

Local regularity for the modified SQG patch equation

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Abstract

We study the patch dynamics for a family of active scalars called modified SQG equations, on the whole plane and on the half-plane. These involve a parameter α which appears in the power of the kernel in their Biot-Savart laws and describes the degree of regularity of the equation. The values $\alpha = 0$ and $\alpha = \frac{1}{2}$ correspond to the 2D Euler and SQG equations, respectively. We establish here local-in-time existence and uniqueness results for these models, for all $\alpha \in (0, \frac{1}{2})$ on the whole plane and for all small $\alpha > 0$ on the half-plane. The main novelty of this work is both in showing existence of local patch solutions on the half-plane and in proving their uniqueness on both domains.

1 Introduction

Two of the most important models in two-dimensional fluid dynamics are the (incompressible) 2D Euler equation, modeling motion of inviscid fluids, and the surface quasi-geostrophic (SQG) equation, which is used in atmospheric science models, appearing for instance in Pedlosky [24]. In the mathematical literature, the SQG equation was first

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discussed in the work of Constantin, Majda, and Tabak [6]. Both these equations (the former in the vorticity formulation) can be written in the form

$$\partial_t \omega + (u \cdot \nabla) \omega = 0, \tag{1.1}$$

along with initial condition $\omega(\cdot, 0) = \omega_0$ and the Biot-Savart law $u := \nabla^\perp (-\Delta)^{-1+\alpha} \omega$. Here $\nabla^\perp := (\partial_{x_2}, -\partial_{x_1})$, and the Euler and SQG cases are obtained by taking $\alpha = 0$ and $\alpha = \frac{1}{2}$, respectively. Note that the Biot-Savart law for the 2D Euler equation is therefore more regular (by one derivative) than that of the SQG equation.

Global regularity of solutions to the 2D Euler equation has been known since the works of Wolibner [30] and Hölder [15]. The necessary estimates barely close, and the upper bound on the growth of the derivatives of the vorticity is double exponential in time. Recently, Kiselev and Šverák showed that this upper bound is sharp by constructing an example of a solution to the 2D Euler equation on a disk whose gradient indeed grows double exponentially in time [18]. Some earlier examples of unbounded growth are due to Yudovich [16, 31], Nadirashvili [23], and Denisov [9, 10], and exponential growth on a domain without a boundary (the torus \mathbb{T}^2) was recently shown to be possible by Zlatoš [33]. On the other hand, while existence of global weak solutions for the SQG equation (which shares many of its features with the 3D Euler equation — see, e.g., [6, 20, 28]) was proved by Resnick [27], the global regularity vs finite time blow-up question for it is a major open problem.

Both the 2D Euler and SQG equations belong to the class of active scalars, equations of the form (1.1) where the fluid velocity is determined from the advected scalar ω itself. A natural family of active scalars which interpolates between the 2D Euler and SQG equations is given by (1.1) with $\alpha \in (0, \frac{1}{2})$ in the above Biot-Savart law. This family has been called modified or generalized SQG equations in the literature (see, e.g., Constantin et al [5], or Pierrehumbert et al [25] and Smith et al [29] for geophysical literature references). The global regularity vs finite time blow-up question is still open for all $\alpha > 0$.

While the above works studied active scalars with sufficiently smooth initial data, an important class of solutions to these equations arises from rougher initial data. Of particular interest is the case of characteristic functions of domains with smooth boundaries, or more generally, sums of characteristic functions of such domains multiplied by some coupling constants. Solutions originating from such initial data are called vortex patches, and they model flows with abrupt variations in their vorticity. The latter, including hurricanes and tornados, are common in nature. Existence and uniqueness of vortex patch solutions to the 2D Euler equation on the whole plane goes back to the work of Yudovich [32], and regularity in this setting refers to a sufficient smoothness of the patch boundaries as well as to a lack of both self-intersections of each patch boundary and

touching of different patches.

The vortex patch problem can be viewed as an interface evolution problem, and singularity formation for 2D Euler patches had initially been conjectured by Majda [19] based on relevant numerical simulations by Buttké [2]. Later, simulations by Dritschel, McIntyre, and Zabusky [11, 12] questioning the singularity formation prediction appeared, and we refer to [26] for a review of these and related works. This controversy was settled in 1993, when Chemin [4] proved that the boundary of a 2D Euler patch remains regular for all times, with a double exponential upper bound on the temporal growth of its curvature (see also the work by Bertozzi and Constantin [1] for a different proof).

The patch problem for the SQG equation is comparatively more involved to set up rigorously. Local existence and uniqueness in the class of weak solutions of the special type $\omega(x, t) = \chi_{\{x_2 < \varphi(x_1, t)\}}$ with $\varphi \in C^\infty$ and periodic in x_1 , corresponding to (single patch) initial data with the same property, was proved by Rodrigo [28]. For SQG and modified SQG patches with boundaries which are simple closed H^3 curves, local existence was obtained by Gancedo [13] via solving a related contour equation whose solutions are some parametrizations of the patch boundary (uniqueness of solutions was also proved for the contour equation when $\alpha \in (0, \frac{1}{2})$, although not for the original modified SQG patch equation). Local existence of such contour solutions in the more singular case $\alpha \in (\frac{1}{2}, 1]$ was obtained by Chae, Constantin, Córdoba, Gancedo, and Wu [3]. Finally, existence of splash singularities (touch of exactly two segments of a patch boundary, which remains uniformly H^3) for the SQG equation was ruled out by Gancedo and Strain [14].

A computational study of the SQG and modified SQG patches by Córdoba, Fontelos, Mancho, and Rodrigo [8] (where the patch problem for the modified SQG equation first appeared) suggested finite time singularity formation, with two patches touching each other and simultaneously developing corners at the point of touch. A more careful numerical study by Mancho [21] suggests self-similar elements involved in this singularity formation process, but its rigorous confirmation and understanding is still lacking.

In this paper, we consider the patch evolution for the modified SQG equations, both on the whole plane and on the half-plane, and prove local-in-time existence and uniqueness for these models (for all $\alpha \in (0, \frac{1}{2})$ on the plane and for all sufficiently small $\alpha > 0$ on the half-plane). Our motivation is, in fact, primarily the half-plane case because in the companion paper [17] we show existence of finite time blow-up for patch solutions to the modified SQG equation with small $\alpha > 0$ on the half-plane. To the best of our knowledge, this is the first rigorous proof of finite time blow-up in this type of fluid dynamics models.

In order to obtain a proper blow-up result, we first need to show local well-posedness of these models and this is our primary focus here. In the fluid interface models, even

local well-posedness results are sometimes far from trivial, see for instance the recent treatment of the Muskat problem in [7]. This is also true for the modified SQG patch equation. In the half-plane case, we will obtain local existence in a natural Sobolev space H^3 for all $\alpha \in (0, \frac{1}{24})$. The proof of this existence result follows the whole plane approach in [13], although the presence of the boundary creates additional significant difficulties. These, in particular, limit the range of the α which we can handle. The second main contribution of this paper is our proposal of a new natural definition of patch solutions to the modified SQG equations. This definition applies to the solutions studied in earlier works but is more general, and it is also not linked to the specific choice of the contour equation. Finally, we also prove uniqueness of patch solutions in this more general class. The main difficulty one has to contend with in this effort is the lack of regularity of the velocity field inherent in models more singular than the 2D Euler equations. It is quite straightforward to check that the fluid velocity is only Hölder regular near the patch boundaries and does not possess higher regularity. However, we are able to overcome this complication by using a novel uniqueness argument, which may have further applications in other models as well.

Let us now turn to the specifics of the model we will study. We only consider here the case $\alpha \in (0, \frac{1}{2})$, and we concentrate on the half-plane case $D := \mathbb{R} \times \mathbb{R}^+$. This is both because this case is our main motivation, and because the proofs are more involved (in fact, the whole-plane proofs are essentially contained in the half-plane ones). The corresponding patch evolution can then be formally defined via the Biot-Savart law

$$u(x, t) := \int_D \left(\frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} - \frac{(x - \bar{y})^\perp}{|x - \bar{y}|^{2+2\alpha}} \right) \omega(y, t) dy \quad (1.2)$$

for $x \in \bar{D}$, along with the requirement that ω is advected by the flow given by u , that is,

$$\omega(x, t) = \omega(\Phi_t^{-1}(x), 0), \quad (1.3)$$

where

$$\frac{d}{dt} \Phi_t(x) = u(\Phi_t(x), t) \quad \text{and} \quad \Phi_0(x) = x. \quad (1.4)$$

Here $v^\perp := (v_2, -v_1)$ and $\bar{v} := (v_1, -v_2)$ for $v = (v_1, v_2)$, and we note that the integral in (1.2) equals $\nabla^\perp(-\Delta)^{-1+\alpha}\omega$ (up to a positive pre-factor, which can be dropped without loss due to scaling), with the Dirichlet Laplacian on D . The vector field u is then divergence free and tangential to the boundary ∂D (i.e., $u_2(x, t) = 0$ when $x_2 = 0$).

We have to be careful, however, with the rigorous definition of the evolution because the low regularity of the fluid velocity u need not allow for a unique definition of trajectories from (1.4) when $\alpha > 0$ (existence will not be an issue here because u is continuous for $\alpha < \frac{1}{2}$). We introduce here the following Definition 1.2 which, as we discuss below,

encompasses various previously used definitions. We start with a definition of some norms of boundaries of domains in \mathbb{R}^2 , letting here $\mathbb{T} := [-\pi, \pi]$ with $\pm\pi$ identified.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set whose boundary $\partial\Omega$ is a simple closed C^1 curve with arc-length $|\partial\Omega|$. We call a constant speed parametrization of $\partial\Omega$ any counter-clockwise parametrization $z : \mathbb{T} \rightarrow \mathbb{R}^2$ of $\partial\Omega$ with $|z'| \equiv \frac{|\partial\Omega|}{2\pi}$ on \mathbb{T} (all such z are translations of each other), and we define $\|\Omega\|_{C^{m,\gamma}} := \|z\|_{C^{m,\gamma}}$ and $\|\Omega\|_{H^m} := \|z\|_{H^m}$.

Remark. It is not difficult to see (using Lemma 3.4 below), that an Ω as above satisfies $\|\Omega\|_{C^{m,\gamma}} < \infty$ (resp. $\|\Omega\|_{H^m} < \infty$) precisely when for some $r > 0$, $M < \infty$, and each $x \in \partial\Omega$, the set $\partial\Omega \cap B(x, r)$ is (in the coordinate system centered at x and with axes given by the tangent and normal vectors to $\partial\Omega$ at x) the graph of a function with $C^{m,\gamma}$ (resp. H^m) norm less than M .

Next, let $d_H(\Gamma, \tilde{\Gamma})$ be the Hausdorff distance of sets $\Gamma, \tilde{\Gamma}$, and for a set $\Gamma \subseteq \mathbb{R}^2$, vector field $v : \Gamma \rightarrow \mathbb{R}^2$, and $h \in \mathbb{R}$, we let

$$X_v^h[\Gamma] := \{x + hv(x) : x \in \Gamma\}.$$

Our definition of patch solutions to (1.1)-(1.2) on the half-plane is now as follows.

Definition 1.2. Let $D := \mathbb{R} \times \mathbb{R}^+$, let $\theta_1, \dots, \theta_N \in \mathbb{R} \setminus \{0\}$, and for each $t \in [0, T)$, let $\Omega_1(t), \dots, \Omega_N(t) \subseteq D$ be bounded open sets whose boundaries $\partial\Omega_k(t)$ are pairwise disjoint simple closed curves, such that each $\partial\Omega_k(t)$ is also continuous in $t \in [0, T)$ with respect to d_H . Denote $\partial\Omega(t) := \bigcup_{k=1}^N \partial\Omega_k(t)$ and let

$$\omega(\cdot, t) := \sum_{k=1}^N \theta_k \chi_{\Omega_k(t)}. \quad (1.5)$$

If for each $t \in (0, T)$ we have

$$\lim_{h \rightarrow 0} \frac{d_H\left(\partial\Omega(t+h), X_{u(\cdot, t)}^h[\partial\Omega(t)]\right)}{h} = 0, \quad (1.6)$$

with u from (1.2), then ω is a patch solution to (1.1)-(1.2) on the time interval $[0, T)$. If we also have $\sup_{t \in [0, T']}] \|\Omega_k(t)\|_{C^{m,\gamma}} < \infty$ (resp. $\sup_{t \in [0, T']}] \|\Omega_k(t)\|_{H^m} < \infty$) for each k and $T' \in (0, T)$, then ω is a $C^{m,\gamma}$ (resp. H^m) patch solution to (1.1)-(1.2) on $[0, T)$.

Remarks. 1. Continuity of u (which is not hard to show, see the last claim in the elementary Lemma 3.1 below) and (1.6) mean that for patch solutions, $\partial\Omega$ is moving with velocity $u(x, t)$ at $t \in [0, T)$ and $x \in \partial\Omega(t)$.

2. We note that our definition encompasses well-known definitions for the 2D Euler equation in terms of (1.4) and in terms of the normal velocity at $\partial\Omega$. Indeed, if ω satisfies $\partial\Omega_k(t) = \Phi_t(\partial\Omega_k(0))$ for each k and $t \in [0, T)$ and the patch boundaries remain pairwise disjoint simple closed curves, then continuity of u , compactness of $\partial\Omega(t)$, and (1.4) show that ω is a patch solution to (1.1)-(1.2) on $[0, T)$. Moreover, if $\partial\Omega(t)$ is C^1 and $n_{x,t}$ is the outer unit normal vector at $x \in \partial\Omega(t)$, then it is easy to see that (1.6) is equivalent to motion of $\partial\Omega(t)$ with outer normal velocity $u(x, t) \cdot n_{x,t}$ at each $x \in \partial\Omega(t)$ (which can be defined in a natural way by (1.6) with $u(\cdot, t)$ replaced by $(u(\cdot, t) \cdot n_{\cdot, t})n_{\cdot, t}$). However, Definition 1.2 can be stated even if $\Phi_t(x)$ cannot be uniquely defined for some $x \in \partial\Omega(0)$ (when $\alpha > 0$, this might even be the case for $x \notin \partial\Omega(0)$, as the hypotheses of Proposition 1.3(a) below suggest) or when $\partial\Omega(t)$ is not C^1 .

3. As we show at the end of this introduction, C^1 patch solutions to (1.1)-(1.2) are also weak solutions to (1.1) in the sense that for each $f \in C^1(\bar{D})$ we have

$$\frac{d}{dt} \int_D \omega(x, t) f(x) dx = \int_D \omega(x, t) [u(x, t) \cdot \nabla f(x)] dx \quad (1.7)$$

for all $t \in (0, T)$, with both sides continuous in t . Also, weak solutions to (1.1)-(1.2) which are of the form (1.5) and have C^1 boundaries $\partial\Omega_k(t)$ which move with some continuous velocity $v : \mathbb{R}^2 \times (0, T) \rightarrow \mathbb{R}^2$ (in the sense of (1.6) with v in place of u), do satisfy (1.6) with u (hence they are patch solutions if those boundaries remain pairwise disjoint simple closed curves). Finally, $|\Omega_k(t)| = |\Omega_k(0)|$ holds for each k and $t \in [0, T)$.

4. In the 2D Euler case $\alpha = 0$, it is not difficult to show using standard results of Yudovich theory that if $\omega(x, 0) = \omega_0(x)$ is as in Definition 1.2, then there exists a unique global weak solution ω to (1.1), and it is of the form (1.5), with $\partial\Omega_k(t) = \Phi_t(\partial\Omega_k(0))$. Remark 2 then shows that if the patch boundaries remain disjoint simple closed curves, ω is also a patch solution to (1.1)-(1.2) on $[0, \infty)$. Moreover, ω must be unique in the class of C^1 patch solutions (if it belongs there) because these are also weak solutions. In [17] we prove that the $C^{1,\gamma}$ patch solutions in the 2D Euler case are globally regular. Therefore, since the Euler case is well-understood, we will only consider $\alpha > 0$ here.

Note that Definition 1.2 automatically requires patch boundaries to not touch each other or themselves. If this happens, the solution develops a singularity. Also note that the definition allows for, e.g., $\Omega_2(t) \subseteq \Omega_1(t)$ and $\theta_2 = -\theta_1$, and then $\sum_{k=1}^2 \theta_k \chi_{\Omega_k(t)}$ represents a non-simply connected patch. Finally, we will say that ω is a patch solution to (1.1)-(1.2) on $[0, T]$ if it is a patch solution to (1.1)-(1.2) on $[0, T')$ for some $T' > T$.

Before we turn to our main results, let us address the relationship of the flow maps from (1.4) to the patch solution definition (1.6). Note that since u is smooth away from $\partial\Omega$ (see Lemma 3.2 below), $\Phi_t(x)$ remains unique at least until it hits $\partial\Omega$ (in the Euler

case, $\Phi_t(x)$ is always unique because u is log-Lipschitz), after which it still exists but need not be unique. The following result shows, in particular, that for $\alpha < \frac{1}{4}$ and patch solutions with sufficiently smooth boundaries, this remains true for any $x \in \bar{D} \setminus \partial\Omega(0)$ until time T .

Proposition 1.3. *Let ω be as in the first paragraph of Definition 1.2. For $x \in \bar{D} \setminus \partial\Omega(0)$, let $t_x \in [0, T]$ be the maximal time such that the solution of (1.4) with u from (1.2) satisfies $\Phi_t(x) \in \bar{D} \setminus \partial\Omega(t)$ for each $t \in [0, t_x]$. Then we have the following.*

- (a) *If $\alpha \in (0, \frac{1}{4})$, $\gamma \in (\frac{2\alpha}{1-2\alpha}, 1]$, and ω is a $C^{1,\gamma}$ patch solution to (1.1)-(1.2) on $[0, T]$, then $t_x = T$ for each $x \in \bar{D} \setminus \partial\Omega(0)$ and $\Phi_t : [\bar{D} \setminus \partial\Omega(0)] \rightarrow [\bar{D} \setminus \partial\Omega(t)]$ is a bijection for each $t \in [0, T]$.*
- (b) *If $\alpha \in (0, \frac{1}{2})$, $t_x = T$ for each $x \in \bar{D} \setminus \partial\Omega(0)$, and $\Phi_t : [\bar{D} \setminus \partial\Omega(0)] \rightarrow [\bar{D} \setminus \partial\Omega(t)]$ is a bijection for each $t \in [0, T]$, then ω is a patch solution to (1.1)-(1.2) on $[0, T]$. Moreover, Φ_t is measure preserving on $\bar{D} \setminus \partial\Omega(0)$ and it also preserves the connected components of $\bar{D} \setminus \partial\Omega$. Finally, we have*

$$\Phi_t(\partial\Omega_k(0)) = \partial\Omega_k(t) \tag{1.8}$$

for each k and $t \in [0, T]$, in the sense that any solution of (1.4) with $x \in \partial\Omega_k(0)$ has $\Phi_t(x) \in \partial\Omega_k(t)$, as well as that for each $y \in \partial\Omega_k(t)$, there is $x \in \partial\Omega_k(0)$ and a solution of (1.4) such that $\Phi_t(x) = y$.

Remarks. 1. Since $H^3(\mathbb{T}) \subseteq C^{1,1}(\mathbb{T})$, we see that when $\alpha < \frac{1}{4}$, this result applies to the H^3 patch solutions from our main result below.

2. We do not know whether this result holds for $\gamma \leq \frac{2\alpha}{1-2\alpha}$.

Let us call the initial data ω_0 for the problem (1.1)-(1.2) *patch-like* if

$$\omega_0 = \sum_{k=1}^N \theta_k \chi_{\Omega_{0k}},$$

with $\theta_1, \dots, \theta_N \in \mathbb{R} \setminus \{0\}$ and $\Omega_{01}, \dots, \Omega_{0N} \subseteq D$ bounded open sets whose boundaries are pairwise disjoint simple closed curves. That is, ω_0 is as $\omega(\cdot, 0)$ in Definition 1.2. Notice also that if $\omega(\cdot, 0) = \sum_{k=1}^{N'} \theta'_k \chi_{\Omega_k(0)}$ is as in Definition 1.2 and $\omega(\cdot, 0) = \omega_0$, then $N' = N$, and (up to a permutation) $\theta'_k = \theta_k$ and $\Omega_k(0) = \Omega_{0k}$ for each k .

Here is our first main result, local existence and uniqueness of H^3 patch solutions to (1.1)-(1.2) on the half-plane $D = \mathbb{R} \times \mathbb{R}^+$ for small $\alpha > 0$. Recall that uniqueness for patch solutions with $\alpha > 0$ was previously only proved within a special class of SQG patches on \mathbb{R}^2 with C^∞ boundaries in [28].

Theorem 1.4. *Let $\alpha \in (0, \frac{1}{24})$. Then for each H^3 patch-like initial data ω_0 , there exists a unique local H^3 patch solution ω to (1.1)-(1.2) with $\omega(\cdot, 0) = \omega_0$. Moreover, if the maximal time T_ω of existence of ω is finite, then at T_ω either two patch boundaries touch, or a patch boundary touches itself, or a patch boundary loses H^3 regularity.*

Remarks. 1. The last claim means that either $\partial\Omega_k(T_\omega) \cap \partial\Omega_i(T_\omega) \neq \emptyset$ for some $k \neq i$, or $\partial\Omega_k(T_\omega)$ is not a simple closed curve for some k , or $\lim_{t \nearrow T_\omega} \|\Omega_k(t)\|_{H^3} = \infty$ for some k . Note that the sets $\partial\Omega_k(T_\omega) := \lim_{t \nearrow T_\omega} \partial\Omega_k(t)$ (with the limit taken in Hausdorff distance) are well defined if $T_\omega < \infty$ because u is uniformly bounded (see (3.1)). In fact, an argument from Lemma 4.10 yields $d_H(\partial\Omega(t), \partial\Omega(s)) \leq \|u\|_{L^\infty} |t - s|$ for $t, s \in [0, T_\omega]$.

2. The last claim further justifies our definition of H^3 patch solutions because it shows that a solution cannot stay regular up to (and including) the time T_ω but stop existing due to some artificial limitation stemming from the definition of solutions.

3. We show (see Corollary 4.7) that T_ω is bounded below by a constant depending on $\alpha, N, \|\omega\|_{L^\infty}$, and the quantity $\|\{\Omega_{0k}\}_{k=1}^N\|_{H^3}$ from Definition 4.6 (the latter expresses how close the initial patch boundaries are to touching each other or themselves, and how large their H^3 norms are).

4. Note that Remark 3 after Definition 1.2 shows that the above solution is also the unique weak solution to (1.1)-(1.2) from the class of functions which are of the form (1.5) and have H^3 boundaries $\partial\Omega_k(t)$ which are disjoint simple closed curves and move with some continuous velocity $v : \mathbb{R}^2 \times (0, T) \rightarrow \mathbb{R}^2$ (in the sense of (1.6) with v in place of u).

5. The hypothesis $\alpha < \frac{1}{24}$ may well be an artifact of the proof, as it only appears in the application of the technical Lemma 2.3 in the existence part. The rest of the proof applies to all $\alpha \in (0, \frac{1}{4})$, so it is possible that the result extends to at least this range.

As we mentioned above, our method also works on the whole plane, where non-existence of a boundary allows us to treat all $\alpha \in (0, \frac{1}{2})$. In this case we again use Definition 1.2 but with $D := \mathbb{R}^2$, and the flow is given by

$$u(x, t) = \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} \omega(y, t) dy \quad (1.9)$$

instead of (1.2). Our second main result is the corresponding version of Theorem 1.4.

Theorem 1.5. *With $D := \mathbb{R}^2$ and (1.2) replaced by (1.9), Proposition 1.3 holds as stated and Theorem 1.4 holds with $\alpha \in (0, \frac{1}{2})$.*

The paper is organized as follows. In Section 2 we derive a contour equation corresponding to the patch dynamics for (1.1)-(1.2) on the half-plane $D = \mathbb{R} \times \mathbb{R}^+$ and prove

that it is locally well-posed for all $\alpha \in (0, \frac{1}{24})$. The proof largely follows that in [13] for the whole plane, but the presence of the boundary ∂D will require us to introduce several non-trivial new arguments. We therefore present it in detail. In Section 3, we prove some auxiliary estimates on the fluid velocity and on the geometry of the boundaries of sufficiently regular patches. These are then used in Section 4 to show that the contour solution in fact yields a unique patch solution to (1.1)-(1.2), and Theorem 1.4 will follow. Some more delicate estimates on the fluid velocity generated by sufficiently regular patches are obtained in Section 5, and these are then used to prove Proposition 1.3. We conclude Section 5 and the paper with the proof of Theorem 1.5.

Proof of Remark 3 after Definition 1.2. Since $\nabla \cdot u = 0$ in $\bar{D} \setminus \partial\Omega(t)$ and u is continuous, the right-hand side of (1.7) is

$$\sum_{k=1}^N \int_{\partial\Omega_k(t)} \theta_k [u(x, t) \cdot n_{x,t}] f(x) d\sigma(x).$$

This equals the left-hand side of (1.7), which can be seen by noticing that the area of the rectangle with vertices $x, x', x' + hu(x', t), x + hu(x, t)$ is $|x - x'| [u(x, t) \cdot n_{x,t}] h + o(|h||x - x'|)$ if $x, x' \in \partial\Omega_k(t)$ (because u is continuous), and then taking x, x' to be successive points on a progressively finer mesh of $\partial\Omega_k(t)$ (as well as letting $h \rightarrow 0$). Finally, continuity of the right-hand side of (1.7) in time follows from continuity of $\partial\Omega_k$ in time in Hausdorff distance and from boundedness of $u \cdot \nabla f$, which is due to $f \in C^1(\bar{D})$ and (3.1) below.

Next, if a weak solution of the form (1.5) satisfies (1.6) with some continuous v in place of u , then the above computation applied to smooth characteristic functions f of successively smaller squares centered at any fixed $x \in \partial\Omega(t)$ yields

$$|x - x'| [v(x, t) \cdot n_{x,t}] h + o(|h||x - x'|) = |x - x'| [u(x, t) \cdot n_{x,t}] h + o(|h||x - x'|).$$

Hence $v(x, t) \cdot n_{x,t} = u(x, t) \cdot n_{x,t}$, so $\partial\Omega(t)$ being C^1 shows that (1.6) holds with u as well.

Now fix any $\tau \in [0, T]$ and let f in (1.7) be 1 on some open neighborhood of $\partial\Omega_k(\tau)$ and with its support having positive distance from $\partial\Omega(\tau) \setminus \partial\Omega_k(\tau)$. Then continuity of $\partial\Omega$ in t and $\nabla \cdot u = 0$ show that for all t close to τ , the right-hand side of (1.7) equals

$$\int_{\Omega_k(t)} \theta_k [u(x, t) \cdot \nabla f(x)] dx = \theta_k \int_{\partial\Omega_k(t)} [u(x, t) \cdot n_{x,t}] d\sigma(x) = 0.$$

Thus $\int_{\Omega_k(t)} f(x) dx$ is constant in all t near τ because it equals $\theta_k^{-1} \int_D \omega(x, t) f(x) dx$. Since also $\int_{\Omega_k(t)} (1 - f(x)) dx$ is constant in all t near τ (because the support of $1 - f(x)$ has positive distance from $\partial\Omega_k(\tau)$), we obtain that $|\Omega_k(t)|$ is constant on an open interval containing τ . This holds for all $\tau \in [0, T]$, hence $|\Omega_k(t)|$ is constant on $[0, T]$.

