alpha \equiv p-1$ and $n \equiv 1$. □

Now we turn to the proof of Theorem 1.3. Hence T and Φ will be the solutions of

$$T_t = \Delta T + Au(y)T_x + f(T) \tag{3.9}$$

$$\Phi_t = \Delta\Phi + Au(y)\Phi_x \tag{3.10}$$

in $\mathbb{R} \times \mathbb{T}^m$. We will consider the initial condition

$$T_0(x,y) \equiv \Phi_0(x,y) \equiv \chi_{[-L,L]}(x), \tag{3.11}$$

since by comparison theorems, u is quenching if and only if for every L the solution T quenches when $|A|$ is large enough. We will again use Lemma 2.1 but to prove part (ii) we need to estimate the decay of Φ

without the help of Lemma 2.3, since the constants in it may not be uniform in A.

Instead, we express the solution of (3.10) in terms of the Brownian motion. Following [11] we obtain $\Phi(t,x,y) = \mathbb{E}(\Phi(0, X_t^x, Y_t^y))$ where $\mathbb{E}$ is the expectation with respect to the random process (X_t^x, Y_t^y) starting at (x,y) and satisfying

$$\begin{aligned} dX_t^x &= \sqrt{2}\, dW_t^x + Au(Y_t^y)dt, \\ dY_t^y &= \sqrt{2}\, dW_t^y. \end{aligned}$$

Here (W_t^x, W_t^y) is the normalized Brownian motion on $\mathbb{R}\times\mathbb{T}^m$ starting at (x,y). Thus, $Y_t^y = y + \sqrt{2}(W_t^y - y) = W_{2t}^y$ and

$$X_t^x = x + \sqrt{2}(W_t^x - x) + \int_0^t Au(Y_s^y)ds = W_{2t}^x + \frac{A}{2}\int_0^{2t} u(W_s^y)ds.$$

Then we have by (3.10), (3.11), and Lemma 7.8 in [15],

$$\Phi(t,x,y) = \mathbb{P}\left(W_{2t}^x + \frac{A}{2}\int_0^{2t} u(W_s^y)ds \in [-L, L] \right). \tag{3.12}$$

To evaluate this probability we employ Lemmas 2.1 and 2.2 in [11]. There they are proved for $m = 1$ but the proof in the general case is identical.

Lemma 3.2 (Kiselev-Zlatoš)**.** *If $u \in C^1(\mathbb{T}^m)$ and $S \subset (0,\infty)$ is compact, then*

$$\lim_{\varepsilon\to 0} \sup_{(t,a,y)\in S\times\mathbb{R}\times\mathbb{T}^m} \mathbb{P}\left(\int_0^{2t} u(W_s^y)ds \in [a, a+\varepsilon] \setminus \{2tu(y)\} \right) = 0,$$

while $\mathbb{P}(\int_0^{2t} u(W_s^y)ds = 2tu(y))$ equals the probability of $\{W_s^y\}_{s\in[0,2t]}$ staying entirely inside a plateau of u, and is zero unless y is in the interior of a plateau.

In other words, if $0 < t_0 < t_1 < \infty$, then by making A large, $\Phi(t,x,y)$ can be made as small as we want for $t \in [t_0, t_1]$ and y not in a plateau of u, since (3.12) and independence of W_{2t}^x and W_s^y imply

$$\Phi(t,x,y) \le \sup_{a\in\mathbb{R}} \mathbb{P}\left(\int_0^{2t} u(W_s^y)ds \in \left[a, a + \frac{4L}{A}\right] \right). \tag{3.13}$$

This is in line with the intuition that, outside of plateaux of u, strong wind quickly extinguishes the flame by stretching it and exposing it to diffusion [5, 11]. This takes care of estimating $\Phi(t,x,y)$ within any finite time interval and for y not in a plateau. When y is inside a plateau, we also need the following estimate.

Lemma 3.3. *Let $y \in B(0,\varepsilon) \subseteq \mathbb{T}^m$ and $C \equiv 4/\pi$. Then*

$$\mathbb{P}\big(W_s^y \in B(0,\varepsilon) \text{ for all } s \in [0,2t]\big) \le Ce^{-\pi^2 t/4\varepsilon^2}.$$

Proof. This probability is obviously largest when $m = 1$ and $y = 0$. The Feynman-Kac formula says that this is

$$\big(e^{-2tH_\varepsilon}\chi\big)(0),$$

where $H_\varepsilon \equiv -\frac{1}{2}\Delta$ on $[-\varepsilon,\varepsilon]$ with Dirichlet boundary conditions at $\pm\varepsilon$ and $\chi(y) \equiv \chi_{[-\varepsilon,\varepsilon]}(y)$. Since on $[-\varepsilon,\varepsilon]$

$$\chi(y) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)\pi} \cos\left(\frac{(2n+1)\pi}{2\varepsilon} y\right)$$

and

$$H_\varepsilon \cos\left(\frac{(2n+1)\pi}{2\varepsilon} y\right) = \frac{(2n+1)^2\pi^2}{8\varepsilon^2} \cos\left(\frac{(2n+1)\pi}{2\varepsilon} y\right),$$

we get

$$\big(e^{-2tH_\varepsilon}\chi\big)(0) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)\pi} e^{-2t(2n+1)^2\pi^2/8\varepsilon^2} \le \frac{4}{\pi} e^{-t\pi^2/4\varepsilon^2},$$

since the sum is alternating. □

Finally, large t can be handled by the estimate

$$\Phi(t,x,y) \le \sup_{a\in\mathbb{R}} \mathbb{P}\big(W_{2t}^x \in [a, a+2L]\big) \le Dt^{-1/2} \tag{3.14}$$

for $D \equiv L\pi^{-1/2}$. The first inequality again follows from (3.12) and the independence of W_{2t}^x and W_s^y, the second because the density function of the random variable W_{2t}^x is $\varphi(z) = (4\pi t)^{-1/2} e^{-|z-x|^2/4t} \le (4\pi t)^{-1/2}$.