We note that for the 2D Euler equation, the definition of weak solutions via (1.7) can be found, for instance, in [22, Theorem 3.2], and that it easily implies

$$\int_D \omega(x, T')g(x, T')dx - \int_D \omega(x, 0)g(x, 0)dx = \int_{D \times (0, T')} \omega(x, t)[\partial_t g(x, t) + u(x, t) \cdot \nabla g(x, t)]dxdt$$

for all $T' \in [0, T)$ and $g \in C^1(\bar{D} \times [0, T'])$. \square

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2 Local regularity for a contour equation and small α

This section is the first step towards the proof of Theorem 1.4. We will derive a PDE whose solutions are time-dependent H^3 curves on the half-plane $\bar{D} = \mathbb{R} \times \mathbb{R}_0^+$, and one expects the latter to be some parametrizations of the patch boundaries $\partial\Omega_k(t)$. We will then prove local regularity for this *contour equation* in the main result of this section, Theorem 2.8 (which is the half-plane analog of its whole plane version from [13]). We will later show in Section 4, using some crucial estimates derived in Section 3, that the solutions of the contour equation indeed yield H^3 patch solutions to (1.1)-(1.2).

2.1 Derivation of the contour equation

Let us first derive our contour equation. Assuming that we have an H^3 patch solution to (1.1)-(1.2), let us parametrize the patch boundaries by

$$\partial\Omega_k(t) = \{z_k(\xi, t) = (z_k^1(\xi, t), z_k^2(\xi, t)) : \xi \in \mathbb{T}\} \subseteq \bar{D}, \quad (2.1)$$

with each $z_k(\cdot, t)$ running once counter-clockwise along $\partial\Omega_k(t)$. We do this so that at $t = 0$, the curves $z_k(\cdot, 0)$ all belong to $H^3(\mathbb{T})$, and are non-degenerate in the sense of the right-hand side of (2.8) below being finite for $t = 0$. Of course, even when all $z_k(\cdot, 0)$ are given, the choice of $z_k(\cdot, t)$ is not unique. Hence, we will have to be careful when choosing our contour equation for the z_k . While our choice is similar to the case of the whole plane in [13], the boundary ∂D creates some new terms, so we present the derivation below for the sake of completeness.

Let $x \in \partial\Omega_k(t)$ and let $n(x, t)$ denote the outer unit normal vector for $\Omega_k(t)$ at x . We use the Biot-Savart law to compute the outer normal velocity at x as follows:

$$\begin{aligned}
u_n(x, t) &= u(x, t) \cdot n(x, t) \\
&= - \sum_{i=1}^N \theta_i \int_{\Omega_i} \left[\frac{(x-y) \cdot n(x, t)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-\bar{y}) \cdot n(x, t)^\perp}{|x-\bar{y}|^{2+2\alpha}} \right] dy \quad (\text{since } u^\perp \cdot v = -u \cdot v^\perp) \\
&= - \sum_{i=1}^N \theta_i \int_{\Omega_i} \left[\frac{(x-y) \cdot n(x, t)^\perp}{|x-y|^{2+2\alpha}} - \frac{(\bar{x}-y) \cdot \overline{n(x, t)^\perp}}{|\bar{x}-y|^{2+2\alpha}} \right] dy \quad (\text{since } u \cdot \bar{v} = \bar{u} \cdot v) \\
&= - \sum_{i=1}^N \frac{\theta_i}{2\alpha} \int_{\Omega_i} \nabla_y \cdot \left[\frac{n(x, t)^\perp}{|x-y|^{2\alpha}} + \frac{\bar{n}(x, t)^\perp}{|\bar{x}-y|^{2\alpha}} \right] dy \quad (\text{since } \bar{n}^\perp = -\overline{n^\perp}) \\
&= \sum_{i=1}^N \frac{\theta_i}{2\alpha} \int_{\partial\Omega_i} \left[\frac{n(y, t)^\perp}{|x-y|^{2\alpha}} + \frac{\overline{n(y, t)^\perp}}{|x-\bar{y}|^{2\alpha}} \right] \cdot n(x, t) d\sigma(y).
\end{aligned} \tag{2.2}$$

Using (2.1), we conclude that the normal velocity at $x = z_k(\xi, t) \in \partial\Omega_k(t)$ is

$$u_n(x, t) = - \sum_{i=1}^N \frac{\theta_i}{2\alpha} \int_{\mathbb{T}} \left[\frac{\partial_\xi z_i(\xi - \eta, t)}{|z_k(\xi, t) - z_i(\xi - \eta, t)|^{2\alpha}} + \frac{\partial_\xi \bar{z}_i(\xi - \eta, t)}{|z_k(\xi, t) - \bar{z}_i(\xi - \eta, t)|^{2\alpha}} \right] \cdot n(x, t) d\eta. \tag{2.3}$$

Intuitively, one can add any multiple of the tangent vector $\partial_\xi z_k(\xi, t)$ to the velocity without changing the evolution of the patch itself (this does affect the particular parametrization z_k , though). Hence, we will use as the *contour equation* for $\partial\Omega_k(t)$ the equation

$$\partial_t z_k(\xi, t) = \sum_{i=1}^N \frac{\theta_i}{2\alpha} \int_{\mathbb{T}} \left[\frac{\partial_\xi z_k(\xi, t) - \partial_\xi z_i(\xi - \eta, t)}{|z_k(\xi, t) - z_i(\xi - \eta, t)|^{2\alpha}} + \frac{\partial_\xi z_k(\xi, t) - \partial_\xi \bar{z}_i(\xi - \eta, t)}{|z_k(\xi, t) - \bar{z}_i(\xi - \eta, t)|^{2\alpha}} \right] d\eta. \tag{2.4}$$

(This particular choice of the tangential component of $\partial_t z_k$ will allow us to derive (2.12) below.) To simplify the notation, we let $y_i^1 := z_i$ and $y_i^2 := \bar{z}_i$, so (2.4) becomes

$$\partial_t z_k(\xi, t) = \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} \int_{\mathbb{T}} \frac{\partial_\xi z_k(\xi, t) - \partial_\xi y_i^m(\xi - \eta, t)}{|z_k(\xi, t) - y_i^m(\xi - \eta, t)|^{2\alpha}} d\eta. \tag{2.5}$$

Note that while our v^\perp is the negative of v^\perp from [13], we have parametrized the curve $\partial\Omega_i$ in the opposite direction as well. Therefore our half-plane contour equation (2.5) is similar to that in the whole space $D = \mathbb{R}^2$ [13], which however only contains the $m = 1$ terms.

2.2 A priori estimates for the contour equation and small α

Let us first define some norms and functionals of the patch boundaries $Z(t) = \{z_k(\cdot, t)\}_{k=1}^N$ which we will need in order to establish local well-posedness for the contour equation:

$$\|Z(t)\|_{H^3}^2 := \sum_{k=1}^N \left(\|z_k(\cdot, t)\|_{L^\infty}^2 + \|\partial_\xi^3 z_k(\cdot, t)\|_{L^2}^2 \right). \quad (2.6)$$

$$\|Z(t)\|_{C^2} := \max_{1 \leq k \leq N} \max_{0 \leq j \leq 2} \|\partial_\xi^j z_k(\cdot, t)\|_{L^\infty},$$

$$\delta[Z(t)] := \min \left\{ \min_{i \neq k} \min_{\xi, \eta \in \mathbb{T}} |z_i(\xi, t) - z_k(\eta, t)|, 1 \right\}, \quad (2.7)$$

$$F[Z(t)] := \max \left\{ \max_{1 \leq k \leq N} \sup_{\xi, \eta \in \mathbb{T}, \eta \neq 0} \frac{|\eta|}{|z_k(\xi, t) - z_k(\xi - \eta, t)|}, 1 \right\}. \quad (2.8)$$

Note that the $H^3(\mathbb{T})$ -norm above is equivalent to the usual definition, where $\|z_k(\cdot, t)\|_{L^\infty}^2$ is replaced by $\|z_k(\cdot, t)\|_{L^2}^2$. We also let $\delta[Z(t)] = 1$ if $N = 1$. Finally, let us define

$$\| \| Z(t) \| \| := \|Z(t)\|_{H^3} + \delta[Z(t)]^{-1} + F[Z(t)]. \quad (2.9)$$

Note that $\| \| \cdot \| \|$ is not a norm, but this will not affect our arguments. Our goal is to obtain an a priori control on the growth of $\| \| Z(t) \| \|$ for smooth solutions. We will show that if $\alpha \in (0, \frac{1}{24})$ and $Z(t)$ solves (2.5) with $\| \| Z(0) \| \| < \infty$, then $\| \| Z(t) \| \|$ will remain finite for a short time. This follows from the main result here, the estimate (2.31) below. To prove it, we will now obtain bounds on the growth of the terms constituting $\| \| Z(t) \| \|$.

The evolution of $\|z_k(\cdot, t)\|_{L^\infty}$ and $\delta[Z(t)]^{-1}$

The evolution of these terms is controlled via the following lemma. (A better bound, on the velocity u rather than on $\partial_t z_k$, will be provided in Lemma 3.1 below. However, since we will also need to work with regularizations of (2.5), for which we do not have (1.2), Lemma 3.1 will not be sufficient here.) Let us denote $\Theta := \sum_{k=1}^N |\theta_k|$.

Lemma 2.1. *For $\alpha \in (0, \frac{1}{2})$ and $S_k[Z(t)](\xi)$ the right-hand side of (2.5) we have*

$$\|S_k[Z(t)]\|_{L^\infty} \leq \frac{4\pi\Theta}{\alpha(1-2\alpha)} (\delta[Z(t)]^{-1} + F[Z(t)])^{2\alpha} \|Z(t)\|_{C^2}.$$

Proof. In the integrands of $S_k[Z(t)](\xi)$, the numerators are bounded above by $2\|Z(t)\|_{C^2}$, and the denominators are bounded below by either $\delta[Z(t)]^{2\alpha}$ or $F[Z(t)]^{-2\alpha}|\eta|^{2\alpha}$. The claim now follows by a simple computation. \square

Thus

$$\left| \frac{d}{dt} \max_k \|z_k(\cdot, t)\|_{L^\infty} \right| \leq \frac{4\pi\Theta}{\alpha(1-2\alpha)} (\delta[Z(t)]^{-1} + F[Z(t)])^{2\alpha} \|Z(t)\|_{C^2}, \quad (2.10)$$

while $|\frac{d}{dt}\delta[Z(t)]|$ is bounded by twice that, so we have

$$\left| \frac{d}{dt} \delta[Z(t)]^{-1} \right| \leq \frac{8\pi\Theta}{\alpha(1-2\alpha)} (\delta[Z(t)]^{-1} + F[Z(t)])^{2+2\alpha} \|Z(t)\|_{C^2}. \quad (2.11)$$

The evolution of $\|\partial_\xi^3 z_k(\cdot, t)\|_{L^2}$

In the following computation, let us assume that each $z_k \in C^{4,1}(\mathbb{T} \times [0, T])$ for each $T < \infty$ (by which we mean that $\partial_{abcd}^4 z_k \in C(\mathbb{T} \times [0, T])$ whenever $a, b, c, d \in \{\xi, t\}$ and at most one of them is t). This will be sufficient because we will eventually apply the obtained estimates to a family of regularized solutions which do possess this regularity. Then for $1 \leq k \leq N$ we have

$$\frac{d}{dt} \|\partial_\xi^3 z_k(\cdot, t)\|_{L^2}^2 = \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{\alpha} \int_{\mathbb{T}^2} \partial_\xi^3 z_k(\xi, t) \cdot \partial_\xi^3 \left(\frac{\partial_\xi z_k(\xi, t) - \partial_\xi y_i^m(\xi - \eta, t)}{|z_k(\xi, t) - y_i^m(\xi - \eta, t)|^{2\alpha}} \right) d\eta d\xi. \quad (2.12)$$

Here we used that

$$\partial_\xi^3 \int_{\mathbb{T}} \frac{\partial_\xi z_k(\xi, t) - \partial_\xi y_i^m(\xi - \eta, t)}{|z_k(\xi, t) - y_i^m(\xi - \eta, t)|^{2\alpha}} d\eta = \int_{\mathbb{T}} \partial_\xi^3 \left(\frac{\partial_\xi z_k(\xi, t) - \partial_\xi y_i^m(\xi - \eta, t)}{|z_k(\xi, t) - y_i^m(\xi - \eta, t)|^{2\alpha}} \right) d\eta,$$

which is obvious for $i \neq k$ as long as $\delta[Z(t)] > 0$, and for $i = k$ it follows from $z_k \in C^{4,1}$ and $2\alpha < 1$ via a triple application of the Leibnitz integral rule.

We will analyze the right hand side of (2.12) separately for $i \neq k$ and $i = k$. For the sake of clarity we will omit the t dependence in the rest of this argument, since all the estimates of the right hand side of (2.12) are done at a fixed time t . We will also omit Z in $\delta[Z(t)]$ and $F[Z(t)]$, instead writing just δ and F .

Step 1: Contribution to (2.12) from the $i \neq k$ terms. For $i \neq k$ and $m = 1, 2$, the integral on the right hand side of (2.12) can be written as $\sum_{j=0}^3 \binom{3}{j} I_{k,i,j}^m$, where

$$I_{k,i,j}^m := \int_{\mathbb{T}^2} \underbrace{\partial_\xi^3 z_k(\xi) \cdot \partial_\xi^{3-j} \left(\partial_\xi z_k(\xi) - \partial_\xi y_i^m(\xi - \eta) \right)}_{T_1(\xi, \eta)} \underbrace{\partial_\xi^j \left(|z_k(\xi) - y_i^m(\xi - \eta)|^{-2\alpha} \right)}_{T_2(\xi, \eta)} d\eta d\xi.$$

The $j = 2$ term is the easiest to control, where we directly use

$$|T_1| \leq 2\|Z\|_{C^2} \leq C\|Z\|_{H^3} \quad \text{and} \quad |T_2| \leq C(\alpha)\delta^{-2-2\alpha}(\|Z\|_{C^2} + 1)^2$$

to obtain

$$|I_{k,i,2}^m| \leq C(\alpha)\delta^{-2-2\alpha}(\|Z\|_{C^2} + 1)^2\|Z\|_{H^3}^2.$$

For $j = 1$ we have a similar estimate for T_2 , so

$$\begin{aligned} |I_{k,i,1}^m| &\leq C(\alpha)\delta^{-2-2\alpha}(\|Z\|_{C^2} + 1)^2 \int_{\mathbb{T}^2} \left[|\partial_\xi^3 z_k(\xi)|^2 + |\partial_\xi^3 z_k(\xi) \cdot \partial_\xi^3 y_i^m(\xi - \eta)| \right] d\eta d\xi \\ &\leq C(\alpha)\delta^{-2-2\alpha}(\|Z\|_{C^2} + 1)^2\|Z\|_{H^3}^2. \end{aligned}$$

For $j = 3$ we have $|T_1| \leq 2\|Z\|_{C^2}$ and

$$|T_2| \leq C(\alpha)\delta^{-1-2\alpha}|\partial_\xi^3 z_k(\xi) - \partial_\xi^3 y_i^m(\xi - \eta)| + C(\alpha)\delta^{-3-2\alpha}(\|Z\|_{C^2} + 1)^3,$$

so that we obtain (also using $\|Z\|_{C^2} \leq C\|Z\|_{H^3}$)

$$\begin{aligned} |I_{k,i,3}^m| &\leq C(\alpha)\|Z\|_{C^2} (\delta^{-1-2\alpha}\|Z\|_{H^3}^2 + \delta^{-3-2\alpha}(\|Z\|_{C^2} + 1)^3\|Z\|_{H^3}) \\ &\leq C(\alpha)\delta^{-3-2\alpha}(\|Z\|_{C^2} + 1)^3\|Z\|_{H^3}^2. \end{aligned}$$

For $j = 0$, we split the integral as follows:

$$I_{k,i,0}^m = \underbrace{\int_{\mathbb{T}^2} \partial_\xi^3 z_k(\xi) \cdot \frac{\partial_\xi^4 z_k(\xi)}{|z_k(\xi) - y_i^m(\xi - \eta)|^{2\alpha}} d\eta d\xi}_{I_{01}} - \underbrace{\int_{\mathbb{T}^2} \partial_\xi^3 z_k(\xi) \cdot \frac{\partial_\xi^4 y_i^m(\xi - \eta)}{|z_k(\xi) - y_i^m(\xi - \eta)|^{2\alpha}} d\eta d\xi}_{I_{02}}.$$

Integration by parts in ξ yields

$$|I_{01}| = \frac{1}{2} \left| \int_{\mathbb{T}^2} |\partial_\xi^3 z_k(\xi)|^2 \partial_\xi (|z_k(\xi) - y_i^m(\xi - \eta)|^{-2\alpha}) d\eta d\xi \right| \leq C(\alpha)\delta^{-1-2\alpha}\|Z\|_{C^2}\|Z\|_{H^3}^2. \quad (2.13)$$

Since $\partial_\xi^4 y_i^m(\xi - \eta) = \frac{d^4}{d\eta^4} y_i^m(\xi - \eta)$, integration by parts in η also yields

$$\begin{aligned} |I_{02}| &= \left| \int_{\mathbb{T}^2} \partial_\xi^3 z_k(\xi) \cdot \frac{\frac{d^4}{d\eta^4} y_i^m(\xi - \eta)}{|z_k(\xi) - y_i^m(\xi - \eta)|^{2\alpha}} d\eta d\xi \right| \\ &= \left| \int_{\mathbb{T}^2} \partial_\xi^3 z_k(\xi) \cdot \frac{d^3}{d\eta^3} y_i^m(\xi - \eta) \frac{d}{d\eta} (|z_k(\xi) - y_i^m(\xi - \eta)|^{-2\alpha}) d\eta d\xi \right| \\ &\leq C(\alpha)\delta^{-1-2\alpha}\|Z\|_{C^2}\|Z\|_{H^3}^2. \end{aligned} \quad (2.14)$$

Thus for $i \neq k$, the integral on the right-hand side of (2.12) is bounded by

$$C(\alpha)\delta^{-3-2\alpha}(\|Z\|_{C^2} + 1)^3\|Z\|_{H^3}^2.$$

Step 2: Contribution to (2.12) from the $i = k$ terms. An argument as in [13] (see the bound just below (24) in [13]) shows that the integral on the right-hand side of (2.12) with $i = k$ and $m = 1$ (that is, $y_i^m = z_k$), is bounded by

$$C(\alpha)F^{3+2\alpha}(\|Z\|_{C^2} + 1)^3\|Z\|_{H^3}^2.$$

However, the term with $i = k$ and $m = 2$ (that is, $y_i^m = \bar{z}_k$), creates some new difficulties. Nevertheless, we will be able to obtain for it the almost identical bound (2.27) below. (Also, as the reader can easily check, the argument in the case $m = 1$ is essentially a subset of the argument below for $m = 2$.)

Using the notation from [13] and writing $z = z_k$, the integral in (2.12) with $i = k$ and $m = 2$ becomes $I_0 + 3I_1 + 3I_2 + I_3$, where

$$I_j := I_{k,k,j}^2 = \int_{\mathbb{T}^2} \partial_\xi^3 z(\xi) \cdot \partial_\xi^{3-j} \left(\partial_\xi z(\xi) - \partial_\xi \bar{z}(\xi - \eta) \right) \partial_\xi^j \left(|z(\xi) - \bar{z}(\xi - \eta)|^{-2\alpha} \right) d\eta d\xi.$$

For $j = 0$ we have using $u \cdot v = \bar{u} \cdot \bar{v}$ and a change of variables,

$$\begin{aligned} I_0 &= \int_{\mathbb{T}^2} \partial_\xi^3 z(\xi) \cdot \frac{\partial_\xi^4 z(\xi) - \partial_\xi^4 \bar{z}(\xi - \eta)}{|z(\xi) - \bar{z}(\xi - \eta)|^{2\alpha}} d\eta d\xi = \int_{\mathbb{T}^2} \partial_\xi^3 \bar{z}(\xi) \cdot \frac{\partial_\xi^4 \bar{z}(\xi) - \partial_\xi^4 z(\xi - \eta)}{|\bar{z}(\xi) - z(\xi - \eta)|^{2\alpha}} d\eta d\xi \\ &= \int_{\mathbb{T}^2} \partial_\xi^3 \bar{z}(\xi - \eta) \cdot \frac{\partial_\xi^4 \bar{z}(\xi - \eta) - \partial_\xi^4 z(\xi)}{|\bar{z}(\xi - \eta) - z(\xi)|^{2\alpha}} d\eta d\xi. \end{aligned}$$

This, an integration by parts in ξ , and a change of variables now yield

$$\begin{aligned} |I_0| &= \frac{1}{2} \left| \int_{\mathbb{T}^2} \left(\partial_\xi^3 z(\xi) - \partial_\xi^3 \bar{z}(\xi - \eta) \right) \cdot \frac{\partial_\xi (\partial_\xi^3 z(\xi) - \partial_\xi^3 \bar{z}(\xi - \eta))}{|z(\xi) - \bar{z}(\xi - \eta)|^{2\alpha}} d\eta d\xi \right| \\ &\leq \frac{2\alpha}{4} \int_{\mathbb{T}^2} \left| \partial_\xi^3 z(\xi) - \partial_\xi^3 \bar{z}(\xi - \eta) \right|^2 \frac{|\partial_\xi z(\xi) - \partial_\xi \bar{z}(\xi - \eta)|}{|z(\xi) - \bar{z}(\xi - \eta)|^{1+2\alpha}} d\eta d\xi \\ &\leq \alpha \int_{\mathbb{T}^2} \left(|\partial_\xi^3 z(\xi)|^2 + |\partial_\xi^3 \bar{z}(\eta)|^2 \right) \underbrace{\frac{|\partial_\xi z(\xi) - \partial_\xi \bar{z}(\eta)|}{|z(\xi) - \bar{z}(\eta)|^{1+2\alpha}}}_{T_3(\xi, \eta)} d\eta d\xi \\ &\leq 2\alpha \|Z\|_{H^3}^2 \max_{\xi \in \mathbb{T}} \int_{\mathbb{T}} T_3(\xi, \eta) d\eta. \end{aligned} \tag{2.15}$$

The above computation is similar to that in [13], with the latter having z in place of \bar{z} (which is our case $m = 1$). In that case the numerator of T_3 is bounded above

by $\|Z\|_{C^2}|\xi - \eta|$, and the denominator is bounded below by $F^{-1-2\alpha}|\xi - \eta|^{1+2\alpha}$, giving $|T_3(\xi, \eta)| \leq F^{1+2\alpha}\|Z\|_{C^2}|\xi - \eta|^{-2\alpha}$. Since $2\alpha < 1$, this now yields a bound on I_0 .

In our case $m = 2$, the lower bound for the denominator continues to hold because

$$|z(\xi) - \bar{z}(\eta)| \geq |z(\xi) - z(\eta)| \geq F^{-1}|\xi - \eta|. \quad (2.16)$$

However, we no longer have the same estimate for the numerator. With the notation $z(\xi) = (z^1(\xi), z^2(\xi))$, the second component of the numerator becomes $\partial_\xi z^2(\xi) + \partial_\xi z^2(\eta)$, which need not converge to 0 as $\xi - \eta \rightarrow 0$. The following lemma will help us instead.

Lemma 2.2. *If $\gamma \in [0, 1]$ and $0 \leq f \in C^{1,\gamma}(\mathbb{T})$, then for any $\xi \in \mathbb{T}$ we have*

$$|f'(\xi)| \leq 2\|f\|_{C^{1,\gamma}}^{1/(1+\gamma)} f(\xi)^{\gamma/(1+\gamma)}. \quad (2.17)$$

We present the proof in Section 2.4. Note that the power of $f(\xi)$ is sharp, and that Sobolev embedding and (2.17) show for $0 \leq f \in H^3(\mathbb{T})$, $\xi \in \mathbb{T}$, and a universal $C < \infty$,

$$|f'(\xi)| \leq C\|f\|_{H^3}^{1/(1+\gamma)} f(\xi)^{\gamma/(1+\gamma)}. \quad (2.18)$$

Lemma 2.2 with $f(\eta) = z^2(\eta) \geq 0$ (together with $|\xi - \eta| \leq \pi$) now yields

$$\begin{aligned} |\partial_\xi z(\xi) - \partial_\xi \bar{z}(\eta)| &\leq |\partial_\xi z(\xi) - \partial_\xi z(\eta)| + |\partial_\xi z(\eta) - \partial_\xi \bar{z}(\eta)| \\ &= |\partial_\xi z(\xi) - \partial_\xi z(\eta)| + 2|\partial_\xi z^2(\eta)| \\ &\leq \|Z\|_{C^2}|\xi - \eta| + 2\|Z\|_{C^2}^{1/2}\sqrt{z^2(\eta)} \\ &\leq C(\|Z\|_{C^2} + 1)\left(\sqrt{|\xi - \eta|} + \sqrt{|z(\xi) - \bar{z}(\eta)|}\right) \\ &\leq 2C(\|Z\|_{C^2} + 1)F^{1/2}|z(\xi) - \bar{z}(\eta)|^{1/2}. \end{aligned} \quad (2.19)$$

Then

$$\begin{aligned} T_3(\xi, \eta) &\leq 2CF^{1/2}(\|Z\|_{C^2} + 1)|z(\xi) - \bar{z}(\eta)|^{-2\alpha-1/2} \\ &\leq 2CF^{1+2\alpha}(\|Z\|_{C^2} + 1)|\xi - \eta|^{-2\alpha-1/2}, \end{aligned} \quad (2.20)$$

which is integrable in η (uniformly in ξ) when $\alpha < \frac{1}{4}$. Plugging this into (2.15) yields

$$|I_0| \leq C(\alpha)F^{1+2\alpha}(\|Z\|_{C^2} + 1)\|Z\|_{H^3}^2$$

for all $\alpha < \frac{1}{4}$.

For $j = 1$ we notice that

$$\begin{aligned} I_1 &= -2\alpha \int_{\mathbb{T}^2} \partial_\xi^3 z(\xi) \cdot (\partial_\xi^3 z(\xi) - \partial_\xi^3 \bar{z}(\xi - \eta)) \frac{(\partial_\xi z(\xi) - \partial_\xi \bar{z}(\xi - \eta)) \cdot (z(\xi) - \bar{z}(\xi - \eta))}{|z(\xi) - \bar{z}(\xi - \eta)|^{2+2\alpha}} d\eta d\xi \\ &= -2\alpha \int_{\mathbb{T}^2} \partial_\xi^3 \bar{z}(\xi - \eta) \cdot (\partial_\xi^3 \bar{z}(\xi - \eta) - \partial_\xi^3 z(\xi)) \frac{(\partial_\xi z(\xi) - \partial_\xi \bar{z}(\xi - \eta)) \cdot (z(\xi) - \bar{z}(\xi - \eta))}{|z(\xi) - \bar{z}(\xi - \eta)|^{2+2\alpha}} d\eta d\xi, \end{aligned}$$

where we used a change of variables (switching ξ and $\xi - \eta$) and $\bar{u} \cdot \bar{v} = u \cdot v$. Thus

$$I_1 = -\alpha \int_{\mathbb{T}^2} |\partial_\xi^3 z(\xi) - \partial_\xi^3 \bar{z}(\xi - \eta)|^2 \frac{(\partial_\xi z(\xi) - \partial_\xi \bar{z}(\xi - \eta)) \cdot (z(\xi) - \bar{z}(\xi - \eta))}{|z(\xi) - \bar{z}(\xi - \eta)|^{2+2\alpha}} d\eta d\xi.$$

So $|I_1|$ is bounded by twice the second line of (2.15), and it obeys the same bound as $|I_0|$.

The estimates for I_2 and I_3 will be slightly more involved. For $j = 2$ we have

$$\begin{aligned} |I_2| &= \left| \int_{\mathbb{T}^2} \partial_\xi^3 z(\xi) \cdot (\partial_\xi^2 z(\xi) - \partial_\xi^2 \bar{z}(\xi - \eta)) \partial_\xi^2 (|z(\xi) - \bar{z}(\xi - \eta)|^{-2\alpha}) d\eta d\xi \right| \\ &\leq 2 \|Z\|_{C^2} \|Z\|_{H^3} \underbrace{\left\| \int_{\mathbb{T}} |\partial_\xi^2 (|z(\xi) - \bar{z}(\xi - \eta)|^{-2\alpha})| d\eta \right\|_{L^2}}_{T_4(\xi)}, \end{aligned} \quad (2.21)$$

so we need to bound $\|T_4\|_{L^2}$. A simple change of variables and $\bar{z} = (z^1, -z^2)$ yield

$$\begin{aligned} |T_4| &\leq C(\alpha) \left[\int_{\mathbb{T}} \frac{|\partial_\xi z(\xi) - \partial_\xi \bar{z}(\eta)|^2}{|z(\xi) - \bar{z}(\eta)|^{2+2\alpha}} d\eta + \int_{\mathbb{T}} \frac{|\partial_\xi^2 z(\xi) - \partial_\xi^2 \bar{z}(\eta)|}{|z(\xi) - \bar{z}(\eta)|^{1+2\alpha}} d\eta \right] \\ &\leq C(\alpha) \left[\int_{\mathbb{T}} \frac{|\partial_\xi z(\xi) - \partial_\xi z(\eta)|^2}{|z(\xi) - \bar{z}(\eta)|^{2+2\alpha}} d\eta + \underbrace{\int_{\mathbb{T}} \frac{|\partial_\xi z^2(\eta)|^2}{|z(\xi) - \bar{z}(\eta)|^{2+2\alpha}} d\eta}_{T_5(\xi)} \right. \\ &\quad \left. + \int_{\mathbb{T}} \frac{|\partial_\xi^2 z(\xi) - \partial_\xi^2 z(\eta)|}{|z(\xi) - \bar{z}(\eta)|^{1+2\alpha}} d\eta + \underbrace{\int_{\mathbb{T}} \frac{|\partial_\xi^2 z^2(\eta)|}{|z(\xi) - \bar{z}(\eta)|^{1+2\alpha}} d\eta}_{T_6(\xi)} \right]. \end{aligned} \quad (2.22)$$

The first and third of the last four integrals can be controlled in the same way as in [13]. Indeed, the numerator in the first term is bounded by $\|Z\|_{C^2}^2 |\xi - \eta|^2$, so the integral is bounded uniformly in ξ by $C(\alpha) F^{2+2\alpha} \|Z\|_{C^2}^2$ due to $2\alpha < 1$ (its L^2 -norm, as a function of ξ , then satisfies the same bound). As for the third term, let us change the η variable back to $\xi - \eta$, so that the numerator equals $|\eta \int_0^1 \partial_\xi^3 z(\xi - s\eta) ds|$. The Minkowski inequality for integrals then shows that that term's L^2 -norm is bounded by

$$\left\| F^{1+2\alpha} \int_{\mathbb{T} \times [0,1]} |\partial_\xi^3 z(\xi - s\eta)| \frac{ds d\eta}{|\eta|^{2\alpha}} \right\|_{L^2(\mathbb{T})} \leq F^{1+2\alpha} \int_{\mathbb{T} \times [0,1]} \left(\int_{\mathbb{T}} |\partial_\xi^3 z(\xi - s\eta)|^2 d\xi \right)^{1/2} \frac{ds d\eta}{|\eta|^{2\alpha}},$$

that is, by $C(\alpha)F^{1+2\alpha}\|Z\|_{H^3}$ for all $\alpha < \frac{1}{2}$.

To deal with T_5 and T_6 , let us first define their regularizations

$$T_5^\epsilon(\xi) := \int_{\mathbb{T}} \frac{|\partial_\xi z^2(\eta)|^2}{(|z(\xi) - \bar{z}(\eta)| + \epsilon)^{2+2\alpha}} d\eta \quad \text{and} \quad T_6^\epsilon(\xi) := \int_{\mathbb{T}} \frac{|\partial_\xi^2 z^2(\eta)|}{(|z(\xi) - \bar{z}(\eta)| + \epsilon)^{1+2\alpha}} d\eta.$$

We will show that $\|T_5^\epsilon\|_{L^2}$ and $\|T_6^\epsilon\|_{L^2}$ are uniformly bounded as $0 < \epsilon \rightarrow 0$, so that the monotone convergence theorem then yields the same bounds for $\|T_5\|_{L^2}$ and $\|T_6\|_{L^2}$.

Let us start with T_5^ϵ . From (2.16) and $|z(\xi) - \bar{z}(\eta)| \geq z^2(\eta) \geq 0$ we have

$$T_5^\epsilon(\xi) \leq C(\alpha) \int_{\mathbb{T}} (F^{-1}|\xi - \eta|)^{-\frac{11}{12}-2\alpha} \underbrace{\frac{|\partial_\xi z^2(\eta)|^2}{(z^2(\eta) + \epsilon)^{\frac{13}{12}}}}_{T_7^\epsilon(\eta)} d\eta$$

Young's inequality for convolutions now yields

$$\|T_5^\epsilon\|_{L^2} \leq C(\alpha)F^{\frac{11}{12}+2\alpha} \left\| \xi^{-\frac{11}{12}-2\alpha} \right\|_{L^1} \|T_7^\epsilon\|_{L^2},$$

with the L^1 -norm finite provided $\alpha < \frac{1}{24}$. The following will help us estimate $\|T_7^\epsilon\|_{L^2}$.

Lemma 2.3. *There is $C < \infty$ such that if $\beta \in [0, \frac{1}{6}]$ and $0 < f \in H^3(\mathbb{T})$, then*

$$\int_{\mathbb{T}} \frac{|f'(\xi)|^n}{f(\xi)^{\beta+n/2}} d\xi \leq C^n \|f\|_{H^3}^{\frac{n}{2}-\beta} \quad (2.23)$$

for any $n \geq 1$, as well as

$$\int_{\mathbb{T}} \frac{f''(\xi)^2}{f(\xi)^\beta} d\xi \leq C \|f\|_{H^3}^{2-\beta}. \quad (2.24)$$

The proof is also presented in Section 2.4. Now (2.23) with $n = 4$ and $\beta = \frac{1}{6}$ yields

$$\|T_7^\epsilon\|_{L^2}^2 = \int \frac{|\partial_\xi z^2(\xi)|^4}{(z^2(\xi) + \epsilon)^{\frac{13}{6}}} d\xi \leq C^4 \|Z\|_{H^3}^{\frac{11}{6}},$$

so that

$$\|T_5^\epsilon\|_{L^2} \leq C(\alpha)F^{\frac{11}{12}+2\alpha} \|Z\|_{H^3}^{\frac{11}{12}}$$

for all $\alpha \in (0, \frac{1}{24})$ and $\epsilon > 0$. Thus the same bound holds for $\|T_5\|_{L^2}$.

An almost identical argument for T_6^ϵ gives

$$\|T_6^\epsilon\|_{L^2} \leq C(\alpha)F^{\frac{11}{12}+2\alpha} \left\| \xi^{-\frac{11}{12}-2\alpha} \right\|_{L^1} \|T_8^\epsilon\|_{L^2},$$

with $T_8^\epsilon(\xi) := |\partial_\xi^2 z^2(\xi)|(z^2(\xi) + \epsilon)^{-1/12}$. From (2.24) we again obtain

$$\|T_8^\epsilon\|_{L^2}^2 \leq C\|Z\|_{H^3}^{\frac{11}{6}},$$

which yields for $\|T_6\|_{L^2}$ the same estimate as for $\|T_5\|_{L^2}$. We therefore have

$$\|T_4\|_{L^2} \leq C(\alpha)F^{2+2\alpha}(\|Z\|_{C^2} + 1)(\|Z\|_{H^3} + 1) \quad (2.25)$$

(also using $\|Z\|_{C^2} \leq C\|Z\|_{H^3}$), so that (2.21) finally yields for all $\alpha < \frac{1}{24}$,

$$|I_2| \leq C(\alpha)F^{2+2\alpha}(\|Z\|_{C^2} + 1)^2(\|Z\|_{H^3} + 1)\|Z\|_{H^3}.$$

Finally, for $j = 3$ we obtain after differentiating inside I_3 and changing variables,

$$\begin{aligned} |I_3| \leq C(\alpha) & \left[\int_{\mathbb{T}^2} |\partial_\xi z(\xi) - \partial_\xi \bar{z}(\eta)| \frac{|\partial_\xi^3 z(\xi)|^2 + |\partial_\xi^3 \bar{z}(\eta)|^2}{|z(\xi) - \bar{z}(\eta)|^{1+2\alpha}} d\eta d\xi \right. \\ & + \int_{\mathbb{T}^2} |\partial_\xi^3 z(\xi)| \frac{|\partial_\xi^2 z(\xi) - \partial_\xi^2 \bar{z}(\eta)| |\partial_\xi z(\xi) - \partial_\xi \bar{z}(\eta)|^2}{|z(\xi) - \bar{z}(\eta)|^{2+2\alpha}} d\eta d\xi \\ & \left. + \int_{\mathbb{T}^2} |\partial_\xi^3 z(\xi)| \frac{|\partial_\xi z(\xi) - \partial_\xi \bar{z}(\eta)|^4}{|z(\xi) - \bar{z}(\eta)|^{3+2\alpha}} d\eta d\xi \right]. \end{aligned} \quad (2.26)$$

The first integral already appeared in (2.15) and hence obeys the same bound as I_0 . We then apply (2.19) squared to each of the other two integrals (removing $|\partial_\xi z(\xi) - \partial_\xi \bar{z}(\eta)|^2$ from the numerator and $|z(\xi) - \bar{z}(\eta)|$ from the denominator) and find out that they are bounded by $\|Z\|_{H^3} F(\|Z\|_{C^2} + 1)^2$ times the L^2 -norm of the expression in the middle of (2.22). That latter norm is bounded by the right-hand side of (2.25), so that in the end we obtain

$$|I_3| \leq C(\alpha)F^{3+2\alpha}(\|Z\|_{C^2} + 1)^3(\|Z\|_{H^3} + 1)\|Z\|_{H^3}$$

for all $\alpha < \frac{1}{24}$.

Thus for $i = k$, the integral on the right-hand side of (2.12) is bounded by

$$C(\alpha)F^{3+2\alpha}(\|Z\|_{C^2} + 1)^3(\|Z\|_{H^3} + 1)\|Z\|_{H^3}. \quad (2.27)$$

This, the analogous estimate for $i = k$ and $m = 1$ from the beginning of Step 2, the estimate from Step 1, and Lemma 2.1 now yield for any $\alpha \in (0, \frac{1}{24})$ and $\Theta := \sum_{k=1}^N |\theta_k|$,

$$\frac{d}{dt} \|Z(t)\|_{H^3} \leq C(\alpha)N\Theta (\delta[Z(t)]^{-1} + F[Z(t)])^{3+2\alpha} (\|Z(t)\|_{C^2} + 1)^3 (\|Z(t)\|_{H^3} + 1). \quad (2.28)$$

The evolution of $F[Z(t)]$

For any $k = 1, \dots, N$ and any $\xi, \lambda \in \mathbb{T}$ with $\lambda \neq 0$, we have (again dropping t from z_k)

$$\frac{d}{dt} \frac{|\lambda|}{|z_k(\xi) - z_k(\xi - \lambda)|} \leq \frac{|\lambda| |\partial_t z_k(\xi) - \partial_t z_k(\xi - \lambda)|}{|z_k(\xi) - z_k(\xi - \lambda)|^2} \leq F[Z(t)]^2 \frac{|\partial_t z_k(\xi) - \partial_t z_k(\xi - \lambda)|}{|\lambda|}. \quad (2.29)$$

Using (2.5) and the mean value theorem, we can estimate

$$\begin{aligned} |\partial_t z_k(\xi) - \partial_t z_k(\xi - \lambda)| &\leq \sum_{i=1}^N \sum_{m=1}^2 \frac{|\theta_i|}{2\alpha} \int_{\mathbb{T}} |\lambda| \sup_{\xi \in \mathbb{T}} \left\{ \left| \partial_\xi \left(\frac{\partial_\xi z_k(\xi) - \partial_\xi y_i^m(\xi - \eta)}{|z_k(\xi) - y_i^m(\xi - \eta)|^{2\alpha}} \right) \right| \right\} d\eta \\ &\leq \sum_{i=1}^N \sum_{m=1}^2 \frac{|\theta_i|}{2\alpha} |\lambda| \int_{\mathbb{T}} \sup_{\xi \in \mathbb{T}} \left\{ \frac{2\|Z(t)\|_{C^2}}{|z_k(\xi) - y_i^m(\xi - \eta)|^{2\alpha}} + \frac{2\alpha |\partial_\xi z_k(\xi) - \partial_\xi y_i^m(\xi - \eta)|^2}{|z_k(\xi) - y_i^m(\xi - \eta)|^{1+2\alpha}} \right\} d\eta \\ &\leq C(\alpha)\Theta |\lambda| (\delta[Z(t)]^{-1} + F[Z(t)])^{1+2\alpha} (\|Z(t)\|_{C^2} + 1)^2, \end{aligned}$$

where in the last inequality we used (2.19) to control the last term on the second line for $i = k$ and $m = 2$. Plugging this into (2.29) now yields

$$\frac{d}{dt} F[Z(t)] \leq C(\alpha)\Theta (\delta[Z(t)]^{-1} + F[Z(t)])^{3+2\alpha} (\|Z(t)\|_{C^2} + 1)^2. \quad (2.30)$$

Finally, this, (2.28), (2.10), and (2.11) imply for $\alpha \in (0, \frac{1}{24})$,

$$\frac{d}{dt} \|||Z(t)\||| \leq C(\alpha)N\Theta \|||Z(t)\|||^{7+2\alpha}. \quad (2.31)$$

2.3 Local H^3 well-posedness for the contour equation and small α

Uniqueness of solutions

Let $W = (w_1, \dots, w_N) = Z - \tilde{Z}$ for classical solutions Z and \tilde{Z} to (2.5) on some time interval $[0, T]$, with $\sup_{t \in [0, T]} (\|||Z(t)\||| + \|||\tilde{Z}(t)\|||) < \infty$ and $Z(0) = \tilde{Z}(0)$ (here we require that $\partial_t Z$ is continuous in (ξ, t) for classical solutions). Then for any $k = 1, \dots, N$ and $t \in [0, T]$ we have (with the argument t again dropped)

$$\begin{aligned} \frac{d}{dt} \|w_k\|_{L^2}^2 &= 2 \int_{\mathbb{T}} w_k(\xi) \cdot \partial_t w_k(\xi) d\xi \\ &= \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{\alpha} \int_{\mathbb{T}^2} w_k(\xi) \cdot \left(\frac{\partial_\xi z_k(\xi) - \partial_\xi y_i^m(\xi - \eta)}{|z_k(\xi) - y_i^m(\xi - \eta)|^{2\alpha}} - \frac{\partial_\xi \tilde{z}_k(\xi) - \partial_\xi \tilde{y}_i^m(\xi - \eta)}{|\tilde{z}_k(\xi) - \tilde{y}_i^m(\xi - \eta)|^{2\alpha}} \right) d\eta d\xi. \end{aligned} \quad (2.32)$$

The last integral equals $A_{k,i}^m + B_{k,i}^m$, where with $w_i^1 := w_i$ and $w_i^2 := \bar{w}_i$,

$$A_{k,i}^m := \int_{\mathbb{T}^2} w_k(\xi) \cdot (\partial_\xi z_k(\xi) - \partial_\xi y_i^m(\xi - \eta)) \left(\frac{1}{|z_k(\xi) - y_i^m(\xi - \eta)|^{2\alpha}} - \frac{1}{|\tilde{z}_k(\xi) - \tilde{y}_i^m(\xi - \eta)|^{2\alpha}} \right) d\eta d\xi,$$

$$B_{k,i}^m := \int_{\mathbb{T}^2} w_k(\xi) \cdot \frac{\partial_\xi w_k(\xi) - \partial_\xi w_i^m(\xi - \eta)}{|\tilde{z}_k(\xi) - \tilde{y}_i^m(\xi - \eta)|^{2\alpha}} d\eta d\xi.$$

Let us first estimate $A_{k,i}^m$. When $i \neq k$, the term inside the parentheses is easily bounded by $C(\alpha) \min\{\delta[Z], \delta[\tilde{Z}]\}^{-1-2\alpha} (|w_k(\xi)| + |w_i(\xi - \eta)|)$, so

$$|A_{k,i}^m| \leq C(\alpha) (\|Z\| + \|\tilde{Z}\|)^{2+2\alpha} \|W\|_{L^2}^2.$$

For $i = k$ and $m = 1$, $A_{k,i}^m$ is controlled the same way as in [13, p. 13-14], yielding

$$|A_{k,k}^1| \leq C(\alpha) (\|Z\| + \|\tilde{Z}\|)^{2+2\alpha} \|W\|_{L^2}^2$$

for $\alpha < \frac{1}{2}$. Finally, for $i = k$ and $m = 2$, the following almost identical computation, (2.19), and $|x^{2\alpha} - 1| \leq |x - 1|$ for $x \geq 0$ yield the same bound for $\alpha < \frac{1}{4}$:

$$\begin{aligned} |A_{k,k}^2| &\leq \|Z\|^{\frac{3}{2}} \|\tilde{Z}\|^{2\alpha} \int_{\mathbb{T}^2} |w_k(\xi)| |z_k(\xi) - \bar{z}_k(\xi - \eta)|^{1/2} \left| \left(\frac{|\tilde{z}_k(\xi) - \bar{\tilde{z}}_k(\xi - \eta)|}{|z_k(\xi) - \bar{z}_k(\xi - \eta)|} \right)^{2\alpha} - 1 \right| |\eta|^{-2\alpha} d\eta d\xi \\ &\leq \|Z\|^{\frac{3}{2}} \|\tilde{Z}\|^{2\alpha} \int_{\mathbb{T}^2} |w_k(\xi)| |z_k(\xi) - \bar{z}_k(\xi - \eta)|^{1/2} \left| \frac{|\tilde{z}_k(\xi) - \bar{\tilde{z}}_k(\xi - \eta)|}{|z_k(\xi) - \bar{z}_k(\xi - \eta)|} - 1 \right| |\eta|^{-2\alpha} d\eta d\xi \\ &\leq \|Z\|^2 \|\tilde{Z}\|^{2\alpha} \int_{\mathbb{T}^2} |w_k(\xi)| \left| |\tilde{z}_k(\xi) - \bar{\tilde{z}}_k(\xi - \eta)| - |z_k(\xi) - \bar{z}_k(\xi - \eta)| \right| |\eta|^{-\frac{1}{2}-2\alpha} d\eta d\xi \\ &\leq \|Z\|^2 \|\tilde{Z}\|^{2\alpha} \int_{\mathbb{T}^2} |w_k(\xi)| (|w_k(\xi)| + |w_k(\xi - \eta)|) |\eta|^{-\frac{1}{2}-2\alpha} d\eta d\xi \\ &\leq C(\alpha) (\|Z\| + \|\tilde{Z}\|)^{2+2\alpha} \|W\|_{L^2}^2. \end{aligned} \tag{2.33}$$

Next we control $B_{k,i}^m$. When $i \neq k$, we split it into two integrals: with $\partial_\xi w_k(\xi)$ and with $\partial_\xi w_i^m(\xi - \eta)$, respectively. After integrating these by parts in ξ and in η , respectively (similarly to (2.13) and (2.14)), we obtain

$$|B_{k,i}^m| \leq C(\alpha) \delta[\tilde{Z}]^{-1-2\alpha} \|\tilde{Z}\|_{C^2} \|W\|_{L^2}^2 \leq C(\alpha) \|\tilde{Z}\|^{2+2\alpha} \|W\|_{L^2}^2.$$

For $i = k$, we symmetrize the integrand similarly to the equation before (2.15) and obtain

$$B_{k,k}^m = \frac{1}{4} \int_{\mathbb{T}^2} \frac{\partial_\xi (|w_k(\xi) - w_k^m(\xi - \eta)|^2)}{|\tilde{z}_k(\xi) - \tilde{y}_k^m(\xi - \eta)|^{2\alpha}} d\eta d\xi.$$

Integration by parts now yields

$$\begin{aligned}
|B_{k,k}^m| &\leq \frac{\alpha}{2} \int_{\mathbb{T}^2} |w_k(\xi) - w_k^m(\xi - \eta)|^2 \frac{|\partial_\xi \tilde{z}_k(\xi) - \partial_\xi \tilde{y}_k^m(\xi - \eta)|}{|\tilde{z}_k(\xi) - \tilde{y}_k^m(\xi - \eta)|^{1+2\alpha}} d\eta d\xi \\
&\leq \alpha \int_{\mathbb{T}^2} \left(|w_k(\xi)|^2 + |w_k^m(\eta)|^2 \right) \underbrace{\frac{|\partial_\xi \tilde{z}_k(\xi) - \partial_\xi \tilde{y}_k^m(\eta)|}{|\tilde{z}_k(\xi) - \tilde{y}_k^m(\eta)|^{1+2\alpha}}}_{\tilde{T}_k^m(\xi, \eta)} d\eta d\xi \\
&\leq 2\alpha \|W\|_{L^2}^2 \max_{\xi \in \mathbb{T}} \int_{\mathbb{T}} \tilde{T}_k^m(\xi, \eta) d\eta.
\end{aligned}$$

The bounds on T_3 from the discussion after (2.15) (for both $m = 1, 2$) equally apply to \tilde{T}_k^m and yield for $m = 1$ and $\alpha < \frac{1}{2}$, as well as for $m = 2$ and $\alpha < \frac{1}{4}$,

$$|B_{k,k}^m| \leq C(\alpha) \|\tilde{Z}\|^{2+2\alpha} \|W\|_{L^2}^2.$$

Combining all the obtained bounds on $A_{k,i}^m$ and $B_{k,i}^m$ now yields

$$\frac{d}{dt} \|W(t)\|_{L^2}^2 \leq C(\alpha) N \Theta (\|Z(t)\| + \|\tilde{Z}(t)\|)^{2+2\alpha} \|W(t)\|_{L^2}^2 \quad (2.34)$$

for $\alpha < \frac{1}{4}$. Gronwall's inequality then shows $\|W(t)\|_{L^2} = 0$ for $t \in [0, T]$, hence $Z = \tilde{Z}$.

Existence of solutions

Similarly to [20, Chapter 3], once we have the a priori control (2.31) on the growth of $\|Z(t)\|$, solutions to the contour equation (2.5) will be obtained as limits of solutions to an appropriate family of mollified equations. We will need to be careful, however, that the solutions of the latter do not exit D .

Consider any initial condition $Z_0 = \{z_{0k}\}_{k=1}^N$ with $z_{0k} : \mathbb{T} \rightarrow \bar{D}$ for all $k = 1, \dots, N$ and $M := \|Z_0\| < \infty$ (then also $M \geq 2$). We let $\phi_\epsilon(\xi) := \epsilon^{-1} \phi(\epsilon^{-1}\xi)$ for some mollifier ϕ which is smooth, even, non-negative on \mathbb{T} , supported in $[-1, 1]$, and satisfies $\int_{\mathbb{T}} \phi(\xi) d\xi = 1$. For $k = 1, \dots, N$, we regularize (2.5) to

$$\partial_t z_k^\epsilon(\xi, t) = \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} \phi_\epsilon * \int_{\mathbb{T}} \frac{\partial_\xi(\phi_\epsilon * z_k^\epsilon)(\xi, t) - \partial_\xi(\phi_\epsilon * y_i^{m,\epsilon})(\xi - \eta, t)}{|(\phi_\epsilon * z_k^\epsilon)(\xi, t) - (\phi_\epsilon * y_i^{m,\epsilon})(\xi - \eta, t)|^{2\alpha}} d\eta + \tilde{C}(\alpha) \Theta \epsilon M^{3+2\alpha} e_2, \quad (2.35)$$

with $e_2 := (0, 1)$, a large constant $\tilde{C}(\alpha) > 0$ to be chosen later, and initial condition

$$Z^\epsilon(0) = \{z_k^\epsilon(\cdot, 0) + \epsilon e_2\}_{k=1}^N = \{\phi_\epsilon * z_{0k} + \epsilon e_2\}_{k=1}^N = \phi_\epsilon * Z_0 + \epsilon e_2.$$

The convolutions are all taken in the first variable only, and the last term in (2.35) will ensure containment in D .

Step 1. We now prepare the setup for an application of the Picard theorem to find a solution of (2.35). Consider the Banach space $B := H^3(\mathbb{T})^N$ with the norm $\|\cdot\|_{H^3}$ and let $h[Z] := \inf_{1 \leq k \leq N \text{ \& } \xi \in \mathbb{T}} z_k^2(\xi)$ be the infimum of the x_2 -coordinates for $Z \in B$. Then

$$O^A := \{Z \in B : \|Z\| < A \text{ and } h[Z] > 0\}$$

(with $A > 2$) and its closure (in B) $\overline{O^A}$ satisfy the following claims.

Lemma 2.4. *Each O^A is an open set in B .*

Proof. This follows from $\|Z - \tilde{Z}\|_{L^\infty} \leq C\|Z - \tilde{Z}\|_{H^3}$ and

$$\begin{aligned} \left| F[Z]^{-1} - F[\tilde{Z}]^{-1} \right| &\leq \left| \inf_{\substack{\xi, \eta \in \mathbb{T} \\ 1 \leq k \leq N}} \frac{|z_k(\xi) - z_k(\eta)|}{|\xi - \eta|} - \inf_{\substack{\xi, \eta \in \mathbb{T} \\ 1 \leq k \leq N}} \frac{|\tilde{z}_k(\xi) - \tilde{z}_k(\eta)|}{|\xi - \eta|} \right| \\ &\leq \sup_{\substack{\xi, \eta \in \mathbb{T} \\ 1 \leq k \leq N}} \frac{|(z_k(\xi) - \tilde{z}_k(\xi)) - (z_k(\eta) - \tilde{z}_k(\eta))|}{|\xi - \eta|} \\ &\leq \|Z - \tilde{Z}\|_{C^1} \leq C\|Z - \tilde{Z}\|_{H^3}, \end{aligned} \quad (2.36)$$

for some universal $C > 0$. □

Notice that

$$\{Z \in B : \|Z\| < A \text{ and } h[Z] \geq 0\} \subseteq \overline{O^A} \subseteq \{Z \in B : \|Z\| \leq A \text{ and } h[Z] \geq 0\}$$

Indeed, the second inclusion follows from the proof of Lemma 2.4. To see the first inclusion, notice that any Z with $\|Z\| < A$ and $h[Z] = 0$ can be approximated by $Z + \sigma e_2 \in O^A$ with $\sigma > 0$, which converges to Z in B as $\sigma \rightarrow 0$.