Proof of Theorem 1.3. (i) For $m = 1$ this is a result from [5], where a radially symmetric subsolution of (2.1) supported on $\mathbb{R} \times I$ (for some plateau I) is constructed using Bessel functions. When $m \ge 2$, the same construction can be applied, with an extra technical difficulty. This stems from the fact that the fundamental solution of $\Delta T = 0$ in $\mathbb{R}^n$ is bounded below when $n \ge 3$. It can be overcome and the result will follow.

(ii) Again we can change c to get $f(T) \le cT^{1+\alpha}$ for all T, with $\alpha \equiv p-1$. First assume that u has no plateaux and let $0 < t_0 < t_1 < \infty$ and

$$s \equiv \sup_{t\in[t_0,t_1]} \|\Phi(t,\cdot,\cdot)\|_\infty^\alpha.$$

Then by $\|\Phi(t,\cdot,\cdot)\|_\infty \le 1$ and (3.14) we have (with $D \equiv L\pi^{-1/2}$)

$$I \equiv \int_0^\infty \|\Phi(t,\cdot,\cdot)\|_\infty^\alpha \, dt \le t_0 + s(t_1 - t_0) + \int_{t_1}^\infty D^\alpha t^{-\alpha/2} \, dt.$$

Now (3.13) and Lemma 3.2 show that by taking $|A|$ large, on can make t_0, t_1^{-1}, and s small enough so that $I < (c\alpha)^{-1}$. The result then follows by taking $\delta_0 \equiv 1$ in Lemma 2.1(i).

Let us now assume that u has plateaux, each contained in a ball of radius less than $\varepsilon < \frac{\pi}{2} d^{-1/2}$, where d is such that $f(T) \le dT$. Such d exists because $f(0) = 0$ and f is Lipschitz. Define $\omega \equiv \pi^2/4\varepsilon^2 - d > 0$ and let $0 < t_0 < t_1 < \infty$ be such that $\int_{t_0}^\infty (2Ce^{-\omega t})^\alpha dt < (2c\alpha)^{-1}$ (with $C \equiv 4/\pi$) and $\int_{t_1}^\infty (e^{dt_0} D t^{-1/2})^\alpha dt < (2c\alpha)^{-1}$. Lemmas 3.2 and 3.3 show that if $|A|$ is large enough, then

$$\|\Phi(t,\cdot,\cdot)\|_\infty \le 2Ce^{-\pi^2 t/4\varepsilon^2} \tag{3.15}$$

for $t \in [t_0, t_1]$. By the maximum principle, $T(t,x,y) \le e^{dt}\Phi(t,x,y)$ and so with $\tilde{T}(t,x,y) \equiv T(t+t_0,x,y)$ and $\tilde{\Phi}(t,x,y) \equiv e^{dt_0}\Phi(t+t_0,x,y)$ we have $0 \le \tilde{T}(0,x,y) \le \tilde{\Phi}(0,x,y)$. By (3.14) and (3.15), we also have

$$\begin{aligned}
\int_0^\infty \|\tilde{\Phi}(t,\cdot,\cdot)\|_\infty^\alpha dt &= \int_{t_0}^\infty \Big(e^{dt_0} \|\Phi(t,\cdot,\cdot)\|_\infty \Big)^\alpha dt \\
&\le \int_{t_0}^{t_1} \Big(2Ce^{-\omega t} \Big)^\alpha dt + \int_{t_1}^\infty \Big(e^{dt_0} D t^{-1/2} \Big)^\alpha dt \\
&< (c\alpha)^{-1}.
\end{aligned}$$

Now Lemma 2.1(i) with $\delta_0 \equiv 1$ gives $\lim_{t\to\infty} \|\tilde{T}(t,\cdot,\cdot)\|_\infty = 0$, and so the same holds for T.

(iii) Follows from Theorem 3.1(ii) with $\alpha \equiv p - 1$ and $n \equiv 1$. □

4. Conclusion

In the present paper we studied the phenomenon of quenching (extinction of reaction) in reaction-diffusion-advection equations. Previous works on the topic dealt with reaction terms with ignition temperature cutoff, thus removing the so-called *cold boundary difficulty* in the analytic treatment of the problem. We dispense with this constraint and, in particular, our results apply to *Arrhenius reaction* terms which often appear in physical and chemical literature.

We show that quenching happens in the presence of a periodic flow when the initial datum is small in L^1 norm (Theorem 1.1), as well as in the presence of a shear flow without plateaux when the initial datum is just compactly supported but the flow strength is comparatively large (Theorem 1.3), both given that the reaction strength is small at low temperatures. On the other hand, quenching never happens when this strength is not small (this is called *hair trigger effect*). We exhibit a sharp transition between these two modes, represented by reaction functions that have at low temperatures a power law behavior with respect to the temperature, with a precisely computed power.

Our main contribution is in obtaining long time control of the solutions of the treated non-linear equations by employing good intermediate and large time estimates on the associated (linear) passive scalar evolution.

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