Lemma 2.5. *If $Z \in \overline{O^A}$ and $\epsilon \in (0, c_0 A^{-2})$ (with a universal $c_0 > 0$), then $\phi_\epsilon * Z \in \overline{O^{2A}}$.*

Proof. We obviously have $h[\phi_\epsilon * Z] \geq h[Z]$ and $\|\phi_\epsilon * Z\|_{H^3} \leq \|Z\|_{H^3}$ for all $\epsilon > 0$. Since ϕ_ϵ is supported in $[-\epsilon, \epsilon]$, we have also (with a universal $C > 0$)

$$\|\phi_\epsilon * Z - Z\|_{L^\infty} \leq \epsilon \|Z\|_{C^1} \leq C\epsilon \|Z\|_{H^3}.$$

Then $\delta[\phi_\epsilon * Z] \geq \delta[Z] - 2C\epsilon A > \frac{1}{2}\delta[Z]$ for $Z \in \overline{O^A}$ and $\epsilon \in (0, \frac{1}{4CA^2})$. Also, (2.36) yields

$$\left| F[\phi_\epsilon * Z]^{-1} - F[Z]^{-1} \right| \leq \|\phi_\epsilon * Z - Z\|_{C^1} \leq \epsilon \|Z\|_{C^2} \leq C\epsilon \|Z\|_{H^3}, \quad (2.37)$$

so again $F[\phi_\epsilon * Z] < 2F[Z]$ for $Z \in \overline{O^A}$ and $\epsilon \in (0, \frac{1}{2CA^2})$. □

Let us denote the right hand side of (2.35) by $G_k^\epsilon[Z^\epsilon(t)]$. In general, for any $Z \in B$ with $\|Z\| < \infty$ define

$$G_k^\epsilon[Z] = \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} \phi_\epsilon * H_{k,i}^m[\phi_\epsilon * Z] + \tilde{C}(\alpha) \Theta_\epsilon M^{3+2\alpha} e_2, \quad (2.38)$$

where

$$H_{k,i}^m[Z](\xi) := \int_{\mathbb{T}} \frac{\partial_\xi z_k(\xi) - \partial_\xi y_i^m(\xi - \eta)}{|z_k(\xi) - y_i^m(\xi - \eta)|^{2\alpha}} d\eta. \quad (2.39)$$

Note that the parameter M in (2.38) is independent of Z and is tied to the initial data for which we are trying to establish existence.

We have the following estimates for these operators.

Lemma 2.6. *There is $C(\alpha) > 0$ such that for any $Z, \tilde{Z} \in \overline{O^A}$, any k, i, m , and $\alpha < \frac{1}{4}$,*

$$\left\| H_{k,i}^m[Z] - H_{k,i}^m[\tilde{Z}] \right\|_{L^\infty} \leq C(\alpha) A^{2+2\alpha} \|Z - \tilde{Z}\|_{C^1}. \quad (2.40)$$

Proof. Let $w_k := z_k - \tilde{z}_k$, as well as $v_k^1 := w_k$ and $v_k^2 := \bar{w}_k$. Then

$$\begin{aligned} \left| H_{k,i}^m[Z](\xi) - H_{k,i}^m[\tilde{Z}](\xi) \right| &\leq \int_{\mathbb{T}} \frac{|\partial_\xi w_k(\xi) - \partial_\xi v_i^m(\xi - \eta)|}{|\tilde{z}_k(\xi) - \tilde{y}_i^m(\xi - \eta)|^{2\alpha}} d\eta \\ &\quad + \int_{\mathbb{T}} \frac{|\partial_\xi z_k(\xi) - \partial_\xi y_i^m(\xi - \eta)|}{|\tilde{z}_k(\xi) - \tilde{y}_i^m(\xi - \eta)|^{2\alpha}} \left| \left(\frac{|\tilde{z}_k(\xi) - \tilde{y}_i^m(\xi - \eta)|}{|z_k(\xi) - y_i^m(\xi - \eta)|} \right)^{2\alpha} - 1 \right| d\eta \\ &\leq C(\alpha) A^{2\alpha} \|Z - \tilde{Z}\|_{C^1} + C(\alpha) A^{2+2\alpha} \|Z - \tilde{Z}\|_{L^\infty} \\ &\leq C(\alpha) A^{2+2\alpha} \|Z - \tilde{Z}\|_{C^1}, \end{aligned}$$

where similarly to (2.33), we used $|x^{2\alpha} - 1| \leq |x - 1|$ for $x \geq 0$ and (2.19) (for $m = 2$) to bound the second integral by

$$\int_{\mathbb{T}} \frac{C A^{3/2} \|Z - \tilde{Z}\|_{L^\infty}}{|\tilde{z}_k(\xi) - \tilde{y}_i^m(\xi - \eta)|^{2\alpha} |z_k(\xi) - y_i^m(\xi - \eta)|^{1/2}} d\eta \leq C(\alpha) A^{2+2\alpha} \|Z - \tilde{Z}\|_{L^\infty}$$

for $\alpha < \frac{1}{4}$. □

Lemma 2.7. *If $Z, \tilde{Z} \in \overline{O^A}$ and $\epsilon \in (0, c_0 A^{-2})$ (with c_0 from Lemma 2.5), then*

$$\|G_k^\epsilon[Z] - G_k^\epsilon[\tilde{Z}]\|_{C^n} \leq C(\alpha, n) \Theta_\epsilon^{-n-1} A^{2+2\alpha} \|Z - \tilde{Z}\|_{L^\infty}$$

for any k , integer $n \geq 0$, and $\alpha \in (0, \frac{1}{4})$. In particular, $G_k^\epsilon : \overline{O^A} \rightarrow B$ is Lipschitz.

Proof. It is easy to check that G_k^ϵ maps $\overline{O^A}$ to B for any $\epsilon > 0$. The properties of ϕ_ϵ and Lemma 2.6 now yield

$$\begin{aligned} \|G_k^\epsilon[Z] - G_k^\epsilon[\tilde{Z}]\|_{C^n} &\leq C(n)\epsilon^{-n} \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} \|H_{k,i}^m[\phi_\epsilon * Z] - H_{k,i}^m[\phi_\epsilon * \tilde{Z}]\|_{L^\infty} \\ &\leq C(\alpha, n)\Theta\epsilon^{-n}(2A)^{2+2\alpha} \|\phi_\epsilon * Z - \phi_\epsilon * \tilde{Z}\|_{C^1} \\ &\leq C(\alpha, n)\Theta\epsilon^{-n-1}A^{2+2\alpha} \|Z - \tilde{Z}\|_{L^\infty}. \end{aligned}$$

The last claim follows from taking $n = 3$ and using $\|Z - \tilde{Z}\|_{L^\infty} \leq \|Z - \tilde{Z}\|_{H^3}$. \square

Step 2. Let $\epsilon_0 := \min\{c_0(4M)^{-2}, 1\}$, with c_0 from Lemma 2.5 and $M = \|Z_0\| \geq 1$. We then have $\|Z_0^\epsilon\| \leq \|\phi_\epsilon * Z_0\| + \epsilon < 3M$ for any $\epsilon \in (0, \epsilon_0)$, hence $Z_0^\epsilon \in O^{3M}$. Also, Lemma 2.7 shows that G_k^ϵ is Lipschitz on O^{4M} for any k and $\epsilon \in (0, \epsilon_0)$. Lemma 2.4 and Picard's Theorem applied in Banach space B thus gives us a solution $Z^\epsilon(t) \in O^{4M}$ to (2.35) with initial data Z_0^ϵ , on some short time interval $[0, t']$ and in the integral sense. Then Lemma 2.7 with $n = 0$ shows

$$\sup_{t \in [0, t']} \|G_k^\epsilon[Z^\epsilon(t)]\|_{L^\infty} \leq C(\alpha)\Theta\epsilon^{-1}M^{3+2\alpha}$$

for each $\epsilon > 0$, so that $Z^\epsilon : [0, t'] \rightarrow L^\infty(\mathbb{T})$ is Lipschitz. Another application of Lemma 2.7, with Z and \tilde{Z} being Z^ϵ at different times shows that $\partial_\xi^n G_k^\epsilon[Z^\epsilon(\cdot)](\cdot)$ ($= \partial_n^\xi \partial_t z_k^\epsilon(\cdot, \cdot)$) is Lipschitz on $\mathbb{T} \times [0, t']$ for each $n \geq 0$. Since $z_k^\epsilon(\xi, t) = z_k^\epsilon(\xi, 0) + \int_0^t \partial_t z_k^\epsilon(\xi, s) ds$ and $Z^\epsilon(0) \in C^\infty$, we have that $Z^\epsilon \in C^{\infty, 1}(\mathbb{T} \times [0, t'])$.

This Z^ϵ can then be continued in time as long as it stays in O^{4M} , and we let t_ϵ be the maximal such time. We have $Z^\epsilon \in C^{\infty, 1}(\mathbb{T} \times [0, t_\epsilon])$ as above. We will therefore be able to apply the a priori estimates from the previous sub-section to Z^ϵ , and show that t_ϵ is bounded below by some $T(\alpha, N, \Theta, M) > 0$, uniformly in $\epsilon \in (0, \epsilon_0)$ (for all small $\alpha > 0$).

Using $\int f(\psi * g) d\xi = \int (\psi * f) g d\xi$ for any even ψ , we now obtain (dropping t)

$$\frac{d}{dt} \|\partial_\xi^3 z_k^\epsilon\|_{L^2}^2 = \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{\alpha} \int_{\mathbb{T}^2} \partial_\xi^3(\phi_\epsilon * z_k^\epsilon)(\xi) \cdot \partial_\xi^3 \left(\frac{\partial_\xi(\phi_\epsilon * z_k^\epsilon)(\xi) - \partial_\xi(\phi_\epsilon * y_i^{m, \epsilon})(\xi - \eta)}{|(\phi_\epsilon * z_k^\epsilon)(\xi) - (\phi_\epsilon * y_i^{m, \epsilon})(\xi - \eta)|^{2\alpha}} \right) d\eta d\xi$$

instead of (2.12). The estimates from the previous sub-section thus apply to Z^ϵ and lead to the analog of (2.28). Namely, for $t \in [0, t_\epsilon]$ and $\alpha \in (0, \frac{1}{24})$ we obtain

$$\frac{d}{dt} \|Z^\epsilon(t)\|_{H^3} \leq C(\alpha)N\Theta \|(\phi_\epsilon * Z^\epsilon)(t)\|^{7+2\alpha} \leq C(\alpha)N\Theta(8M)^{7+2\alpha},$$

where the last inequality holds due to Lemma 2.5 and $\epsilon < \epsilon_0$. The estimates for $\|Z(t)\|_{L^\infty}$, $\delta[Z(t)]^{-1}$, and $F[Z(t)]$ also extend to $Z^\epsilon(t)$, with the first gaining an additional term due to the extra term in (2.35):

$$\frac{d}{dt}\|Z^\epsilon(t)\|_{L^\infty} \leq C(\alpha)\Theta\|Z^\epsilon(t)\|^{1+2\alpha} + \tilde{C}(\alpha)\Theta\epsilon M^{3+2\alpha}.$$

So for $\epsilon \in (0, \epsilon_0)$ (recall that $\epsilon_0 \leq 1 \leq M$) and $t \in [0, t_\epsilon]$ we have (with some $\bar{C}(\alpha) < \infty$ which also depends on our choice of $\tilde{C}(\alpha)$)

$$\frac{d}{dt}\|Z^\epsilon(t)\| \leq \bar{C}(\alpha)N\Theta M^{7+2\alpha}. \quad (2.41)$$

Since $\|Z_0^\epsilon\| < 3M$, it will follow that we have a uniform in $\epsilon \in (0, \epsilon_0)$ estimate $t_\epsilon > T(\alpha, N\Theta, M) := (\bar{C}(\alpha)N\Theta)^{-1}M^{-6-2\alpha} > 0$, as long as we can show that $h[Z^\epsilon(t)] = 0$ cannot happen for $t \in (0, T(\alpha, N\Theta, M)]$. This will be ensured by choosing $\tilde{C}(\alpha)$ large enough (keeping in mind that (2.41) yields $\|Z^\epsilon(t)\| < 4M$ for these t).

Indeed, let us assume that $Z^\epsilon(t) \in \overline{O^{4M}}$ and t is the minimal time for which we have $z_k^\epsilon(\xi, t) \in \partial D$ for some k, ξ . Symmetry shows $(H_{k,i}^1[Z^\epsilon(t)](\xi) + H_{k,i}^2[Z^\epsilon(t)](\xi)) \cdot e_2 = 0$ for any i , thus (dropping t)

$$\begin{aligned} \partial_t z_k^\epsilon(\xi) \cdot e_2 &= \left(\sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} (\phi_\epsilon * H_{k,i}^m[\phi_\epsilon * Z^\epsilon])(\xi) \right) \cdot e_2 + \tilde{C}(\alpha)\Theta\epsilon M^{3+2\alpha} \\ &\geq - \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} |(\phi_\epsilon * H_{k,i}^m[\phi_\epsilon * Z^\epsilon])(\xi) - H_{k,i}^m[\phi_\epsilon * Z^\epsilon](\xi)| \\ &\quad - \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} |H_{k,i}^m[\phi_\epsilon * Z^\epsilon](\xi) - H_{k,i}^m[Z^\epsilon](\xi)| + \tilde{C}(\alpha)\Theta\epsilon M^{3+2\alpha} \\ &=: -T_1 - T_2 + \tilde{C}(\alpha)\Theta\epsilon M^{3+2\alpha}. \end{aligned}$$

We use $\|\phi_\epsilon * f - f\|_{L^\infty} \leq \epsilon\|f'\|_{L^\infty}$, (2.19), and $\|\phi_\epsilon * Z^\epsilon\| \leq 8M$ to bound

$$\begin{aligned} T_1 &\leq \epsilon \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} \|\partial_\xi H_{k,i}^m[\phi_\epsilon * Z^\epsilon]\|_{L^\infty} \\ &\leq \epsilon \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} \int_{\mathbb{T}} \frac{\|\phi_\epsilon * Z^\epsilon\|_{C^2}}{|(\phi_\epsilon * z_k^\epsilon)(\xi) - (\phi_\epsilon * y_i^{m,\epsilon})(\xi - \eta)|^{2\alpha}} + \frac{|\partial_\xi(\phi_\epsilon * z_k^\epsilon)(\xi) - \partial_\xi(\phi_\epsilon * y_i^{m,\epsilon})(\xi - \eta)|^2}{|(\phi_\epsilon * z_k^\epsilon)(\xi) - (\phi_\epsilon * y_i^{m,\epsilon})(\xi - \eta)|^{1+2\alpha}} d\eta \\ &\leq C(\alpha)\Theta\epsilon M^{3+2\alpha}. \end{aligned}$$

On the other hand, Lemma 2.6 yields

$$T_2 \leq C(\alpha)\Theta\|\phi_\epsilon * Z^\epsilon - Z^\epsilon\|_{C^1} \leq C(\alpha)\Theta\epsilon M^{2+2\alpha}\|Z^\epsilon\|_{C^2} \leq C(\alpha)\Theta\epsilon M^{3+2\alpha}.$$

Hence

$$\partial_t z_k^\epsilon(\xi, t) \cdot e_2 \geq \left(\tilde{C}(\alpha) - C(\alpha) \right) \Theta \epsilon M^{3+2\alpha},$$

which is positive if we choose $\tilde{C}(\alpha) > C(\alpha)$. This yields a contradiction with our assumption that t is the first time of touch, so this choice of $\tilde{C}(\alpha)$ indeed ensures $h[Z^\epsilon(t)] > 0$ for $\epsilon \in (0, \epsilon_0)$ and $t \leq T(\alpha, N\Theta, M)$. Thus $t_\epsilon > T(\alpha, N\Theta, M)$ for these ϵ (with $\alpha < \frac{1}{24}$).

Step 3. To obtain local existence of solutions to (2.5), we take $\epsilon \rightarrow 0$ (with $\alpha < \frac{1}{24}$). Let $\epsilon, \tilde{\epsilon} \in (0, \epsilon_0)$, consider $Z := Z^\epsilon$ and $\tilde{Z} := Z^{\tilde{\epsilon}}$ solving (2.35), and let $W := Z - \tilde{Z}$. We have for $t \in [0, T]$ (with $T = T(\alpha, N\Theta, M) > 0$ from Step 2, and dropping t from the arguments)

$$\frac{d}{dt} \|w_k\|_{L^2}^2 = 2 \int_{\mathbb{T}} w_k(\xi) \cdot \partial_t w_k(\xi) d\xi = K_k + \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{\alpha} (I_{k,i}^m + J_{k,i}^m),$$

where

$$K_k := 2\tilde{C}(\alpha)\Theta(\epsilon - \tilde{\epsilon})M^{3+2\alpha} \int_{\mathbb{T}} w_k(\xi) \cdot e_2 d\xi,$$

while (by again using $\int f(\psi * g)d\xi = \int (\psi * f)gd\xi$ for even ψ) $I_{k,i}^m$ equals

$$\int_{\mathbb{T}^2} (\phi_{\tilde{\epsilon}} * w_k)(\xi) \cdot \left(\frac{\partial_\xi(\phi_{\tilde{\epsilon}} * z_k)(\xi) - \partial_\xi(\phi_{\tilde{\epsilon}} * y_i^m)(\xi - \eta)}{|(\phi_{\tilde{\epsilon}} * z_k)(\xi) - (\phi_{\tilde{\epsilon}} * y_i^m)(\xi - \eta)|^{2\alpha}} - \frac{\partial_\xi(\phi_{\tilde{\epsilon}} * \tilde{z}_k)(\xi) - \partial_\xi(\phi_{\tilde{\epsilon}} * \tilde{y}_i^m)(\xi - \eta)}{|(\phi_{\tilde{\epsilon}} * \tilde{z}_k)(\xi) - (\phi_{\tilde{\epsilon}} * \tilde{y}_i^m)(\xi - \eta)|^{2\alpha}} \right) d\eta d\xi$$

and $J_{k,i}^m$ equals

$$\int_{\mathbb{T}} w_k(\xi) \cdot \left(\phi_\epsilon * \int_{\mathbb{T}} \frac{\partial_\xi(\phi_\epsilon * z_k)(\xi) - \partial_\xi(\phi_\epsilon * y_i^m)(\xi - \eta)}{|(\phi_\epsilon * z_k)(\xi) - (\phi_\epsilon * y_i^m)(\xi - \eta)|^{2\alpha}} d\eta - \phi_{\tilde{\epsilon}} * \int_{\mathbb{T}} \frac{\partial_\xi(\phi_{\tilde{\epsilon}} * z_k)(\xi) - \partial_\xi(\phi_{\tilde{\epsilon}} * y_i^m)(\xi - \eta)}{|(\phi_{\tilde{\epsilon}} * z_k)(\xi) - (\phi_{\tilde{\epsilon}} * y_i^m)(\xi - \eta)|^{2\alpha}} d\eta \right) d\xi.$$

We obviously have $|K_k| \leq C(\alpha)\Theta(\epsilon + \tilde{\epsilon})M^{3+2\alpha}\|W\|_{L^2}$ (with a new $C(\alpha)$). As in the above uniqueness argument estimating the right hand side of (2.32), only with all functions mollified by $\phi_{\tilde{\epsilon}}$, we obtain

$$|I_{k,i}^m| \leq C(\alpha)(\|\phi_{\tilde{\epsilon}} * Z\| + \|\phi_{\tilde{\epsilon}} * \tilde{Z}\|)^{2+2\alpha} \|\phi_{\tilde{\epsilon}} * W\|_{L^2}^2 \leq C(\alpha)M^{2+2\alpha}\|W\|_{L^2}^2$$

for all k, i, m, t . On the other hand, the bound

$$\|\phi_\epsilon * f - \phi_{\tilde{\epsilon}} * g\|_{L^\infty} \leq \|(\phi_\epsilon - \phi_{\tilde{\epsilon}}) * f\|_{L^\infty} + \|\phi_{\tilde{\epsilon}} * (f - g)\|_{L^\infty} \leq (\epsilon + \tilde{\epsilon})\|f'\|_{L^\infty} + \|f - g\|_{L^\infty}$$

can be repeatedly used to show that the term in the parentheses in the definition of $J_{k,i}^m$ is bounded uniformly in ξ by

$$C(\alpha)(\epsilon + \tilde{\epsilon}) (\|Z\|_{C^1}\|Z\|_{C^2}\|\phi_\epsilon * Z\|^{1+2\alpha} + \|Z\|_{C^2}\|\phi_\epsilon * Z\|^{2\alpha} + \|Z\|_{C^1}\|Z\|_{C^2}(\|\phi_\epsilon * Z\| + \|\phi_{\tilde{\epsilon}} * Z\|)^{1+2\alpha})$$

(here we also used (2.19) for $\phi_\epsilon * z_k$). Hence

$$|J_{k,i}^m| \leq C(\alpha)(\epsilon + \tilde{\epsilon})M^{3+2\alpha}\|W\|_{L^2}$$

for all k, i, m, t , so we obtain

$$\frac{d}{dt}\|W(t)\|_{L^2} \leq C(\alpha)N\Theta M^{3+2\alpha}(\epsilon + \tilde{\epsilon} + \|W(t)\|_{L^2})$$

for all $t \in [0, T]$. Since also $\|W(0)\|_{L^2} \leq C(\epsilon + \tilde{\epsilon})(\|Z_0\|_{C^1} + 1) \leq CM(\epsilon + \tilde{\epsilon})$, we get for any $\epsilon, \tilde{\epsilon} \in (0, \epsilon_0)$,

$$\sup_{t \in [0, T]} \|Z^\epsilon(t) - Z^{\tilde{\epsilon}}(t)\|_{L^2} \leq C(\alpha, N\Theta, M)(\epsilon + \tilde{\epsilon}).$$

Hence Z^ϵ converges in $L^\infty([0, T]; L^2(\mathbb{T}^N))$ to some Z as $\epsilon \rightarrow 0$. This and the estimate $\sup_{t \in [0, T]} \|Z^\epsilon(t)\| \leq 4M$ for all $\epsilon \in (0, \epsilon_0)$ then show that $\sup_{t \in [0, T]} \|Z(t)\| \leq 4M$ and

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [0, T]} \|Z^\epsilon(t) - Z(t)\|_{H^s} = 0$$

for $s < 3$. We also obtain the same convergence in C^2 . This and $\sup_{t \in [0, T]} \|Z^\epsilon(t)\| \leq 4M$ yield that the integrands in (2.35) converge to the integrands in (2.5) as $\epsilon \rightarrow 0$, uniformly on compact subsets of $\mathbb{T} \times (\mathbb{T} \setminus \{0\}) \times [0, T]$ (i.e., with $\eta \neq 0$; note that the denominators in (2.5) are uniformly bounded below by $C(M)^{-1}\eta^{2\alpha}$ due to $\sup_{t \in [0, T]} \|Z(t)\| \leq 4M$). Since the integrands are also uniformly bounded above by $C(M)\eta^{-2\alpha}$ for all small ϵ (and $2\alpha < 1$), it follows that as $\epsilon \rightarrow 0$, the integrals in (2.35) converge to those in (2.5) uniformly on $\mathbb{T} \times [0, T]$. Lemma 2.6 applied to $Z(t)$ and its translate in ξ , together with $\sup_{t \in [0, T]} \|Z(t)\| \leq 4M$, shows that the integrals in (2.5) are Lipschitz in ξ , uniformly in $t \in [0, T]$. Hence the integrals in (2.35) convolved with ϕ_ϵ also converge to them uniformly on $\mathbb{T} \times [0, T]$ as $\epsilon \rightarrow 0$. Thus $\partial_t z_k^\epsilon$ (which is continuous) converges to the right-hand side of (2.5) as $\epsilon \rightarrow 0$, uniformly on $\mathbb{T} \times [0, T]$. The latter is then also continuous on $\mathbb{T} \times [0, T]$. But since $z_k^\epsilon \rightarrow z_k$ as $\epsilon \rightarrow 0$, we see that $\partial_t z_k$ exists and equals the (continuous) right-hand side of (2.5). Hence Z is a classical solution to (2.5) and obviously $Z(0) = Z_0$.

The above proves the following local regularity result for the contour equation (2.5) corresponding to the half-plane case $D = \mathbb{R} \times \mathbb{R}^+$.

Theorem 2.8. *Let $\theta_1, \dots, \theta_N \in \mathbb{R} \setminus \{0\}$ and $Z_0 = \{z_{0k}\}_{k=1}^N \in H^3(\mathbb{T}, \bar{D})^N$ be a collection of (counter-clockwise) initial parameterizations of patch boundaries with $\|Z_0\| < \infty$ (and $\|\cdot\|$ from (2.9)). For any $\alpha \in (0, \frac{1}{24})$, there is $T = T(\alpha, N \sum_{k=1}^N |\theta_k|, \|Z_0\|) > 0$ such that there exists a unique solution $Z = \{z_k\}_{k=1}^N$ to (2.5) in $L^\infty([0, T]; H^3(\mathbb{T}, \bar{D})^N) \cap C^1([0, T]; C(\mathbb{T}, \bar{D})^N)$ with $Z(0) = Z_0$ and $\sup_{t \in [0, T]} \|Z(t)\| < \infty$. This T can be chosen to be decreasing in the last two arguments and so that $\sup_{t \in [0, T]} \|Z(t)\| \leq 4\|Z_0\|$.*

We note that by using an argument similar to that in [20, Chapter 3], we could prove that $Z \in C([0, T]; H^3(\mathbb{T}, \bar{D})^N)$, but we will not need this.

2.4 Lemmas on non-negative functions in $H^3(\mathbb{T})$

We now prove the results about nonnegative H^3 functions, used in the proof of local well-posedness for the patch equation in H^3 .

Proof of Lemma 2.2. This is obvious for $\gamma = 0$ so let us consider $\gamma \in (0, 1]$. Reflection $\xi \mapsto -\xi$ shows that it suffices to consider $\xi \in \mathbb{T}$ such that $f'(\xi) > 0$. For such ξ let

$$\xi' := \xi - \left(\frac{f'(\xi)}{2\|f\|_{C^{1,\gamma}}} \right)^{1/\gamma}.$$

Then for any $\eta \in [\xi', \xi]$ we have

$$|f'(\eta) - f'(\xi)| \leq \|f\|_{C^{1,\gamma}} |\xi - \eta|^\gamma \leq \frac{f'(\xi)}{2}.$$

It follows that $f'(\eta) \geq \frac{1}{2}f'(\xi)$ for all $\eta \in [\xi', \xi]$, so

$$f(\xi) \geq f(\xi) - f(\xi') \geq \frac{f'(\xi)}{2}(\xi - \xi') = \frac{f'(\xi)^{1+1/\gamma}}{2^{1+1/\gamma}\|f\|_{C^{1,\gamma}}^{1/\gamma}}.$$

The result follows. □

Let us now prove Lemma 2.3. We start with showing that if $0 < f \in H^3(\mathbb{T})$, then $f^{\frac{1}{2}-\beta} \in W^{1,1}(\mathbb{T})$ for small $\beta \geq 0$. We were unable to find this result in the literature.

Lemma 2.9. *If $\beta \in [0, \frac{1}{6}]$ and $0 < f \in H^3(\mathbb{T})$, then*

$$\int_{\mathbb{T}} \frac{|f'(\xi)|}{f(\xi)^{\beta+1/2}} d\xi \leq 10\|f\|_{H^3}^{\frac{1}{2}-\beta}. \quad (2.42)$$

Proof. The integral on the left side of (2.42) is, up to a constant, the BV -norm of $f^{\frac{1}{2}-\beta}$, which we will bound as follows. Let $\xi_0 \leq \xi_1 \leq \dots \leq \xi_{n-1} \leq \xi_n = \xi_0 + 2\pi$ be a (finite) sequence of local extrema of f . If for any such sequence we can show that

$$\sum_{i=1}^n \left| f(\xi_i)^{\frac{1}{2}-\beta} - f(\xi_{i-1})^{\frac{1}{2}-\beta} \right|$$

is bounded above by the right hand side of (2.42), we will be done.

Using first $f > 0$ and concavity of the function $s^{1/2-\beta}$ on $(0, \infty)$, and then Hölder's inequality with $p = 2(\frac{1}{2} - \beta)^{-1}$ and $\frac{1}{q} = 1 - \frac{1}{p} = \frac{3}{4} + \frac{\beta}{2}$, we obtain

$$\begin{aligned} \sum_{i=1}^n \left| f(\xi_i)^{\frac{1}{2}-\beta} - f(\xi_{i-1})^{\frac{1}{2}-\beta} \right| &\leq \sum_{i=1}^n |h_i|^{\frac{1}{2}-\beta} \\ &= \sum_{i=1}^n \left(|h_i|^{\frac{1}{2}-\beta} d_i^{-\frac{5}{2}(\frac{1}{2}-\beta)} \right) d_i^{\frac{5}{2}(\frac{1}{2}-\beta)} \\ &\leq \left(\sum_{i=1}^n h_i^2 d_i^{-5} \right)^{\frac{1}{p}} \left(\sum_{i=1}^n d_i^{\frac{5}{2}(\frac{1}{2}-\beta)q} \right)^{\frac{1}{q}}, \end{aligned} \quad (2.43)$$

where $h_i := f(\xi_i) - f(\xi_{i-1})$ and $d_i := \xi_i - \xi_{i-1}$. Since $\sum_{i=1}^n d_i = 2\pi$, the second sum in the last expression is bounded above by $(2\pi)^{\frac{5}{2}(\frac{1}{2}-\beta)q}$ as long as $\frac{5}{2}(\frac{1}{2} - \beta)q \geq 1$, which is equivalent to $\beta \leq \frac{1}{6}$. Since $\beta \geq 0$, we have $\frac{5}{2}(\frac{1}{2} - \beta) \leq \frac{5}{4}$, so it suffices to prove that

$$\sum_{i=1}^n h_i^2 d_i^{-5} \leq [10(2\pi)^{-5/4}]^p \|f\|_{H^3}^2. \quad (2.44)$$

Since $\max_{\xi \in [\xi_{i-1}, \xi_i]} |f'(\xi)| \geq h_i d_i^{-1}$ and $f'(\xi_i) = 0 = f'(\xi_{i-1})$, we have

$$\max_{\xi \in [\xi_{i-1}, \xi_i]} |f''(\xi)| \geq 2h_i d_i^{-2}.$$

Hölder inequality and Rolle's theorem for f' (i.e., $f''(\xi) = 0$ for some $\xi \in [\xi_{i-1}, \xi_i]$) yield

$$\int_{\xi_{i-1}}^{\xi_i} f'''(\xi)^2 d\xi \geq \frac{1}{d_i} \left(\int_{\xi_{i-1}}^{\xi_i} |f'''(\xi)| d\xi \right)^2 \geq \frac{1}{d_i} \left(\max_{\xi \in [\xi_{i-1}, \xi_i]} |f''(\xi)| \right)^2 \geq \frac{4h_i^2}{d_i^5}.$$

Summing this up over $i = 1, \dots, n$ and using that $\frac{1}{4} < [10(2\pi)^{-5/4}]^p$ gives (2.44). \square

Proof of Lemma 2.3. Multiplying $(n-1)$ -st power of (2.18) with $\gamma = 1$ by $|f'(\xi)|f(\xi)^{-\beta-n/2}$, then integrating and using Lemma 2.9 yields with a universal $C < \infty$,

$$\int_{\mathbb{T}} \frac{|f'(\xi)|^n}{f(\xi)^{\beta+n/2}} dx \leq C^n \|f\|_{H^3}^{\frac{n-1}{2}} \int_{\mathbb{T}} \frac{|f'(\xi)|}{f(\xi)^{\beta+1/2}} d\xi \leq 3C^n \|f\|_{H^3}^{\frac{n}{2}-\beta}.$$

This is (2.23). As for (2.24), we obtain via integration by parts

$$\begin{aligned} \int_{\mathbb{T}} \frac{f''(\xi)^2}{f(\xi)^\beta} dx &\leq \left| \int_{\mathbb{T}} \frac{f'(\xi)f'''(\xi)}{f(\xi)^\beta} d\xi \right| + \beta \left| \int_{\mathbb{T}} \frac{f'(\xi)^2 f''(\xi)}{f(\xi)^{\beta+1}} d\xi \right| \\ &\leq \|f\|_{H^3} \|f' f^{-\beta}\|_{L^2} + \beta \|f\|_{C^2} \left| \int_{\mathbb{T}} \frac{f'(\xi)^2}{f(\xi)^{\beta+1}} d\xi \right|. \end{aligned}$$

For any $\beta \in [0, \frac{1}{6}]$ we have $\|f'f^{-\beta}\|_{L^\infty} \leq C\|f\|_{H^3}^{1-\beta}$ by (2.18) with $\gamma := \beta(1-\beta)^{-1}$, and $\|(f')^2f^{-\beta-1}\|_{L^1} \leq C\|f\|_{H^3}^{1-\beta}$ by (2.23) with $n = 2$. Sobolev embedding now yields (2.24). \square

3 Estimates on velocity fields and $C^{1,\gamma}$ patches

In this section, we prove some basic estimates on the fluid velocities for general ω , as well as on the geometry of $C^{1,\gamma}$ patches (we will always consider $\gamma \in (0, 1]$). Naturally, the latter also apply in the case of the more regular H^3 patches.

Lemma 3.1. *For $D = \mathbb{R} \times \mathbb{R}^+$, $\alpha \in (0, \frac{1}{2})$, and $u(\cdot, t)$ from (1.2) with $\omega(\cdot, t) \in L^1(D) \cap L^\infty(D)$, we have*

$$\|u(\cdot, t)\|_{L^\infty} \leq \frac{2\pi}{1-2\alpha} \|\omega(\cdot, t)\|_{L^\infty} + 2\|\omega(\cdot, t)\|_{L^1} \quad (3.1)$$

and

$$\|u(\cdot, t)\|_{C^{1-2\alpha}} \leq \frac{8\pi}{\alpha(1-2\alpha)} \|\omega(\cdot, t)\|_{L^\infty} + 2\|\omega(\cdot, t)\|_{L^1}. \quad (3.2)$$

Furthermore, if ω is weak-* continuous as a function from some time interval $[a, b]$ to $L^\infty(D)$, and is supported inside some fixed compact subset of \bar{D} for every $t \in [a, b]$, then u is continuous on $\bar{D} \times [a, b]$.

Proof. Let $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the odd extension of $\omega(\cdot, t)$ to the whole plane. The velocity law (1.2) for $x \in D$ then becomes

$$u(x, t) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \eta(y) dy, \quad (3.3)$$

and (3.1) follows from

$$\begin{aligned} |u(x, t)| &\leq \int_{|x-y| \leq 1} \frac{|\eta(y)|}{|x-y|^{1+2\alpha}} dy + \int_{|x-y| > 1} \frac{|\eta(y)|}{|x-y|^{1+2\alpha}} dy \\ &\leq \|\eta\|_{L^\infty} \int_{|x-y| \leq 1} \frac{1}{|x-y|^{1+2\alpha}} dy + \|\eta\|_{L^1} \\ &\leq \frac{2\pi}{1-2\alpha} \|\omega(\cdot, t)\|_{L^\infty} + 2\|\omega(\cdot, t)\|_{L^1}. \end{aligned}$$

To prove (3.2), consider any $x, z \in \bar{D}$ with $r := |x - z|$. Then

$$\begin{aligned}
|u(x, t) - u(z, t)| &\leq \int_{B(x, 2r)} \frac{1}{|x - y|^{1+2\alpha}} \eta(y) dy + \int_{B(x, 2r)} \frac{1}{|z - y|^{1+2\alpha}} \eta(y) dy \\
&\quad + \int_{\mathbb{R}^2 \setminus B(x, 2r)} \left| \frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} - \frac{(z - y)^\perp}{|z - y|^{2+2\alpha}} \right| \eta(y) dy \\
&\leq 4\pi \|\eta\|_{L^\infty} \int_0^{3r} s^{-2\alpha} ds + 32 \|\eta\|_{L^\infty} \int_{2r}^\infty r s^{-1-2\alpha} ds \\
&\leq \left(\frac{12\pi}{1-2\alpha} + \frac{32}{2\alpha} \right) \|\eta\|_{L^\infty} |x - z|^{1-2\alpha}.
\end{aligned}$$

Combining this with (3.1) yields (3.2).

It remains to prove the last claim. Since the kernel in (3.3) is L^1 on any compact subset of \bar{D} , the assumptions show that u is continuous in $t \in [a, b]$ for any fixed $x \in \bar{D}$. The claim now follows from uniform continuity of u in $x \in \bar{D}$, see (3.2). \square

Remark. As is clear from the proof, the lemma remains valid in the more general case where u is given by (3.3) with $\omega(y, t)$ in place of $\eta(y)$ and ω satisfies the hypotheses of the lemma with D replaced by \mathbb{R}^2 .

The remaining results in this section hold for a single time t , so we drop the time variable from the notation.

Lemma 3.2. *For $\alpha \in (0, \frac{1}{2})$ and $\omega \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, let*

$$v(x) := \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} \omega(y) dy. \quad (3.4)$$

Assume that $\omega \equiv c$ in $B(x, d)$, for some $x \in \mathbb{R}^2$, $d > 0$, and $c \in \mathbb{R}$. Then

$$|\nabla u(x)| \leq C(\alpha) \|\omega\|_\infty d^{-2\alpha},$$

and more generally, for D^n any spatial derivative of order n we have

$$|D^n u(x)| \leq C(\alpha, n) \|\omega\|_\infty d^{1-2\alpha-n}.$$

Proof. Let $\phi : \mathbb{R}^+ \rightarrow [0, 1]$ be a smooth function with $\phi \equiv 0$ on $[0, \frac{1}{3}]$, $\phi \equiv 1$ on $[\frac{1}{2}, \infty)$, and $0 \leq \phi' \leq 10$. Let

$$g(x) := \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} \phi\left(\frac{|x - y|}{d}\right) \omega(y) dy.$$

Then $g \equiv v$ on $B(x, \frac{d}{2})$ due to $\omega \equiv c$ in $B(x, d)$ and the mean zero property of the kernel. Hence

$$\begin{aligned} |\nabla v(x)| &= |\nabla g(x)| \leq C(\alpha) \|\omega\|_\infty \int_{\mathbb{R}^2} \left[\frac{1}{|x-y|^{2+2\alpha}} \phi\left(\frac{|x-y|}{d}\right) + \frac{1}{d|x-y|^{1+2\alpha}} \phi'\left(\frac{|x-y|}{d}\right) \right] dy \\ &\leq C(\alpha) \|\omega\|_\infty \left(\int_{d/3}^\infty r^{-(1+2\alpha)} dr + \int_{d/3}^{d/2} d^{-1} r^{-2\alpha} dr \right) \\ &\leq C(\alpha) \|\omega\|_\infty d^{-2\alpha}. \end{aligned}$$

The proof of the higher derivatives case is analogous. \square

Let us now turn to $C^{1,\gamma}$ patches.

Definition 3.3. For a bounded open $\Omega \subseteq \mathbb{R}^2$ whose boundary is a simple closed $C^{1,\gamma}$ curve with arc-length $|\partial\Omega| =: 2\pi L$, let $\|\Omega\|_{1,\gamma} := \|z\|_{C^{1,\gamma}} + F[z]$, where z is any constant speed parametrization of $\partial\Omega$ from Definition 1.1 and $F[z] := \max\{\sup_{\xi,\eta \in \mathbb{T}, \xi \neq \eta} \frac{|\xi-\eta|}{|z(\xi)-z(\eta)|}, 1\}$. We also denote by n_P the outer unit normal vector for Ω at $P \in \partial\Omega$.

Remark. We clearly have $z'(\xi) = -Ln_{z(\xi)}^\perp$.

Lemma 3.4. Let $\Omega \subseteq \mathbb{R}^2$ be as in Definition 3.3, with $\|\Omega\|_{1,\gamma} \leq A$ for some $A \geq 1$. Let $R := L^{\frac{1}{\gamma}}(4A)^{-\frac{1}{\gamma}-1}$ and consider any $P \in \partial\Omega$. Then we have:

- (a) $\partial\Omega \cap B(P, R)$ is a simply connected curve.
- (b) In the coordinate system (w_1, w_2) centered at P and with axes n_P^\perp and n_P , the set $\partial\Omega \cap B(P, R)$ is a graph $w_2 = f(w_1)$ with $|f(w_1)| \leq 4L^{-1-\gamma} A |w_1|^{1+\gamma}$.
- (c) For any $Q \in \partial\Omega \cap B(P, R)$, we have $|n_P - n_Q| \leq 2L^{-1-\gamma} A |P - Q|^\gamma$.
- (d) If also $\Omega \subseteq D$, then for $P = (p_1, p_2) \in \partial\Omega$ we have $|n_P \cdot (1, 0)| \leq 2L^{-1} A^{\frac{1}{1+\gamma}} p_2^{\frac{\gamma}{1+\gamma}}$.

Proof. (a) Let $\xi_0 \in \partial\Omega$ be arbitrary and let $P := z(\xi_0)$. Since $\|z\|_{C^{1,\gamma}} \leq A$, we have $-z'(\xi) \cdot n_P^\perp \geq \frac{L}{2}$ for all ξ such that $|\xi - \xi_0| \leq \left(\frac{L}{2A}\right)^{1/\gamma}$. Moreover, if $|\xi - \xi_0| > \left(\frac{L}{2A}\right)^{1/\gamma}$, then $z(\xi) \notin B(P, R)$ due to $F[z] \leq A$ and the definition of R .

(b) Let us write $z(\xi) = P + w_1(\xi)n_P^\perp + w_2(\xi)n_P$. Then for any $z(\xi) \in B(P, R)$, the discussion above gives $-w_1'(\xi) \geq \frac{L}{2}$ and thus $|w_1(\xi)| \geq \frac{L}{2}|\xi - \xi_0|$, where $P = z(\xi_0)$. Since

$w_2'(\xi_0) = z'(\xi_0) \cdot n_P = 0$, we also have $|w_2'(\xi)| = |w_2'(\xi) - w_2'(\xi_0)| \leq A|\xi - \xi_0|^\gamma$. Hence when $z \in B(P, R)$ (i.e., when $w_1(\xi)^2 + w_2(\xi)^2 \leq R^2$), we have

$$\left| \frac{dw_2}{dw_1} \right| = \left| \frac{w_2'(\xi)}{w_1'(\xi)} \right| \leq \frac{A|\xi - \xi_0|^\gamma}{L/2} \leq \frac{A|2w_1(\xi)/L|^\gamma}{L/2} \leq 4L^{-1-\gamma}A|w_1|^\gamma.$$

Integrating this inequality gives $|w_2| \leq 4(1 + \gamma)^{-1}L^{-1-\gamma}A|w_1|^{1+\gamma}$.

(c) Let $\xi_1 \in \mathbb{T}$ be such that $Q = z(\xi_1)$. From $n_P^\perp = -\frac{1}{L}z'(\xi_0)$ and $n_Q^\perp = -\frac{1}{L}z'(\xi_1)$ we obtain

$$|n_P - n_Q| = |n_P^\perp - n_Q^\perp| = \frac{|z'(\xi_0) - z'(\xi_1)|}{L} \leq \frac{A|\xi_0 - \xi_1|^\gamma}{L} \leq \frac{A(2|P - Q|/L)^\gamma}{L} \leq 2L^{-1-\gamma}A|P - Q|^\gamma,$$

where we also used $|z(\xi_1) - z(\xi_0)| \geq \frac{L}{2}|\xi_1 - \xi_0|$, due to $-w_1'(\xi) \geq \frac{L}{2}$ when $z(\xi) \in B(P, R)$.

(d) Let again $P = z(\xi_0)$, so that we need to show $|z_2'(\xi_0)| \leq 2A^{\frac{1}{1+\gamma}}p_2^{\frac{\gamma}{1+\gamma}}$. We have $|z_2'(\xi) - z_2'(\xi_0)| \leq \frac{1}{2}|z_2'(\xi_0)|$ when $|\xi - \xi_0| \leq \left(\frac{|z_2'(\xi_0)|}{2A}\right)^{1/\gamma}$, so $\xi_\pm := \xi_0 \pm \left(\frac{|z_2'(\xi_0)|}{2A}\right)^{1/\gamma}$ satisfy

$$0 \leq \min\{z_2(\xi_+), z_2(\xi_-)\} \leq z_2(\xi_0) - \left(\frac{|z_2'(\xi_0)|}{2A}\right)^{1/\gamma} \frac{|z_2'(\xi_0)|}{2},$$

which is $|z_2'(\xi_0)| \leq 2A^{\frac{1}{1+\gamma}}p_2^{\frac{\gamma}{1+\gamma}}$. □

4 Local regularity for the patch equation and small α

In this section we will prove that the solution of the contour equation which we constructed in Section 2 yields a (local) H^3 patch solution to (1.1)-(1.2), and that the latter is also the unique H^3 patch solution to (1.1)-(1.2). This is achieved in Corollary 4.7 and Theorem 4.12, which together with the remark after Corollary 4.7 prove Theorem 1.4. We consider here the half-plane case $D = \mathbb{R} \times \mathbb{R}^+$, but the arguments are identical for the whole plane $D = \mathbb{R}^2$.

4.1 Contour equation solution is a patch solution

We start with using the results from the previous section to show that the solution of the contour equation is also a patch solution to (1.1)-(1.2). The main result of this sub-section is the following proposition.

Proposition 4.1. *Consider the setting of Theorem 2.8 and let $\Omega_k(t)$ be the interior of the contour $z_k(\cdot, t)$. Then $\omega(\cdot, t) := \sum_{k=1}^N \theta_k \chi_{\Omega_k(t)}$ is an H^3 patch solution to (1.1)-(1.2) on $[0, T]$.*

Since $z_k(\cdot, t)$ need not be a constant speed parametrization of $\partial\Omega_k(t)$, we first need to obtain a bound on the latter.

Lemma 4.2. *Let $Z = (z_1, \dots, z_N) : \mathbb{T} \rightarrow (\mathbb{R}^2)^N$ and assume $\|Z\| < \infty$, with $\|\cdot\|$ from (2.9) (thus the z_k are pairwise disjoint simple closed curves). Let Ω_k be the interior of the curve z_k and let y_k be any constant speed parametrization of $\partial\Omega_k$ from Definition 1.1. There is a universal constant $C \geq 1$ such that $Y = (y_1, \dots, y_N)$ satisfies*

$$\|Y\| \leq C \|Z\|^8.$$

Proof. Since all constant speed parametrizations of $\partial\Omega_k$ are translations of each other on \mathbb{T} (and such translation does not affect $\|Y\|$), it suffices to prove the result for one of them. We will therefore assume $Y(0) = Z(0)$. We can also assume without loss that z_k is a counter-clockwise parametrization of $\partial\Omega_k$.

We obviously have $\delta[Y] = \delta[Z]$ and $\|y_k\|_{L^\infty} = \|z_k\|_{L^\infty}$ for each k . Since y_k and z_k are both counter-clockwise parametrizations of $\partial\Omega_k$, with $|y'_k(\xi)| \equiv \frac{1}{2\pi} |\partial\Omega_k|$, there is a bijection $f_k : \mathbb{T} \rightarrow \mathbb{T}$ with $f_k(0) = 0$ such that $y_k(f_k(\xi)) \equiv z_k(\xi)$. Then for $\xi \in \mathbb{T}$,

$$f'_k(\xi) = \frac{2\pi |z'_k(\xi)|}{|\partial\Omega_k|}. \quad (4.1)$$

To simplify notation, let us now drop the index k , and denote $y = (y_1, y_2)$ and $z = (z_1, z_2)$. For any distinct $\eta_1, \eta_2 \in \mathbb{T}$, there are distinct $\xi_1, \xi_2 \in \mathbb{T}$ such that $\eta_1 = f(\xi_1)$ and $\eta_2 = f(\xi_2)$. Then

$$\frac{|\eta_1 - \eta_2|}{|y(\eta_1) - y(\eta_2)|} = \frac{|f(\xi_1) - f(\xi_2)|}{|z(\xi_1) - z(\xi_2)|} \leq \frac{|f(\xi_1) - f(\xi_2)|}{|\xi_1 - \xi_2|} F[Z] \leq \|f'\|_{L^\infty} \|Z\|.$$

Since we have $\|z\|_{C^1} \leq C \|Z\|$ (with a universal $C < \infty$, which may change later) and $|\partial\Omega| \geq 2|z(\pi) - z(0)| \geq \frac{2\pi}{F[Z]}$, it follows from (4.1) that $\|f'\|_{L^\infty} \leq C \|Z\|^2$, yielding $F[Y] \leq C \|Z\|^3$.

Since $\|Z\| \geq 1$ by definition, it thus suffices to show $\|y'''\|_{L^2} \leq C \|z\|_{H^3} \|Z\|^7$. A direct computation yields

$$y'_1(f(\xi)) = \frac{|\partial\Omega|}{2\pi} \frac{z'_1(\xi)}{|z'(\xi)|},$$

$$y_1''(f(\xi)) = \frac{|\partial\Omega|^2}{(2\pi)^2} \left(\frac{z_1''(\xi)}{|z'(\xi)|^2} - \frac{z_1'(\xi)[z_1''(\xi)z_1'(\xi) + z_2''(\xi)z_2'(\xi)]}{|z'(\xi)|^4} \right),$$

$$y_1'''(f(\xi)) = \frac{|\partial\Omega|^3}{(2\pi)^3} \left(\frac{z_1'''}{|z'|^3} - \frac{z_1'''(z_1')^2 + z_2'''z_1'z_2' + 4(z_1'')^2z_1' + 3z_1''z_2''z_2' + (z_2'')^2z_1'}{|z'|^5} + \frac{4z_1'[z_1''z_1' + z_2''z_2']^2}{|z'|^7} \right),$$

where for convenience we dropped ξ in the last expression. Therefore,

$$|y_1'''(f(\xi))| \leq C|\partial\Omega|^3 \left(\frac{|z'''(\xi)|}{|z'(\xi)|^3} + \frac{|z''(\xi)|^2}{|z'(\xi)|^4} \right).$$

This gives the following estimate on $\|y_1'''\|_{L^2}$ (recall that f is a bijection):

$$\begin{aligned} \|y_1'''\|_{L^2}^2 &= \int_{\mathbb{T}} [y_1'''(f(\xi))]^2 f'(\xi) d\xi \\ &\leq \int_{\mathbb{T}} C|\partial\Omega|^6 \left(\frac{|z'''(\xi)|^2}{|z'(\xi)|^6} + \frac{|z''(\xi)|^4}{|z'(\xi)|^8} \right) \frac{|z'(\xi)|}{|\partial\Omega|} d\xi \\ &\leq C|\partial\Omega|^5 \left(\frac{\|z\|_{H^3}^2}{\min_{\xi \in \mathbb{T}} |z'(\xi)|^5} + \frac{\|z\|_{C^2}^4}{\min_{\xi \in \mathbb{T}} |z'(\xi)|^7} \right). \end{aligned}$$

Since $|\partial\Omega| \leq \frac{1}{2\pi} \|z'\|_{C^1} \leq C\|Z\|$ and $\min_{\xi \in \mathbb{T}} |z'(\xi)| \geq \frac{1}{F|Z|} \geq \|Z\|^{-1}$, it follows that $\|y_1'''\|_{L^2} \leq C\|z\|_{H^3}\|Z\|^7$. Since the same estimate holds for y_2 , the proof is finished. \square

Proof of Proposition 4.1. First note that by Theorem 2.8, the boundaries $\partial\Omega_k(t)$ are pairwise disjoint simple closed curves in \bar{D} for each $t \in [0, T]$, which also have parametrizations $z_k(\cdot, t)$ that are uniformly-in-time bounded in H^3 . Due to Lemma 4.2, the latter then also holds for their constant speed parametrizations. Lemma 2.1 shows that each $\partial\Omega_k$ is continuous in time with respect to Hausdorff distance, so it remains to show (1.6).

The derivation of (2.5) shows that its right-hand side $S_k[Z(t)](\xi)$ satisfies

$$S_k[Z(t)](\xi) = u(z_k(\xi, t), t) + \beta_k(\xi, t)\partial_\xi z_k(\xi, t)$$

for some $\beta_k(\xi, t) \in \mathbb{R}$, and Theorem 2.8 shows that $S_k[Z(\cdot)](\cdot)$ is continuous on $\mathbb{T} \times [0, T]$. Since z_k and u are also continuous (the latter by Lemma 3.1) and $\partial_\xi z_k(\xi, t) \geq F[Z(t)]^{-1} > 0$, we will have that β_k is continuous if we show that $\partial_\xi z_k$ is. But the latter holds because $\sup_{t \in [0, T]} \|z_k(\cdot, t)\|_{C^2} < \infty$ and z_k is continuous in $t \in [0, T]$.

This means that for each $\tau \in [0, T]$ and $x \in \partial\Omega_k(\tau)$, the ODE $\zeta'(t) = -\beta_k(\zeta(t), t)$ has a solution $\zeta : [0, T] \rightarrow \mathbb{T}$ with $\zeta(\tau) = \xi$, where $\xi \in \mathbb{T}$ is such that $z_k(\xi, \tau) = x$. Then $\Psi_{x, \tau}(t) := z_k(\zeta(t), t) \in \partial\Omega_k(t)$ solves

$$\frac{d}{dt} \Psi_{x, \tau}(t) = u(\Psi_{x, \tau}(t), t) \quad \text{for } t \in [0, T] \text{ and } \quad \Psi_{x, \tau}(\tau) = x.$$

Given any $t \in (0, T)$, for any small $h \in \mathbb{R}$ and $x \in \partial\Omega(t+h)$, let $y_{x,t,h} := \Psi_{x,t+h}(t) \in \partial\Omega(t)$ and $\tilde{y}_{x,t,h} := y_{x,t,h} + hu(y_{x,t,h}, t) \in X_{u(\cdot,t)}^h[\partial\Omega(t)]$. Then

$$\sup_{x \in \partial\Omega(t+h)} |x - \tilde{y}_{x,t,h}| \leq hw(2|h|\|u\|_{L^\infty})$$

for all small h , where w is the modulus of continuity of u on some neighborhood of $\partial\Omega(t) \times \{t\}$. Note that w satisfies $\lim_{s \searrow 0} w(s) = 0$ because u is continuous and $\partial\Omega(t)$ is compact, which (together with (3.1)) yields

$$\lim_{h \rightarrow 0} \sup_{x \in \partial\Omega(t+h)} \frac{\text{dist}\left(x, X_{u(\cdot,t)}^h[\partial\Omega(t)]\right)}{h} = 0.$$

Similarly, for small $h \in \mathbb{R}$ and $x \in X_{u(\cdot,t)}^h[\partial\Omega(t)]$, there is $y_{x,t,h} \in \partial\Omega(t)$ such that $x = y_{x,t,h} + hu(y_{x,t,h}, t)$ and we let $\tilde{y}_{x,t,h} := \Psi_{y_{x,t,h},t}(t+h) \in \partial\Omega(t+h)$. Then again

$$\sup_{x \in X_{u(\cdot,t)}^h[\partial\Omega(t)]} |x - \tilde{y}_{x,t,h}| \leq hw(2|h|\|u\|_{L^\infty}),$$

so we obtain

$$\lim_{h \rightarrow 0} \sup_{x \in X_{u(\cdot,t)}^h[\partial\Omega(t)]} \frac{\text{dist}(x, \partial\Omega(t+h))}{h} = 0.$$

This proves (1.6), and the proof is finished. \square

4.2 Independence from initial contour parametrization

Next, we will show that the solution obtained in Proposition 4.1 is independent of the chosen contour parametrization Z_0 for a given initial value $\omega(\cdot, 0)$. The main result of this sub-section is Corollary 4.7.

Hence, let us consider two families $\Omega(t) = \{\Omega_k(t)\}_{k=1}^N$ and $\tilde{\Omega}(t) = \{\tilde{\Omega}_k(t)\}_{k=1}^N$ of sets as in Definition 1.2, but with $\Omega(t)$ now the sequence (rather than the union) of the $\Omega_k(t)$. This notation will be more convenient in what follows. We also drop the argument t where we discuss results concerning a fixed time.

We start with an estimate on the area of the symmetric difference $\Omega \Delta \tilde{\Omega} := (\Omega \setminus \tilde{\Omega}) \cup (\tilde{\Omega} \setminus \Omega)$. Recall the functional $\|\cdot\|$ from (2.9).

Lemma 4.3. *Let $\Omega = \{\Omega_k\}_{k=1}^N$ and $\tilde{\Omega} = \{\tilde{\Omega}_k\}_{k=1}^N$ be two families of bounded open subsets of \mathbb{R}^2 whose boundaries are simple closed curves, and let Z and \tilde{Z} be some parametriza-*

tions of $\partial\Omega$ and $\partial\tilde{\Omega}$, respectively (that is, $Z = \{z_k\}_{k=1}^N$, with $z_k : \mathbb{T} \rightarrow \mathbb{R}^2$ a parametrization of $\partial\Omega_k$, and similarly for \tilde{Z}). There exists a universal constant $C < \infty$ such that

$$|\Omega\Delta\tilde{\Omega}| := \sum_{k=1}^N |\Omega_k\Delta\tilde{\Omega}_k| \leq C(\|Z\| + \|\tilde{Z}\|) \sum_{k=1}^N \|z_k - \tilde{z}_k\|_{L^2}. \quad (4.2)$$

Proof. Let $A := \|Z\| + \|\tilde{Z}\|$. It obviously suffices to assume $A < \infty$, and to prove for each $k \in \{1, \dots, N\}$ that $|\Omega_k\Delta\tilde{\Omega}_k| \leq CA\|z_k - \tilde{z}_k\|_{L^1}$ (with some universal C). We will now do this, dropping the index k in the following.

We claim that for any $x \in \Omega\Delta\tilde{\Omega}$, there exists some $(\xi, s) \in \mathbb{T} \times [0, 1]$, such that $x = (1-s)z(\xi) + s\tilde{z}(\xi)$. This is obvious for $x \in \partial\Omega \cup \partial\tilde{\Omega}$, so assume that $x \notin \partial\Omega \cup \partial\tilde{\Omega}$. Let $\Gamma_s(\xi) := (1-s)z(\xi) + s\tilde{z}(\xi)$ for $(\xi, s) \in \mathbb{T} \times [0, 1]$, so that $\Gamma_s : \mathbb{T} \rightarrow \mathbb{R}^2$ is a closed curve for each fixed $s \in [0, 1]$, with $\Gamma_0 = z$ and $\Gamma_1 = \tilde{z}$. Since Γ_0 and Γ_1 have different winding numbers with respect to x and Γ is continuous in (s, ξ) , we must have $x \in \Gamma_s(\mathbb{T})$ for some $s \in (0, 1)$.

Consider now the quadrilateral $Q(\xi; h)$ with vertices at $z(\xi), z(\xi + h), \tilde{z}(\xi + h), \tilde{z}(\xi)$. The above discussion and both z, \tilde{z} being H^3 yield

$$|\Omega\Delta\tilde{\Omega}| \leq |\{(1-s)z(\xi) + s\tilde{z}(\xi) : (\xi, s) \in \mathbb{T} \times [0, 1]\}| \leq \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} \left| Q\left(\frac{2\pi j}{n}; \frac{2\pi}{n}\right) \right|. \quad (4.3)$$

We also have, with C such that $\|f\|_{C^1} \leq C\|f\|_{H^3}$ for each $f \in H^3(\mathbb{T})$,

$$\begin{aligned} |Q(\xi; h)| &\leq \max\{|z(\xi) - z(\xi + h)|, |\tilde{z}(\xi) - \tilde{z}(\xi + h)|\} \max\{|z(\xi) - \tilde{z}(\xi)|, |z(\xi + h) - \tilde{z}(\xi + h)|\} \\ &\leq CAh(|z(\xi) - \tilde{z}(\xi)| + 2CAh). \end{aligned}$$

Hence the limit in (4.3) is bounded above by $CA\|z - \tilde{z}\|_{L^1}$, and an application of Hölder inequality leads to (4.2). \square

Definition 4.4. For two families $\Omega = \{\Omega_k\}_{k=1}^N$ and $\tilde{\Omega} = \{\tilde{\Omega}_k\}_{k=1}^N$ of subsets of \mathbb{R}^2 , we define the Hausdorff distance of their boundaries to be

$$d_H(\partial\Omega, \partial\tilde{\Omega}) := \max_{1 \leq k \leq N} \max \left\{ \sup_{x \in \partial\Omega_k} \inf_{y \in \partial\tilde{\Omega}_k} |x - y|, \sup_{x \in \partial\tilde{\Omega}_k} \inf_{y \in \partial\Omega_k} |x - y| \right\}.$$

Next we prove that if we solve the contour equation (2.5) with two families of H^3 initial curves which parametrize the same simple closed curves $\partial\Omega(0) := \{\partial\Omega_k(0)\}_{k=1}^N$, then the solutions parametrize the same curves $\partial\Omega(t) := \{\partial\Omega_k(t)\}_{k=1}^N$ throughout their common interval of existence.

Proposition 4.5. *Let $\alpha \in (0, \frac{1}{24})$, let $\theta_1, \dots, \theta_N \in \mathbb{R}$, and let Z, \tilde{Z} be both as Z in Theorem 2.8, with initial conditions Z_0, \tilde{Z}_0 , respectively. Let $T' > 0$ be the smaller of their maximal times of existence and for $t \in [0, T']$, let $\Omega_k(t)$ and $\tilde{\Omega}_k(t)$ be the interiors of the contours $z_k(\cdot, t)$ and $\tilde{z}_k(\cdot, t)$, respectively. If $\Omega_k(0) = \tilde{\Omega}_k(0)$ for each k , then $\Omega_k(t) = \tilde{\Omega}_k(t)$ for each k and $t \in [0, T']$.*

Remark. Here T' is largest such that $\sup_{t \in [0, T']} (\|Z(t)\| + \|\tilde{Z}(t)\|) < \infty$ for each $T < T'$.

Proof. Due to the uniqueness claim in Theorem 2.8, it suffices to prove this for the smaller of the $T > 0$ (from the theorem) for Z and \tilde{Z} , instead of for T' . We then have $\sup_{t \in [0, T]} (\|Z(t)\| + \|\tilde{Z}(t)\|) \leq 4(\|Z(0)\| + \|\tilde{Z}(0)\|) =: A$.

Our strategy here is to first prove the claim for a family of regularized equations, and then show that the solutions of the latter converge to those of the original equation (in an appropriate sense) as their parameter $\beta \rightarrow 0$.

Specifically, for any $\beta > 0$, we regularize the Biot-Savart law in (1.2) to

$$u^\beta(x, t) = \int_D \left(\frac{(x - y)^\perp}{(|x - y|^2 + \beta^2)^{1+\alpha}} - \frac{(x - \bar{y})^\perp}{(|x - \bar{y}|^2 + \beta^2)^{1+\alpha}} \right) \omega(y, t) dy \quad (4.4)$$

for $x \in \bar{D}$. For (1.1) with $u = u^\beta$, following the same derivation as in Section 2.1, we obtain the contour equation

$$\partial_t z_k(\xi, t) = \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{2\alpha} \int_{\mathbb{T}} \frac{\partial_\xi z_k(\xi, t) - \partial_\xi y_i^m(\xi - \eta, t)}{(|z_k(\xi, t) - y_i^m(\xi - \eta, t)|^2 + \beta^2)^\alpha} d\eta \quad (4.5)$$

instead of (2.5). Then the same derivation as in Section 2.2 again yields the a priori estimate (2.31) for the solutions to (4.5), with the same (β -independent) constant. We can then use the same arguments as in Section 2.3 to show that there exist unique (local) solutions Z^β and \tilde{Z}^β to (4.5) with initial data $Z(0)$ and $\tilde{Z}(0)$, respectively, and they again satisfy $\sup_{t \in [0, T]} (\|Z^\beta(t)\| + \|\tilde{Z}^\beta(t)\|) \leq A$, for the above (β -independent) time T .

If now $\Omega_k^\beta(t)$ and $\tilde{\Omega}_k^\beta(t)$ are the interiors of the contours $z_k^\beta(\cdot, t)$ and $\tilde{z}_k^\beta(\cdot, t)$, respectively, then we can show as in Proposition 4.1 that $\omega^\beta(\cdot, t) := \sum_{k=1}^N \theta_k \chi_{\Omega_k^\beta(t)}$ and $\tilde{\omega}^\beta(\cdot, t) := \sum_{k=1}^N \theta_k \chi_{\tilde{\Omega}_k^\beta(t)}$ are H^3 patch solutions to (1.1) with $u = u^\beta$ on $[0, T]$. (Note also that since u^β is smooth, $\Phi_t(x)$ from (1.4) is uniquely defined for any $(x, t) \in \bar{D} \times [0, T]$.)

One can now apply standard estimates for the 2D Euler equation (see, e.g., [20, Theorems 8.1 and 8.2]) to (1.1) with the (smooth) velocity $u = u^\beta$ to show that there exists a

unique weak solution to it on $\mathbb{R}^2 \times [0, \infty)$ with initial data $\omega^\beta(\cdot, 0)$ ($= \tilde{\omega}^\beta(\cdot, 0)$) extended oddly to $\mathbb{R} \times \mathbb{R}^-$. Since H^3 patch solutions are also weak solutions (see Remark 3 after Definition 1.2), and obviously remain such when extended oddly to $x \in \mathbb{R} \times \mathbb{R}^-$, it follows that for any $\beta > 0$ we have $\Omega_k^\beta(t) = \tilde{\Omega}_k^\beta(t)$ for each k and $t \in [0, T]$. (See the remark after this proof for an alternative argument.)

Next, we claim that with $\partial\Omega^\beta(t) := \{\partial\Omega_k^\beta(t)\}_{k=1}^N$ we have

$$\lim_{\beta \rightarrow 0} \sup_{t \in [0, T]} d_H(\partial\Omega^\beta(t), \partial\Omega(t)) = 0. \quad (4.6)$$

Since the same result then holds with $\tilde{\Omega}$ in place of Ω , this proves the proposition. The key step in showing (4.6) is the estimate

$$\sup_{t \in [0, T]} \|Z^\beta(t) - Z(t)\|_{L^2} \leq C(\alpha, N\Theta, A, T)\beta, \quad (4.7)$$

with $\Theta := \sum_{k=1}^N |\theta_k|$. Then Lemma 4.3 yields $\sup_{t \in [0, T]} |\Omega^\beta(t) \Delta \Omega(t)| \leq C(\alpha, N, \Theta, A)\beta$ for each $\beta > 0$, which together with the uniform H^3 bound on Z, Z^β implies (4.6). It therefore remains to prove (4.7).

We let $W := Z - Z^\beta$ so that (after dropping the argument t)

$$\frac{d}{dt} \|w_k\|_{L^2}^2 = 2 \int_{\mathbb{T}} w_k(\xi) \cdot \partial_t w_k(\xi) d\xi = G_k + \sum_{i=1}^N \sum_{m=1}^2 \frac{\theta_i}{\alpha} S_{k,i}^m,$$

where G_k is the right hand side of (2.32) with \tilde{Z} is replaced by Z^β , while $S_{k,i}^m$ equals

$$\int_{\mathbb{T}^2} w_k(\xi) \cdot \left[\partial_\xi z_k^\beta(\xi) - \partial_\xi y_i^{m,\beta}(\xi - \eta) \right] \left[\frac{1}{|z_k^\beta(\xi) - y_i^{m,\beta}(\xi - \eta)|^{2\alpha}} - \frac{1}{(|z_k^\beta(\xi) - y_i^{m,\beta}(\xi - \eta)|^2 + \beta^2)^\alpha} \right] d\eta d\xi.$$

Note that the same derivation as in the uniqueness part of Section 2.3 yields

$$G_k \leq C(\alpha)\Theta(\|Z(t)\| + \|Z^\beta(t)\|)^{2+2\alpha} \|W(t)\|_{L^2}^2 \leq C(\alpha)\Theta A^{2+2\alpha} \|W(t)\|_{L^2}^2.$$

To control $S_{k,i}^m$, we first use that for any $x, \beta > 0$ and $\alpha \in [0, 1]$ we have

$$\left| \frac{1}{x^{2\alpha}} - \frac{1}{(x^2 + \beta^2)^\alpha} \right| = x^{-2\alpha} \left| 1 - \left(1 + \left(\frac{\beta}{x} \right)^2 \right)^{-\alpha} \right| \leq x^{-1-2\alpha} \beta, \quad (4.8)$$

where in the last inequality we used that $|1 - (1 + b^2)^{-\alpha}| \leq b$ for $b > 0$ and $\alpha \in [0, 1]$. Applying (4.8) now yields

$$\begin{aligned} |S_{k,i}^m| &\leq \beta \int_{\mathbb{T}^2} |w_k(\xi)| \frac{|\partial_\xi z_k^\beta(\xi) - \partial_\xi y_i^{m,\beta}(\xi - \eta)|}{|z_k^\beta(\xi) - y_i^{m,\beta}(\xi - \eta)|^{1+2\alpha}} d\eta d\xi \\ &\leq \sqrt{2\pi} \beta \|w_k\|_{L^2} \underbrace{\sup_{\xi \in \mathbb{T}} \int_{\mathbb{T}} \frac{|\partial_\xi z_k^\beta(\xi) - \partial_\xi y_i^{m,\beta}(\xi - \eta)|}{|z_k^\beta(\xi) - y_i^{m,\beta}(\xi - \eta)|^{1+2\alpha}} d\eta}_{=: J}. \end{aligned} \quad (4.9)$$

For $i \neq k$, we immediately have $J \leq 4\pi \|Z^\beta\|_{C^1} \delta [Z^\beta]^{-1-2\alpha}$. For $i = k$ and $m = 1$, we have

$$J \leq \int_{\mathbb{T}} \|Z^\beta\|_{C^2} F[Z^\beta]^{1+2\alpha} \eta^{-2\alpha} d\eta \leq C(\alpha) \|Z^\beta\|_{C^2} F[Z^\beta]^{1+2\alpha}$$

for $\alpha < \frac{1}{2}$. When $i = k$ and $m = 2$, then the integrand in J is the same as T_3 in (2.15), only with η replaced by $\xi - \eta$. Hence the same bound as in (2.20) gives

$$J \leq \int_{\mathbb{T}} C(\|Z^\beta\|_{C^2} + 1) F[Z^\beta]^{1+2\alpha} \eta^{-2\alpha-1/2} d\eta \leq C(\alpha) (\|Z^\beta\|_{C^2} + 1) F[Z^\beta]^{1+2\alpha}$$

for $\alpha < \frac{1}{4}$. It now follows that

$$|S_{k,i}^m| \leq C(\alpha) \|Z^\beta\|^{2+2\alpha} \beta \|w_k\|_{L^2} \leq C(\alpha) A^{2+2\alpha} \beta \|w_k\|_{L^2}.$$

Combining the estimates for G_k and $S_{k,i}^m$ gives

$$\frac{d}{dt} \|W(t)\|_{L^2} \leq C(\alpha) N \Theta A^{2+2\alpha} (\|W(t)\|_{L^2} + \beta)$$

for $t \in [0, T]$. Solving this differential inequality with initial condition $\|W(0)\|_{L^2} = 0$ yields $\|W(t)\|_{L^2} \leq (e^{C(\alpha) N \Theta A^{2+2\alpha} t} - 1) \beta$ for $t \in [0, T]$, which implies (4.7). \square

Remark. We note that one can in fact prove $\Omega_k^\beta(t) = \tilde{\Omega}_k^\beta(t)$ (even uniqueness of C^1 patch solutions to (1.1) with $u = u^\beta$) for any $\beta > 0$ without a reference to weak solutions. Indeed, Lemma 4.9 below holds with power 1 (instead of $1 - 2\alpha$) in (4.10), which follows from the first paragraph of its proof because u^β is clearly smooth and (4.11) now holds with $(|x - y|^2 + \beta^2)^{-\alpha-1/2}$ ($\leq \beta^{-1-2\alpha}$) inside the integral. (The constant in (4.10) then becomes $C = C(\alpha, \beta, \sum_{k=1}^N |\theta_k|, |\partial\Omega(t)|) < \infty$.) The argument in Lemma 4.10 below then yields $d'(t) \leq C d(t)$ for $d(t) := d_H(\partial\Omega(t), \partial\tilde{\Omega}(t))$ (as long as $d(t) \leq 1$), so $d(0) = 0$ implies $d(t) = 0$ for all $t \in [0, T]$.

Definition 4.6. For a family $\Omega = \{\Omega_k\}_{k=1}^N$ of bounded open subsets of D whose boundaries $\partial\Omega_k$ are pairwise disjoint simple closed H^3 curves (i.e., $\|\Omega_k\|_{H^3} < \infty$ for each k), let us define $\|\Omega\|_{H^3} := \|Z\|$, where $Z = \{z_k\}_{k=1}^N$ and each z_k is a constant speed parametrization of $\partial\Omega_k$ as in Definition 1.1.

Remarks. 1. Since $z_k \in H^3(\mathbb{T})$ has constant speed and \mathbb{T} is compact, it is not difficult to see that any Ω as in the definition satisfies $\|\Omega\|_{H^3} < \infty$. Indeed, looking at (2.9), the only term that is not clearly finite due to the assumptions of the definition is $F[Z]$ from (2.8). It is clear that the constant speed of parametrization and $\|\Omega_k\|_{H^3} < \infty$ show that there exists $r > 0$ such that if $|\eta| \in (0, r)$, then $|\eta|z_k(\xi + \eta) - z_k(\xi)|^{-1} \leq 2L_k^{-1}$ for all ξ, k . For $|\eta| \geq r$ and any ξ, k , the expression on the right hand side of (2.8) is bounded due to its continuity, compactness of \mathbb{T} , and the assumption that all the $\partial\Omega_k$ are simple closed curves.

2. Similarly, any H^3 patch solution $\omega(\cdot, t) = \sum_{k=1}^N \theta_k \chi_{\Omega_k(t)}$ to (1.1)-(1.2) on $[0, T)$ must satisfy $\sup_{t \in [0, T']} \|\Omega(t)\|_{H^3} < \infty$ for any $T' < T$ (due to continuity of Ω in time and compactness of $\mathbb{T} \times [0, T']$).

Here is a corollary that summarizes the previous results in this section.

Corollary 4.7. Let $\alpha \in (0, \frac{1}{24})$, let $\theta_1, \dots, \theta_N \in \mathbb{R} \setminus \{0\}$, and let $\Omega(0) = \{\Omega_k(0)\}_{k=1}^N$ be as in Definition 4.6. There exists $T_\omega > 0$ and an H^3 patch solution $\omega(\cdot, t) = \sum_{k=1}^N \theta_k \chi_{\Omega_k(t)}$ to (1.1)-(1.2) on $[0, T_\omega)$ which satisfies the following.

- (a) For any $T' \in [0, T_\omega)$ and any parametrization $Z(T')$ of $\partial\Omega(T')$ with $\|Z(T')\| < \infty$, let $T = T(\alpha, N \sum_{k=1}^N |\theta_k|, \|Z(T')\|) > 0$ be from Theorem 2.8. Then the corresponding H^3 patch solution to (1.1)-(1.2) on $[T', T' + T]$ from Proposition 4.1, with initial value $Z(T')$ at time T' , is equal to ω on the time interval $[T', T' + T]$.
- (b) There is a universal $C \geq 1$ such that with $T = T(\alpha, N \sum_{k=1}^N |\theta_k|, \|\Omega(0)\|_{H^3}) > 0$ from Theorem 2.8 we have $T_\omega \geq T$ and $\sup_{t \in [0, T]} \|\Omega(t)\|_{H^3} \leq C \|\Omega(0)\|_{H^3}^8$.
- (c) If $T_\omega < \infty$, then $\lim_{t \nearrow T_\omega} \|\Omega(t)\|_{H^3} = \infty$.

Basically, what the Corollary says is that the patch solution that we can obtain using the contour equation is unique. We cannot obtain different patch solutions by changing the parametrization of the initial data or at any other time. Also this solution may cease to exist only if its norm $\|\Omega(t)\|_{H^3}$ blows up. On the other hand, the corollary does not rule out existence of other H^3 patch solutions with the same initial data, obtained not from the contour equation but in some other way. We will eliminate this possibility in the next section.

Proof. Let Z_0 be a constant speed parametrization of $\partial\Omega(0)$ (which satisfies $\|Z_0\| = \|\Omega(0)\|_{H^3} < \infty$ due to Remark 1 after Definition 4.6) and consider the solution ω from Proposition 4.1 on $[0, T_0]$, with $T_0 := T(\alpha, N \sum_{k=1}^N |\theta_k|, \|\Omega(0)\|_{H^3}) > 0$ from Theorem 2.8. Then $\sup_{t \in [0, T_0]} \|\Omega(t)\|_{H^3} \leq C(4\|\Omega(0)\|_{H^3})^8$, with C from Lemma 4.2, so (b) holds with $T := T_0$. We can also extend ω up to time $T_0 + T_1$, with $T_1 := T(\alpha, N \sum_{k=1}^N |\theta_k|, \|\Omega(T_0)\|_{H^3}) > 0$, by using Proposition 4.1 with initial condition a constant speed parametrization of $\partial\Omega(T_0)$ at time T_0 . We can continue this way to obtain $T_\omega := \sum_{j=0}^{\infty} T_j$, which then must satisfy either $\lim_{t \nearrow T_\omega} \|\Omega(t)\|_{H^3} = \infty$ or $T_\omega = \infty$. This proves (c), while (a) follows from Proposition 4.5 (note also that T in (a) must be less than $T_\omega - T'$ because $\sup_{t \in [T', \min\{T'+T, T_\omega\})} \|\Omega(t)\|_{H^3} < \infty$ by Proposition 4.1). \square

Remark. Now is the natural time to prove the last statement of Theorem 1.4 describing precisely how the blow up may manifest itself. Let us assume that $T_\omega < \infty$, and define $\partial\Omega(T_\omega) := \lim_{t \nearrow T_\omega} \partial\Omega(t)$ (the limit is taken with respect to Hausdorff distance and exists due to (3.1)). Let us also assume that $\min_{k \neq i} \text{dist}(\partial\Omega_k(T_\omega), \partial\Omega_i(T_\omega)) > 0$ and that for each k , the limit $\lim_{t \nearrow T_\omega} \|\Omega_k(t)\|_{H^3}$ is either finite or does not exist. Then there must be some k and $t_j \nearrow T_\omega$ such that for any constant speed parametrization z_j of some $\partial\Omega_k(t_j)$ we have

$$A := \sup_j \|z_j\|_{H^3} < \infty \quad \text{and} \quad \lim_{j \rightarrow \infty} \sup_{\xi, \eta \in \mathbb{T}, \eta \neq 0} \frac{|\eta|}{|z_j(\xi) - z_j(\xi - \eta)|} = \infty.$$

The first of these statements, together with $|\partial\Omega_k(t)|$ being bounded below uniformly in t , due to $|\Omega_k(t)|$ being constant in time, and along with the constant speed property of z_j , implies that there is $r = r(A)$ such that

$$\sup_j \sup_{\xi, \eta \in \mathbb{T}, |\eta| \in (0, r)} \frac{|\eta|}{|z_j(\xi) - z_j(\xi - \eta)|} < \infty.$$

Thus there must be two sequences of points $x_j, y_j \in \partial\Omega_k(t_j)$ with $\lim_{j \rightarrow \infty} |x_j - y_j| = 0$ but distance of x_j and y_j along $\partial\Omega_k(t_j)$ uniformly bounded below by a positive number. Continuity of $\partial\Omega_k$ in time (and its compactness) then implies that $\partial\Omega_k(T_\omega)$ cannot be a simple closed curve. This proves the last statement in Theorem 1.4.

4.3 Uniqueness of H^3 patch solutions

We will now prove (local) uniqueness of H^3 patch solutions to (1.1)-(1.2). Hence the unique solution for a given initial value $\omega(\cdot, 0)$ is the one from Corollary 4.7. The main result of this sub-section is Theorem 4.12.

The next lemma is a simple geometric result, concerning two H^3 patches whose boundaries are close to each other in Hausdorff distance. It will be used in the following lemma to estimate the difference of the velocities from (1.2) corresponding to two sets of H^3 patches whose boundaries are close to each other in Hausdorff distance. As before, we denote by n_P the outer unit normal vector for Ω at $P \in \partial\Omega$.

Lemma 4.8. *Let $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^2$ be two bounded open sets whose boundaries are simple closed curves, and let $\|\Omega\|_{H^3} + \|\tilde{\Omega}\|_{H^3} \leq A$ for some $A \geq 1$. Let $R := (4C_0A)^{-3}$, where $C_0 \geq 1$ is a universal constant such that $\|f\|_{C^2} \leq C_0\|f\|_{H^3}$ for each $f \in H^3(\mathbb{T})$, and let $P \in \partial\Omega$. If $d_H(\partial\Omega, \partial\tilde{\Omega}) \leq \frac{R}{20}$, then in the coordinate system (w_1, w_2) centered at P and with axes $n_{\tilde{P}}$ and n_P , both $\partial\Omega \cap B(P, R)$ and $\partial\tilde{\Omega} \cap B(P, \frac{19}{20}R)$ can be represented as graphs $w_2 = f(w_1)$ and $w_2 = g(w_1)$, respectively, and we have $|f'(w_1)| \leq 1$, $|g'(w_1)| \leq 1$, and $|f(w_1) - g(w_1)| \leq 2d_H(\partial\Omega, \partial\tilde{\Omega})$ for $|w_1| \leq \frac{R}{2}$.*

Proof. Let $h := d_H(\partial\Omega, \partial\tilde{\Omega})$, let $\tilde{P} \in \partial\tilde{\Omega}$ be such that $|\tilde{P} - P| = \text{dist}(P, \partial\tilde{\Omega}) \leq h$, and denote by $\tilde{n}_{\tilde{P}}$ the outer unit normal for $\tilde{\Omega}$ at \tilde{P} . (If \tilde{P} is not unique, we pick one such point.) By Lemma 3.4(a,c) with $\gamma = 1$ and the definition of R (note that if either $\partial\Omega$ or $\partial\tilde{\Omega}$ has arc-length $2\pi L$ and a constant speed parametrization z , then $L \geq \frac{1}{\pi}|z(\pi) - z(0)| \geq F[z]^{-1} \geq \frac{1}{A}$), both $\partial\Omega \cap B(P, R)$ and $\partial\tilde{\Omega} \cap B(\tilde{P}, R)$ are simply connected curves whose (outer to Ω and $\tilde{\Omega}$) unit normal vectors lie in $B(n_P, \frac{1}{32})$ and in $B(\tilde{n}_{\tilde{P}}, \frac{1}{32})$, respectively.

This implies that $\tilde{n}_{\tilde{P}} \cdot n_P \geq \cos \frac{\pi}{6}$. Indeed, otherwise we could find $P' \in \partial\Omega \cap \partial B(P, \frac{R}{2})$ such that $\text{dist}(P', \partial\tilde{\Omega}) \geq \frac{R}{2} \sin(\frac{\pi}{6} - 2 \cdot \arcsin \frac{1}{32}) - h > h$ (since we assume $h \leq \frac{R}{20}$), contradicting $d_H(\partial\Omega, \partial\tilde{\Omega}) = h$.

From $|\tilde{P} - P| \leq h \leq \frac{R}{20}$ and $\tilde{n}_{\tilde{P}} \cdot n_P \geq \cos \frac{\pi}{6}$ (together with the normal vectors of $\partial\Omega \cap B(P, R)$ and $\partial\tilde{\Omega} \cap B(\tilde{P}, R)$ lying in $B(n_P, \frac{1}{32})$ and in $B(\tilde{n}_{\tilde{P}}, \frac{1}{32})$, respectively), we have that in the coordinate system (w_1, w_2) , both $\partial\Omega \cap B(P, R)$ and $\partial\tilde{\Omega} \cap B(P, \frac{19}{20}R)$ are graphs $w_2 = f(w_1)$ and $w_2 = g(w_1)$, respectively, with $|f'(w_1)| \leq \tan \arcsin \frac{1}{32} < 1$ and $|g'(w_1)| \leq \tan(\frac{\pi}{6} + \arcsin \frac{1}{32}) < 1$. Since $19^2 > 12^2 + 13^2$, it follows that the domains of f, g both contain $[-\frac{12}{20}R, \frac{12}{20}R]$.

If now $|f(w_1) - g(w_1)| > 2h$ for some $|w_1| \leq \frac{R}{2}$, then $Q := (w_1, f(w_1)) \in \partial\Omega \cap [-\frac{R}{2}, \frac{R}{2}]^2$ has $\text{dist}(Q, \mathbb{R}^2 \setminus [-\frac{12}{20}R, \frac{12}{20}R]^2) \geq 2h$ and also $\text{dist}(Q, \partial\tilde{\Omega} \cap [-\frac{12}{20}R, \frac{12}{20}R]^2) \geq \sqrt{2}h$ (the latter because $|g'(w_1)| \leq 1$ for $|w_1| \leq \frac{12}{20}R$). This again contradicts $d_H(\partial\Omega, \partial\tilde{\Omega}) = h$. \square

Lemma 4.9. *Let $\alpha \in (0, \frac{1}{2})$, let $\theta_1, \dots, \theta_N \in \mathbb{R}$, let $\Omega = \{\Omega_k\}_{k=1}^N$ and $\tilde{\Omega} = \{\tilde{\Omega}_k\}_{k=1}^N$ be as in Definition 4.6, and let u and \tilde{u} be the velocity fields from (1.2) corresponding to $\omega := \sum_{k=1}^N \theta_k \chi_{\Omega_k}$ and $\tilde{\omega} := \sum_{k=1}^N \theta_k \chi_{\tilde{\Omega}_k}$, respectively. Let also $\|\Omega\|_{H^3} + \|\tilde{\Omega}\|_{H^3} \leq A$*

for some $A \geq 1$. There exists a constant $C = C(\alpha, \sum_{k=1}^N |\theta_k|, A) < \infty$ such that if $d_H(\partial\Omega, \partial\tilde{\Omega}) \leq 1$, then for any $x, \tilde{x} \in \bar{D}$ we have

$$|u(x) - \tilde{u}(\tilde{x})| \leq C \max\{|x - \tilde{x}|, d_H(\partial\Omega, \partial\tilde{\Omega})\}^{1-2\alpha}. \quad (4.10)$$

Proof. First note that (3.2) shows that it is sufficient to consider $\tilde{x} = x$. From (3.1) it follows that it further suffices to restrict the proof to the case $h := d_H(\partial\Omega, \partial\tilde{\Omega}) \leq \frac{R}{20}$, with $R = R(A)$ from Lemma 4.8. We then have

$$|u(x) - \tilde{u}(x)| \leq \sum_{k=1}^N 2|\theta_k| \underbrace{\int_{\Omega_k \Delta \tilde{\Omega}_k} |x - y|^{-1-2\alpha} dy}_{=: I_k}, \quad (4.11)$$

so it finally suffices to show $I_k \leq C(\alpha, A)h^{1-2\alpha}$ for each k and some $C(\alpha, A) < \infty$.

Let $P \in \partial\Omega_k$ be such that $|x - P| = d(x, \partial\Omega_k) =: d_k$. Let us first assume that $d_k \geq \frac{R}{4}$. Since $\partial\Omega_k$ and $\partial\tilde{\Omega}_k$ both have arc-length bounded by CA (for some universal $C < \infty$) and $A \geq 1$, we have $|\Omega_k \Delta \tilde{\Omega}_k| \leq CAh$ (with a different universal C). Thus

$$d(x, \Omega_k \Delta \tilde{\Omega}_k) \geq \frac{R}{4} - h \geq \frac{R}{5}$$

because $h \leq \frac{R}{20}$. Since then also $h \leq 1$, this indeed yields

$$I_k \leq |\Omega_k \Delta \tilde{\Omega}_k| \left(\frac{R}{5}\right)^{-1-2\alpha} \leq C(\alpha, A)h \leq C(\alpha, A)h^{1-2\alpha}. \quad (4.12)$$

Let us now assume that $d_k < \frac{R}{4}$, and split I_k into

$$I_k = \underbrace{\int_{(\Omega_k \Delta \tilde{\Omega}_k) \cap B(P, R/2)} |x - y|^{-1-2\alpha} dy}_{=: I_k^1} + \underbrace{\int_{(\Omega_k \Delta \tilde{\Omega}_k) \cap (D \setminus B(P, R/2))} |x - y|^{-1-2\alpha} dy}_{=: I_k^2}.$$

If $y \notin B(P, \frac{R}{2})$, then $|x - y| \geq |y - P| - |P - x| \geq \frac{R}{2} - \frac{R}{4} \geq \frac{R}{4}$. Hence I_k^2 can be bounded as I_k in (4.12), yielding $I_k^2 \leq C(\alpha, A)h^{1-2\alpha}$.

To bound I_k^1 , we apply Lemma 4.8 to Ω_k and $\tilde{\Omega}_k$. Thus in the coordinate system (w_1, w_2) centered at P and with axes n_P^\perp and n_P (the latter being the outer unit normal for Ω_k at P), both $\partial\Omega_k \cap B(P, R)$ and $\partial\tilde{\Omega}_k \cap B(P, \frac{19}{20}R)$ are graphs $w_2 = f(w_1)$ and $w_2 = g(w_1)$, respectively, such that $|f(w_1) - g(w_1)| \leq 2h$ for $|w_1| \leq \frac{R}{2}$. In this new

coordinate system, x is either $(0, d_k)$ or $(0, -d_k)$, and we can assume the former without loss. Then

$$I_k^1 \leq \int_{-R/2}^{R/2} \underbrace{\left| \int_{f(w_1)}^{g(w_1)} (w_1^2 + (w_2 - d_k)^2)^{-\frac{1}{2}-\alpha} dw_2 \right|}_{=:T(w_1)} dw_1.$$

For $|w_1| \geq h$ we have

$$T(w_1) \leq |g(w_1) - f(w_1)| |w_1|^{-1-2\alpha} \leq 2h |w_1|^{-1-2\alpha},$$

whereas for $|w_1| < h$ we have

$$T(w_1) \leq \left| \int_{f(w_1)}^{g(w_1)} |w_1|^{-\frac{1}{2}-\alpha} |w_2 - d_k|^{-\frac{1}{2}-\alpha} dw_2 \right| \leq 2 |w_1|^{-\frac{1}{2}-\alpha} \int_0^h s^{-\frac{1}{2}-\alpha} ds \leq \frac{4}{1-2\alpha} |w_1|^{-\frac{1}{2}-\alpha} h^{\frac{1}{2}-\alpha}.$$

It follows that

$$I_k^1 \leq 2 \int_h^{R/2} 2h w_1^{-1-2\alpha} dw_1 + 2 \int_0^h \frac{4}{1-2\alpha} w_1^{-\frac{1}{2}-\alpha} h^{\frac{1}{2}-\alpha} dw_1 \leq C(\alpha, A) h^{1-2\alpha}.$$

So we again have $I_k \leq C(\alpha, A) h^{1-2\alpha}$, and the proof is finished. \square

We will now use Lemma 4.9 to show that the Hausdorff distance of two H^3 patch solutions with the same initial data grows (for a short time) at most as $t^{\frac{1}{2\alpha}}$.

Lemma 4.10. *Let $\alpha \in (0, \frac{1}{2})$, let $\omega(\cdot, t) = \sum_{k=1}^N \theta_k \chi_{\Omega_k(t)}$ and $\tilde{\omega}(\cdot, t) = \sum_{k=1}^N \theta_k \chi_{\tilde{\Omega}_k(t)}$ be two H^3 patch solutions to (1.1)-(1.2) on $[0, T]$, and let $\sup_{t \in [0, T]} (\|\Omega(t)\|_{H^3} + \|\tilde{\Omega}(t)\|_{H^3}) \leq A$ for some $A \geq 1$, where $\Omega(t) := \{\Omega_k(t)\}_{k=1}^N$ and $\tilde{\Omega}(t) := \{\tilde{\Omega}_k(t)\}_{k=1}^N$. There is a constant $C = C(\alpha, \sum_{k=1}^N |\theta_k|, A) < \infty$ such that if $\omega(\cdot, 0) = \tilde{\omega}(\cdot, 0)$, then for all $t \in [0, \min\{C^{-2\alpha}, T\}]$ we have*

$$d_H(\partial\Omega(t), \partial\tilde{\Omega}(t)) \leq Ct^{1/2\alpha}. \quad (4.13)$$

Proof. Let u and \tilde{u} be the velocity fields from (1.2) corresponding to ω and $\tilde{\omega}$, respectively. Let $C = C(\alpha, \sum_{k=1}^N |\theta_k|, A)$ be the constant from Lemma 4.9 and let $d(t) := d_H(\partial\Omega(t), \partial\tilde{\Omega}(t))$.

We first claim that d is Lipschitz on $[0, T]$, with some constant $\tilde{C} = \tilde{C}(\alpha, \sum_{k=1}^N |\theta_k|, A) < \infty$ which is three times the right-hand side of (3.1). It is obviously sufficient to prove this for any $[a, b] \subseteq (0, T)$. From (1.6) and (3.1) we have that for each $t \in [a, b]$ there

is $h_t > 0$ such that $|d(t+h) - d(t)| \leq \tilde{C}h$ whenever $|h| < h_t$. Thus we also have $|d(t) - d(s)| \leq \tilde{C}|t-s|$ whenever $(t-h_t, t+h_t) \cap (s-h_s, s+h_s) \neq \emptyset$. Since there is a finite sub-cover of $[a, b]$ from $\{(t-h_t, t+h_t)\}_{t \in [a, b]}$, it follows that d is indeed \tilde{C} -Lipschitz on $[a, b]$. It follows that d is differentiable almost everywhere on $[0, T]$ and $d(t) - d(0) = \int_0^t d'(s)ds$ for $t \in [0, T]$.

Consider any $t > 0$ such that $d(t) \in (0, 1]$ and $d'(t)$ exists. Then (1.6) shows that for small $h > 0$ and any $x \in \partial\Omega(t+h)$, there is $y_x \in \partial\Omega(t)$ such that $|y_x + hu(y_x, t) - x| \leq o(h)$, with $o(h)$ uniform in x . Then there are also $\tilde{y}_x \in \partial\tilde{\Omega}(t)$ and $\tilde{x} \in \partial\tilde{\Omega}(t+h)$ such that $|\tilde{y}_x - y_x| \leq d(t)$ and $|\tilde{y}_x + h\tilde{u}(\tilde{y}_x, t) - \tilde{x}| \leq o(h)$ (with a new uniform $o(h)$). This and Lemma 4.9 applied to y_x and \tilde{y}_x show that

$$|\tilde{x} - x| \leq d(t) + Cd(t)^{1-2\alpha}h + 2o(h)$$

Since $o(h)$ is uniform in $x \in \partial\Omega(t+h)$, and since the same argument applies to $\partial\Omega$ and $\partial\tilde{\Omega}$ swapped, we obtain $d'(t) \leq Cd(t)^{1-2\alpha}$ for each t such that $d(t) \in (0, 1]$ and $d'(t)$ exists. Integrating this differential inequality (recall that d is Lipschitz and $d(0) = 0$) yields $d(t) \leq (4\alpha Ct)^{1/2\alpha}$ on any time interval $[0, T']$ such that $\sup_{t \in [0, T']} d(t) \leq 1$. Hence the theorem holds with (the new) C being (the old) $(4\alpha C)^{1/2\alpha}$. \square

The next lemma, which is our last ingredient for the proof of uniqueness, says that the boundaries of two H^3 patches $\Omega, \tilde{\Omega}$ have constant speed parametrizations which differ (in L^∞) by no more than $O(d_H(\partial\Omega, \partial\tilde{\Omega}))$, with the constant depending on $\|\Omega\|_{H^3} + \|\tilde{\Omega}\|_{H^3}$.

Lemma 4.11. *Let $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^2$ be two bounded open sets whose boundaries are simple closed curves, and let $\|\Omega\|_{H^3} + \|\tilde{\Omega}\|_{H^3} \leq A$ for some $A \geq 1$. There is a universal constant $C < \infty$ such that if $d_H(\partial\Omega, \partial\tilde{\Omega}) \leq 1$, then there exist some constant speed parametrizations z and \tilde{z} of $\partial\Omega$ and $\partial\tilde{\Omega}$, respectively, such that*

$$\|z - \tilde{z}\|_{L^\infty} \leq CA^7 d_H(\partial\Omega, \partial\tilde{\Omega}). \quad (4.14)$$

We postpone the proof of Lemma 4.11 until after the following theorem, which is our main uniqueness result.

Theorem 4.12. *Let $\alpha \in (0, \frac{1}{24})$, let $\theta_1, \dots, \theta_N \in \mathbb{R} \setminus \{0\}$, and let $\omega(\cdot, t) = \sum_{k=1}^N \theta_k \chi_{\Omega_k(t)}$ and $\tilde{\omega}(\cdot, t) = \sum_{k=1}^N \theta_k \chi_{\tilde{\Omega}_k(t)}$ be two H^3 patch solutions to (1.1)-(1.2) on some interval $[0, T]$. If $\omega(\cdot, 0) = \tilde{\omega}(\cdot, 0)$, then $\omega(\cdot, t) = \tilde{\omega}(\cdot, t)$ for all $t \in [0, T]$.*

Let us first provide an overview of the argument (see also Figure 1). Corollary 4.7 shows that there is a unique patch solution (with a given initial data) which can be obtained via the contour equation. It is then sufficient to prove the result with $\tilde{\omega}$ being

this solution. We will assume that there exists another H^3 patch solution $\omega \neq \tilde{\omega}$ and arrive at a contradiction. The idea is to use a sequence of auxiliary H^3 patch solutions $\{\omega^{s_j}\}_{j=1}^J$, obtained via Corollary 4.7 with initial data $\omega^{s_j}(\cdot, s_j) = \omega(\cdot, s_j)$, where $s_j = \frac{j}{J}T_1$ and $T_1 \leq 1$ is a fixed time. The first key step will be to apply Lemma 4.10 to show that $\omega(\cdot, s_j)$ ($= \omega^{s_j}(\cdot, s_j)$) and $\omega^{s_{j-1}}(\cdot, s_j)$ are $J^{-1/2\alpha}$ close (in Hausdorff distance of their patch boundaries, and hence their constant speed parametrizations are also $J^{-1/2\alpha}$ close due to Lemma 4.11). Next, since the ω^{s_j} were obtained via the contour equation, the L^2 stability estimate (2.34) applies to them up to time T_1 and allows us to show that $\omega^{s_j}(\cdot, T_1)$ and $\omega^{s_{j-1}}(\cdot, T_1)$ are $J^{-1/2\alpha}$ close as well. The latter is in terms of the L^2 distance of some parametrizations of the curves, but Lemma 4.3 allows us to transfer this into the same estimate for the area of the symmetric difference of the corresponding patches. After telescoping the latter, we find that the area of such a symmetric difference corresponding to $\omega(\cdot, T_1)$ and $\tilde{\omega}(\cdot, T_1)$ is bounded above by $O(J^{1-1/2\alpha})$. Taking $J \rightarrow \infty$ and using $2\alpha < 1$ (and then applying this argument to arbitrary T_1), we find that $\omega = \tilde{\omega}$, which is a contradiction with our hypothesis.

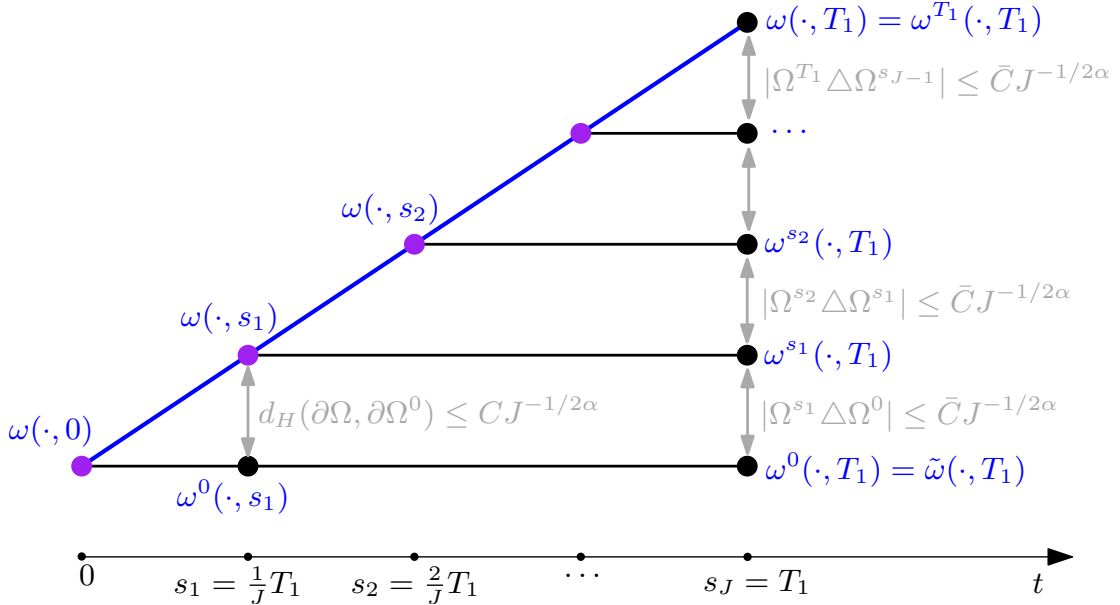


Figure 1: An abstract phase space illustration of the proof of Theorem 4.12.

Proof. Obviously, it suffices to prove the result in the case of $\tilde{\omega}$ being the solution from Corollary 4.7. Assume the contrary and let $T' := \inf\{t \in (0, T) : \omega(\cdot, t) \neq \tilde{\omega}(\cdot, t)\} < T$. Without loss we can assume that $T' = 0$.

With the notation from Definition 4.6, let $B := \sup_{t \in [0, T/2]} \|\Omega(t)\|_{H^3} (\geq 1)$, and for each $s \in [0, \frac{T}{2}]$, let $\omega^s(\cdot, t) = \sum_{k=1}^N \theta_k \chi_{\Omega_k^s(t)}$ be the (unique) solution from Corollary 4.7 with initial condition $\omega^s(\cdot, s) := \omega(\cdot, s)$ at time s . In particular, $\omega^0 = \tilde{\omega}$. Let $A := 2CB^8 \geq B$ (where $C \geq 1$ is from Corollary 4.7(b)), then let $T_0 := T(\alpha, N \sum_{k=1}^N |\theta_k|, A) > 0$ (which is decreasing in the last variable) be from Corollary 4.7(b) and consider any $T_1 \in (0, \min\{T_0, \frac{T}{2}, 1\}]$. Thus Corollary 4.7(b) shows that ω^s exists on $[s, T_1]$ for each $s \in [0, T_1]$ and satisfies $\sup_{t \in [s, T_1]} \|\Omega^s(t)\|_{H^3} \leq \frac{A}{2}$.

For any $J \in \mathbb{N}$, let $s_j := \frac{j}{J}T_1$ for $j = 0, \dots, J$. Let $C = C(\alpha, \sum_{k=1}^N |\theta_k|, A) < \infty$ be from Lemma 4.10 and also larger than the universal C in Lemmas 4.3 and 4.11, and consider any $J \geq C^{2\alpha}$. Then Lemma 4.10 applied to ω and $\omega^{s_{j-1}}$ (with starting time s_{j-1}) and $s_j - s_{j-1} = \frac{T_1}{J} \leq \frac{1}{J}$ imply

$$d_H(\partial\Omega(s_j), \partial\Omega^{s_{j-1}}(s_j)) \leq CJ^{-1/2\alpha}$$

for $j = 1, \dots, J$. Since $\Omega(s_j) = \Omega^{s_j}(s_j)$, it follows from Lemma 4.11 that the families $\partial\Omega^{s_j}(s_j)$ and $\partial\Omega^{s_{j-1}}(s_j)$ have constant speed parametrizations $Z_j(s_j)$ and $\tilde{Z}_j(s_j)$ satisfying

$$\|Z_j(s_j) - \tilde{Z}_j(s_j)\|_{L^2} \leq \sqrt{2\pi N} \|Z_j(s_j) - \tilde{Z}_j(s_j)\|_{L^\infty} \leq \sqrt{2\pi N} C^2 A^7 J^{-1/2\alpha}.$$

We have

$$\|Z_j(s_j)\| + \|\tilde{Z}_j(s_j)\| = \|\Omega^{s_j}(s_j)\|_{H^3} + \|\Omega^{s_{j-1}}(s_j)\|_{H^3} \leq A,$$

so that Theorem 2.8 yields solutions Z_j and \tilde{Z}_j to (2.5) on the time interval $[s_j, s_j + T_0] \supseteq [s_j, T_1]$ and with initial data $Z_j(s_j)$ and $\tilde{Z}_j(s_j)$, respectively. Theorem 2.8 also shows that $\sup_{t \in [s_j, T_1]} (\|Z_j(t)\| + \|\tilde{Z}_j(t)\|) \leq 4A$, and then (2.34) with $W := Z_j - \tilde{Z}_j$ yields

$$\|Z_j(T_1) - \tilde{Z}_j(T_1)\|_{L^2} \leq e^{C(\alpha)N \sum_{k=1}^N |\theta_k| (4A)^3 T_1} \|Z_j(s_j) - \tilde{Z}_j(s_j)\|_{L^2} \leq \tilde{C} J^{-1/2\alpha},$$

where $\tilde{C} := \sqrt{2\pi N} C^2 A^7 e^{C(\alpha)N \sum_{k=1}^N |\theta_k| (4A)^3 T_1}$. This and Lemma 4.3 show that with $\bar{C} := C4A\sqrt{N}\tilde{C}$ we have

$$|\Omega^{s_j}(T_1) \Delta \Omega^{s_{j-1}}(T_1)| \leq \bar{C} J^{-1/2\alpha}.$$

This holds for $j = 1, \dots, J$, hence we obtain by telescoping,

$$|\Omega(T_1) \Delta \tilde{\Omega}(T_1)| = |\Omega^{T_1}(T_1) \Delta \Omega^0(T_1)| \leq \bar{C} J^{1-1/2\alpha}.$$

Since \bar{C} is independent of J and $2\alpha < 1$, we take $J \rightarrow \infty$ to get $|\Omega(T_1) \Delta \tilde{\Omega}(T_1)| = 0$. Hence $\omega(\cdot, T_1) = \tilde{\omega}(\cdot, T_1)$ for each $T_1 \in (0, \min\{T_0, \frac{T}{2}, 1\}]$, which is a contradiction with our hypothesis $\inf\{t \in (0, T) : \omega(\cdot, t) \neq \tilde{\omega}(\cdot, t)\} = 0$. \square

Proof of Lemma 4.11. Let C_0 and $R := (4C_0A)^{-3}$ be from Lemma 4.8. We can assume without loss that $h := d_H(\partial\Omega, \partial\tilde{\Omega}) \leq \frac{R^2}{4}$, because otherwise the result holds with any $C \geq 8(4C_0)^6$ due to $\Omega, \tilde{\Omega} \subseteq B(0, A)$.

Since $\frac{R^2}{4} < \frac{R}{20}$ due to $R \leq \frac{1}{4^3}$, we can apply Lemma 4.8 to Ω and $\tilde{\Omega}$. It shows that in the coordinate system (w_1, w_2) centered at any given $P \in \partial\Omega$ and with axes n_P^\perp and n_P , both $\partial\Omega \cap B(P, R)$ and $\partial\tilde{\Omega} \cap B(P, \frac{19}{20}R)$ are graphs $w_2 = f(w_1)$ and $w_2 = g(w_1)$, respectively, such that for any $|w_1| \leq \frac{R}{2}$ we have $|f'(w_1)| \leq 1$, $|g'(w_1)| \leq 1$, and

$$|f(w_1) - g(w_1)| \leq 2h. \quad (4.15)$$

We also claim that $|f''(w_1)| \leq 2C_0A^3$ and $|g''(w_1)| \leq 2C_0A^3$ for all $|w_1| \leq \frac{R}{2}$. Indeed, let $y(\xi)$ be $z(\xi)$ in the new coordinates (w_1, w_2) . Then for any $\xi \in \mathbb{T}$ such that $y(\xi) \in B(0, R)$ (i.e., $z(\xi) \in B(P, R)$), we have $f(y_1(\xi)) = y_2(\xi)$. Thus $f'(y_1(\xi)) = \frac{y_2'(\xi)}{y_1'(\xi)}$ and

$$f''(y_1(\xi)) = \frac{(y_2'(\xi)/y_1'(\xi))'}{y_1'(\xi)} = \frac{y_2''(\xi)y_1'(\xi) - y_1''(\xi)y_2'(\xi)}{y_1'(\xi)^3} = \frac{y_2''(\xi) - y_1''(\xi)f'(y_1(\xi))}{y_1'(\xi)^2}.$$

If, in addition, $|y_1(\xi)| \leq \frac{R}{2}$, then we have $y_1'(\xi) \geq \frac{1}{\sqrt{2}A}$ because $|y'(\xi)| \geq \frac{1}{A}$ (due to $F[y] = F[z] \leq A$) and $|\frac{y_2'(\xi)}{y_1'(\xi)}| = |f'(y_1(\xi))| \leq 1$. This, $|f'(y_1(\xi))| \leq 1$, and $\|y''\|_{L^\infty} = \|z''\|_{L^\infty} \leq C_0A$ now yield $|f''(y_1(\xi))| \leq 2C_0A^3$. The bound for g is obtained identically.

Next, we claim that for $|w_1| \leq \frac{R}{2}$ we have

$$|f'(w_1) - g'(w_1)| \leq 8C_0A^{3/2}\sqrt{h}. \quad (4.16)$$

If this is violated for some $|w_1^0| \leq \frac{R}{2}$ (without loss we can assume $w_1^0 \leq 0$ as well as $f'(w_1^0) - g'(w_1^0) > 8C_0\sqrt{A^3h}$), the estimate $|f'' - g''| \leq 4C_0A^3$ on $[-\frac{R}{2}, \frac{R}{2}]$ yields

$$f'(w_1) - g'(w_1) > 4C_0A^{3/2}\sqrt{h}$$

for all $w_1 \in [w_1^0, w_1^1]$, where $w_1^1 := w_1^0 + A^{-3/2}\sqrt{h}$ ($\leq \frac{R}{2}$ because $w_1^0 \leq 0$, $A \geq 1$, and $h \leq \frac{R^2}{4}$). Then $C_0 \geq 1$ shows

$$f(w_1^1) - g(w_1^1) > f(w_1^0) - g(w_1^0) + 4C_0A^{3/2}\sqrt{h}A^{-3/2}\sqrt{h} \geq -2h + 4C_0h \geq 2h,$$

contradicting (4.15). Thus (4.16) holds.

For any $P \in \partial\Omega$, let $F(P) \in \partial\tilde{\Omega} \cap B(P, \frac{19}{20}R)$ be such that $(F(P) - P) \cdot n_P^\perp = 0$. Lemma 4.8 shows that such $F(P)$ exists and is unique, $|F(P) - P| \leq 2h$, and $F(P)$ is continuous in P (the latter because of continuity of f' and the bound $|g'| \leq 1$ on $[-\frac{R}{2}, \frac{R}{2}]$).

In addition, F is injective. Indeed, assume that $F(P) = F(Q) =: S$ for some distinct $P, Q \in \partial\Omega$, and also without loss that $|S - P| \geq |S - Q|$. Then $|P - Q| \leq 2|P - S| \leq 4h < \frac{R}{2}$, so Lemma 3.4(c) with $\gamma = 1$ (together with $L \geq \frac{1}{A}$, as before) yields

$$\sin \angle PSQ = n_Q \cdot n_P^\perp = (n_Q - n_P) \cdot n_P^\perp \leq |n_Q - n_P| \leq 2C_0A^3|P - Q|. \quad (4.17)$$

We also have $\angle PSQ \leq \frac{\pi}{4}$ (due to $|f'| \leq 1$ on $[-\frac{R}{2}, \frac{R}{2}]$ in Lemma 4.8), so $|S - P| \geq |S - Q|$ and $|f'| \leq 1$ imply $\angle PQS \in [\frac{3\pi}{8}, \frac{3\pi}{4}]$. The law of sines now yields

$$|S - P| = \frac{|P - Q| \sin \angle PQS}{\sin \angle PSQ} \geq \frac{\sin \angle PQS}{2C_0 A^3} \geq \frac{1}{4C_0 A^3} > R,$$

a contradiction with $|S - P| = |F(P) - P| \leq 2h < R/4$. Hence $F : \partial\Omega \rightarrow \partial\tilde{\Omega}$ is injective. Since it is also continuous and $\partial\Omega, \partial\tilde{\Omega}$ are both simple closed curves, F is a bijection.

Next, we claim that for any distinct $P, Q \in \partial\Omega$ with $|P - Q| \leq h$, we have with $C_1 := 300C_0^2$,

$$\left| \frac{|F(P) - F(Q)|}{|P - Q|} - 1 \right| \leq C_1 A^6 h. \quad (4.18)$$

Without loss assume that P is the origin and $n_P = (0, 1)$ (so that $(F(P))_1 = 0$). Let f, g be from Lemma 4.8 and recall that we proved above that $|f''| \leq 2C_0 A^3$ on $[-\frac{R}{2}, \frac{R}{2}]$. This and $f'(0) = 0$ yield $|f'| \leq 2C_0 A^3 |Q|$ on $[-|Q|, |Q|]$, hence $\frac{|Q_2|}{|Q_1|} \leq 2C_0 A^3 |Q|$ and $\frac{|(n_Q)_1|}{|(n_Q)_2|} \leq 2C_0 A^3 |Q|$. Since $|Q_1| \leq h$ and $|(n_Q)_2| \leq 1$, it follows that

$$(1 - 2C_0 A^3 h)|Q| \leq |Q_1| \leq |Q| \quad \text{and} \quad |(n_Q)_1| \leq 2C_0 A^3 |Q|. \quad (4.19)$$

This and $|F(Q) - Q| \leq 2h$ yield

$$|(F(Q) - Q)_1| = |F(Q) - Q| |(n_Q)_1| \leq 4C_0 A^3 h |Q|.$$

By using $(F(P))_1 = 0$, an elementary inequality $||a| - |b|| \leq |a - c| + ||c| - |b||$, and the first bound in (4.19), we obtain

$$|| (F(P) - F(Q))_1 | - |Q| | \leq |(F(Q) - Q)_1| + ||Q_1| - |Q|| \leq 6C_0 A^3 h |Q|. \quad (4.20)$$

From $f'(0) = 0$ and (4.16) we also have $|g'(0)| \leq 8C_0 A^{3/2} \sqrt{h}$, which together with $|g''| \leq 2C_0 A^3$ on $[-\frac{R}{2}, \frac{R}{2}]$ (proved above) yields $|g'| \leq 18C_0 A^3 \sqrt{h}$ on $[-5h, 5h]$. Since

$$|F(P) - F(Q)| \leq |F(P) - P| + |P - Q| + |Q - F(Q)| \leq 2h + h + 2h = 5h,$$

it follows that $\frac{|(F(P) - F(Q))_2|}{|(F(P) - F(Q))_1|} \leq 18C_0 A^3 \sqrt{h}$. Since $6C_0 A^3 h \leq \frac{1}{10}$ (due to $h \leq \frac{R^2}{4}$, the definition of R , and $C_0, A \geq 1$), it follows from this and (4.20) that

$$|(F(P) - F(Q))_2| \leq 20C_0 A^3 \sqrt{h} |Q|.$$

But this and (4.20) now yield (also using $6C_0 A^3 h \leq \frac{1}{10}$ and $\sqrt{1+b} \leq 1 + \frac{b}{2}$ for $b \geq 0$)

$$||F(P) - F(Q)| - |Q|| \leq \left| (1 + 13C_0 A^3 h + 400C_0^2 A^6 h)^{1/2} - 1 \right| |Q| \leq 207C_0^2 A^6 h |Q|,$$

so (4.18) follows because P is the origin.

For $P, Q \in \partial\Omega$ (or $\partial\tilde{\Omega}$), we now define $L(P, Q)$ (or $\tilde{L}(P, Q)$) to be the arc-length along $\partial\Omega$ (or $\partial\tilde{\Omega}$) from P to Q , in the counter-clockwise direction. For any $P, Q \in \partial\Omega$, one can obtain $L(P, Q)$ as the limit as $J \rightarrow \infty$ of lengths of polygonal paths $P = P_0 \rightarrow P_1 \rightarrow \dots \rightarrow P_J$, with each P_{j+1} lying on the arc $P_j Q$ of $\partial\Omega$ and all the segment lengths $|P_{j+1} - P_j|$ less than some l_J which satisfies $\lim_{J \rightarrow \infty} l_J = 0$. Then $\tilde{L}(F(P), F(Q))$ is the limit of the lengths of the paths $F(P) = F(P_0) \rightarrow F(P_1) \rightarrow \dots \rightarrow F(P_J) = F(Q)$ because $|P_{j+1} - P_j| \leq 2l_J$ for all large J , due to (4.18) and $h \leq \frac{R^2}{4} = \frac{1}{4^7 C_0^6 A^6} < \frac{1}{300 C_0^2 A^6}$.

It follows then from (4.18) that for any $P, Q \in \partial\Omega$ we have

$$\frac{\tilde{L}(F(P), F(Q))}{L(P, Q)} \in [1 - C_1 A^6 h, 1 + C_1 A^6 h].$$

If z is a constant speed parametrization of $\partial\Omega$, then

$$L(P, Q) \leq |\partial\Omega| = \|z'\|_{L^1} \leq 2\pi \|z'\|_{L^\infty} \leq 2\pi \|z\|_{C^2} \leq 2\pi C_0 \|z\|_{H^3},$$

which yields (with $C_2 := 2\pi C_0 C_1$)

$$|\tilde{L}(F(P), F(Q)) - L(P, Q)| \leq C_1 A^6 h |L(P, Q)| \leq C_2 A^7 h.$$

In particular, we have $\left| |\partial\tilde{\Omega}| - |\partial\Omega| \right| \leq C_2 A^7 h$.

Finally, fix z above and let \tilde{z} be the (unique) constant speed parametrization of $\partial\tilde{\Omega}$ satisfying $\tilde{z}(0) = F(z(0))$. Then for any $\xi \in [0, 2\pi)$, we have

$$\begin{aligned} |z(\xi) - \tilde{z}(\xi)| &\leq |z(\xi) - F(z(\xi))| + |F(z(\xi)) - \tilde{z}(\xi)| \\ &\leq 2h + |\tilde{L}(F(z(\xi)), \tilde{z}(0)) - \tilde{L}(\tilde{z}(0), \tilde{z}(\xi))| \\ &\leq 2h + |\tilde{L}(F(z(0)), F(z(\xi))) - L(z(0), z(\xi))| + |L(z(0), z(\xi)) - \tilde{L}(\tilde{z}(0), \tilde{z}(\xi))| \\ &\leq 2h + C_2 A^7 h + \frac{\xi}{2\pi} \left| |\partial\tilde{\Omega}| - |\partial\Omega| \right| \\ &\leq (2 + 2C_2) A^7 h, \end{aligned}$$

which yields (4.14). □

5 Proofs of Proposition 1.3 and Theorem 1.5

Let us start with some estimates on fluid velocities generated by $C^{1,\gamma}$ patches. These results apply at a fixed time, hence we drop the argument t in them. And again, while

we consider here the half-plane case $D = \mathbb{R} \times \mathbb{R}^+$, the arguments are identical for the whole plane $D = \mathbb{R}^2$.

We first consider the setting from Definition 3.3, and will assume that $L \geq 1$ (note that since the area of each evolving patch stays constant, we only need to choose it to be at least π initially so that the arc-length of the patch boundary will always be at least 2π). This is done to simplify our estimates but can be replaced by $L \geq \frac{1}{A}$ for some $A < \infty$. Since $L \geq \frac{1}{\pi}|z(\pi) - z(0)| \geq F[z]^{-1} \geq \|\Omega\|_{1,\gamma}^{-1}$, this assumption can even be omitted (at the expense of changing the constants) because the results below assume $\|\Omega\|_{1,\gamma} \leq A$.

The following is a crucial bound on the gradient of the component of v from (3.4) normal to $\partial\Omega$, that is, on $\nabla(v(x) \cdot n_P) = \nabla v(x)n_P$, with $x = P + rn_P$ for some small r .

Lemma 5.1. *For $\gamma > \frac{2\alpha}{1-2\alpha}$, let $\Omega \subseteq \mathbb{R}^2$ be as in Definition 3.3, with $L \geq 1$ and $\|\Omega\|_{1,\gamma} \leq A$ for some $A \geq 1$. Let also $R := (4A)^{-\frac{1}{\gamma}-1}$ and v be given by (3.4) with $\omega(x) = \chi_\Omega(x)$. Then for any $P \in \partial\Omega$ and any $x = P + rn_P$ with $|r| \in (0, \frac{R}{2})$, we have $|\nabla v(x)n_P| \leq C(\alpha, \gamma)A$.*

Proof. Let $S_P := \{y \in \mathbb{R}^2 : (y - P) \cdot n_P \in (-R, 0)\}$, and let v_{S_P} be given by (3.4) with $\omega = \chi_{S_P}$, evaluated as principal value. By symmetry we have $v_{S_P} \cdot n_P \equiv 0$. Thus

$$\begin{aligned} |\nabla v(x)n_P| &= |\nabla(v(x) - v_{S_P}(x))n_P| \leq |\nabla(v(x) - v_{S_P}(x))| \leq C(\alpha) \int_{\Omega \Delta S_P} \frac{1}{|x - y|^{2+2\alpha}} dy \\ &\leq C(\alpha) \left(\underbrace{\int_{(\Omega \Delta S_P) \cap B(P,R)} \frac{1}{|x - y|^{2+2\alpha}} dy}_{=: I_1} + \underbrace{\int_{\mathbb{R}^2 \setminus B(P,R)} \frac{1}{|x - y|^{2+2\alpha}} dy}_{=: I_2} \right), \end{aligned}$$

where as before $A \Delta B := (A \setminus B) \cup (B \setminus A)$. Using $|x - P| < \frac{R}{2}$, we obtain

$$I_2 \leq 2\pi \int_{R/2}^{\infty} r^{-(1+2\alpha)} dr \leq C(\alpha)R^{-2\alpha} \leq C(\alpha)A, \quad (5.1)$$

where in the last step we used the definition of R , $A \geq 1$, and $2\alpha \frac{1+\gamma}{\gamma} < 1$.

To control I_1 , we change coordinates to (w_1, w_2) from Lemma 3.4(b), which then implies that $(\Omega \Delta S_P) \cap B(P, R)$ lies between the curves $w_2 = \pm 4Aw_1^{1+\gamma}$. Hence

$$I_1 \leq 2 \int_0^R w_1^{-(2+2\alpha)} 8Aw_1^{1+\gamma} dw_1 \leq C(\alpha, \gamma)AR^{\gamma-2\alpha} \leq C(\alpha, \gamma)A,$$

where we first used that the w_1 coordinate of x is 0, and then that $\gamma > 2\alpha$ and $R < 1$. \square

Remarks. 1. The estimate on $\nabla v(x)n_P$ holds not only on the line normal to $\partial\Omega$ at P , but also non-tangentially. Given any $\sigma > 0$, it is easy to see that we can replace the condition on x in the statement of Lemma 5.1 with $|x - P| < c(R, \sigma)$ (for some $c(R, \sigma) > 0$) and $(x - P) \cdot n_P \geq \sigma|x - P|$, with the conclusion being $|\nabla v(x)n_P| \leq C(\alpha, \gamma, \sigma)A$.

2. Note that ∇v is in general not defined at $P \in \partial\Omega$ due to a lack of regularity in the tangential component $v(x) \cdot n_P^\perp$ of v at P . However, the argument in the proof of Lemma 5.1 can be used to show that the normal component $v(x) \cdot n_P$ is sufficiently regular at P , and $\nabla(v(P) \cdot n_P)$ can in fact be defined. We will make this more precise later.

Lemma 5.1 and Lemma 3.2 now yield the following.

Corollary 5.2. *Let Ω and v satisfy the hypotheses of Lemma 5.1. Then for any $x \notin \partial\Omega$ and any $P \in \partial\Omega$ such that $|x - P| = \text{dist}(x, \partial\Omega) =: d(x)$ we have $|\nabla v(x) \frac{x-P}{|x-P|}| \leq C(\alpha, \gamma)A$ and $|(v(x) - v(P)) \cdot \frac{x-P}{|x-P|}| \leq C(\alpha, \gamma)A|x - P|$.*

Proof. Notice that $\frac{x-P}{|x-P|} \in \{n_P, -n_P\}$. If $d(x) < \frac{R}{2}$, then the first claim follows from Lemma 5.1. Otherwise, Lemma 3.2 yields $|\nabla v(x)n_P| \leq |\nabla v(x)| \leq C(\alpha)R^{-2\alpha} \leq C(\alpha)A$ because $A \geq 1$ and $2\alpha \frac{1+\gamma}{\gamma} < 1$.

To prove the second claim, note that if $y_s := x + s(P - x)$ for $s \in [0, 1]$, then $\text{dist}(y_s, P) = \text{dist}(y_s, \partial\Omega)$. Hence the first claim yields $|\nabla(v(y_s) \cdot n_P)| \leq C(\alpha, \gamma)A$ for $s \in [0, 1]$. Integrating this in $s \in [0, 1)$ and using continuity of u yields the second claim. \square

We now extend Lemma 5.1 and Corollary 5.2 to the case of N patches $\Omega_k \subseteq D$ with disjoint boundaries. We let $2\pi L_k := |\partial\Omega_k|$, and if $P \in \partial\Omega_k$ (such k is then unique), we denote by n_P the outer unit normal vector for Ω_k at P . We will again assume that $|L_k| \geq 1$, and also that $|\theta_k| \leq 1$, both of which are not essential but simplify our formulas. Finally, recall that $\partial\Omega := \bigcup_{k=1}^N \partial\Omega_k$.

Proposition 5.3. *For $\gamma > \frac{2\alpha}{1-2\alpha}$, some $A \geq 1$, and $k = 1, \dots, N$, let $\Omega_k \subseteq D$ be as in Definition 3.3, with $L_k \geq 1$ and $\|\Omega_k\|_{1,\gamma} \leq A$. Assume also $\text{dist}(\partial\Omega_i, \partial\Omega_k) \geq \frac{1}{A}$ for all $i \neq k$ and let $R := (4A)^{-\frac{1}{\gamma}-1}$. Finally, let u be given by (1.2) where $\omega = \sum_{k=1}^N \theta_k \chi_{\Omega_k}$ and $|\theta_k| \leq 1$. Then for any $P \in \partial\Omega$ and any $x = P + rn_P \in \bar{D}$ with $|r| \in (0, \frac{R}{2})$, we have $|\nabla u(x)n_P| \leq C(\alpha, \gamma)A$.*

Proof. Denote by $\tilde{\Omega}_k$ the reflection of Ω_k with respect to the x_1 -axis. Since $P \in \partial\Omega_k$ for some k , $\min_{i \neq k} \text{dist}(\partial\Omega_i, \partial\Omega_k) \geq \frac{1}{A} > R$, and $x \in B(P, \frac{R}{2}) \cap \bar{D}$, we have $\text{dist}(x, \Omega_i) > \frac{R}{2}$ and $\text{dist}(x, \tilde{\Omega}_i) > \frac{R}{2}$ for all $i \neq k$. Due to Lemma 3.2, the total contribution to $|\nabla u(x)|$

from all the Ω_i and $\tilde{\Omega}_i$ with $i \neq k$ is bounded by $C(\alpha)R^{-2\alpha}$, and hence also by $C(\alpha)A$, by the definition of R , $A \geq 1$, and $\frac{1+\gamma}{\gamma}2\alpha < 1$.

Moreover, the contribution to $|\nabla u(x)n_P|$ from Ω_k is bounded by $C(\alpha, \gamma)A$ due to Lemma 5.1. Thus it suffices to bound the contribution from $\tilde{\Omega}_k$. Let

$$\tilde{v}(x) := \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \chi_{\tilde{\Omega}_k}(y) dy,$$

so that it suffices to show that $|\nabla \tilde{v}(x)n_P| \leq C(\alpha, \gamma)A$. Let $\tilde{d}_k(x) := \text{dist}(x, \tilde{\Omega}_k)$, where the minimum is achieved at some $Q_x \in \partial \tilde{\Omega}_k$, and let n_{Q_x} be the outer unit normal for $\tilde{\Omega}_k$ at Q_x . We then have

$$\begin{aligned} |\nabla \tilde{v}(x)n_P| &\leq |\nabla \tilde{v}(x)n_{Q_x}| + |\nabla \tilde{v}(x)(n_P + n_{Q_x})| \\ &\leq C(\alpha, \gamma)A + C(\alpha)\tilde{d}_k(x)^{-2\alpha}|n_P + n_{Q_x}|, \end{aligned}$$

where we bounded the first term by Corollary 5.2 for x and $\tilde{\Omega}_k$, and the second term by Lemma 3.2.

Note that if $\tilde{d}_k(x) \geq \frac{R}{4}$, then the needed inequality holds because $R^{2\alpha} \geq \frac{1}{A}$. Hence it suffices to show that $|n_P + n_{Q_x}| \leq C(\alpha, \gamma)A\tilde{d}_k(x)^{2\alpha}$ if $\tilde{d}_k(x) \leq \frac{R}{4}$. Let $\bar{Q}_x \in \partial \Omega_k$ be the reflection of Q_x across the x_1 -axis. Then we have

$$|P - \bar{Q}_x| \leq |x - P| + |x - \bar{Q}_x| \leq 2\text{dist}(x, \partial \Omega_k) + \tilde{d}_k(x) \leq 3\tilde{d}_k(x),$$

where in the second inequality we used that $|x - P| < \frac{R}{2}$ and Lemma 3.4(b) imply $\text{dist}(x, \partial \Omega_k) > \frac{|x-P|}{2}$. Now Lemma 3.4(c), $R < 1$, $L_k \geq 1$ and $\gamma > 2\alpha$ yield

$$|n_P - n_{\bar{Q}_x}| \leq 2A|P - \bar{Q}_x|^\gamma \leq 6A\tilde{d}_k(x)^{2\alpha}.$$

Symmetry, Lemma 3.4(d), $\tilde{d}_k(x) = |x - Q_x|$, $x \in \bar{D}$, and $\frac{\gamma}{1+\gamma} > 2\alpha$ also give (with $\bar{Q}_x =: (q_1, q_2)$ and $Q_x = (q_1, -q_2)$)

$$|n_{Q_x} + n_{\bar{Q}_x}| = 2|n_{\bar{Q}_x} \cdot (1, 0)| \leq 4A^{\frac{1}{1+\gamma}} q_2^{\frac{\gamma}{1+\gamma}} \leq 4A^{\frac{1}{1+\gamma}} \tilde{d}_k(x)^{\frac{\gamma}{1+\gamma}} \leq 4A\tilde{d}_k(x)^{2\alpha}.$$

Thus $|n_P + n_{Q_x}| \leq 10A\tilde{d}_k(x)^{2\alpha}$ and the proof is finished. \square

We now obtain the following analog of Corollary 5.2 (with an identical proof).

Corollary 5.4. *Let ω, u satisfy the hypotheses of Proposition 5.3. Then for any $x \in \bar{D} \setminus \partial \Omega$ and any $P \in \partial \Omega$ such that $|x - P| = \text{dist}(x, \partial \Omega) =: d(x)$ we have*

$$\left| \nabla u(x) \frac{x - P}{|x - P|} \right| \leq C(\alpha, \gamma)A \quad (5.2)$$

and

$$\left| (u(x) - u(P)) \cdot \frac{x - P}{|x - P|} \right| \leq C(\alpha, \gamma) A |x - P|. \quad (5.3)$$

Therefore, the normal component of the velocity u generated by such ω is Lipschitz in the normal direction relative to $\partial\Omega$. We will also need this result for ∂D .

Proposition 5.5. *Let ω, u satisfy the hypotheses of Proposition 5.3. Then for any $x = (x_1, x_2) \in \bar{D}$ we have*

$$|u_2(x)| \leq C(\alpha, \gamma) N A^{\frac{2+\gamma}{1+\gamma}} x_2. \quad (5.4)$$

Proof. We have

$$\begin{aligned} u_2(x) &= \sum_{k=1}^N \theta_k \int_{\Omega_k} \left(\frac{y_1 - x_1}{|x - y|^{2+2\alpha}} - \frac{y_1 - x_1}{|x - \bar{y}|^{2+2\alpha}} \right) dy \\ &= - \sum_{k=1}^N \frac{\theta_k}{2\alpha} \int_{\Omega_k} \partial_{y_1} \left(\frac{1}{|x - y|^{2\alpha}} - \frac{1}{|x - \bar{y}|^{2\alpha}} \right) dy \\ &= - \sum_{k=1}^N \frac{\theta_k}{2\alpha} \int_{\partial\Omega_k} (n_y)_1 \left(\frac{1}{|x - y|^{2\alpha}} - \frac{1}{|x - \bar{y}|^{2\alpha}} \right) d\sigma(y) \\ &= \sum_{k=1}^N \frac{\theta_k}{2\alpha L_k} \int_{\mathbb{T}} (z'_k)_2(\xi) \left(\frac{1}{|x - z_k(\xi)|^{2\alpha}} - \frac{1}{|x - \bar{z}_k(\xi)|^{2\alpha}} \right) d\xi, \end{aligned} \quad (5.5)$$

where $(n_y)_1 = n_y \cdot (1, 0)$ for $y \in \partial\Omega_k$ and $(z_k)_2 = z_k \cdot (0, 1)$, with z_k a constant speed parametrization of $\partial\Omega_k$. Hence for any $x \in \bar{D}$ we have

$$\begin{aligned} |u_2(x)| &\leq \sum_{k=1}^N \frac{\theta_k}{2\alpha L_k} \int_{\mathbb{T}} \frac{|(z'_k)_2(\xi)|}{|x - z_k(\xi)|^{2\alpha}} \left| 1 - \frac{|x - z_k(\xi)|^{2\alpha}}{|x - \bar{z}_k(\xi)|^{2\alpha}} \right| d\xi \\ &\leq \sum_{k=1}^N \frac{\theta_k}{2\alpha L_k} \int_{\mathbb{T}} \frac{|(z'_k)_2(\xi)| 2x_2}{|x - z_k(\xi)|^{2\alpha} |x - \bar{z}_k(\xi)|} d\xi \\ &\leq \sum_{k=1}^N \frac{2\theta_k A^{\frac{1}{1+\gamma}} x_2}{\alpha L_k} \underbrace{\int_{\mathbb{T}} \frac{1}{|x - z_k(\xi)|^{2\alpha + \frac{1}{1+\gamma}}} d\xi}_{=: T_k}, \end{aligned} \quad (5.6)$$

where in the second inequality we used that $|1 - b^{2\alpha}| \leq |1 - b|$ for $b \geq 0$ and $\alpha \in (0, \frac{1}{2})$, as well as that $0 \leq |x - \bar{z}_k(\xi)| - |x - z_k(\xi)| \leq 2x_2$; and in the last inequality we used Lemma 2.2 for $(z_k)_2$ and also that $|x - \bar{z}_k(\xi)| \geq \max\{|(z_k)_2(\xi)|, |x - z_k(\xi)|\}$.

It now remains to show that $T_k \leq C(\alpha, \gamma)A$. Let $d_k(x) := d(x, \partial\Omega_k)$, and consider only the case $d_k(x) \leq 1$ because otherwise clearly $T_k \leq 2\pi$. Let $\xi_0 \in \mathbb{T}$ be such that $|x - z_k(\xi_0)| = d_k(x)$. Using $d_k(x) \leq 1$ and $F[z_k] \leq A$ yields

$$\begin{aligned}
T_k &= \int_{|\xi - \xi_0| \leq 2Ad_k(x)} |x - z_k(\xi)|^{-2\alpha - \frac{1}{1+\gamma}} d\xi + \int_{|\xi - \xi_0| > 2Ad_k(x)} |x - z_k(\xi)|^{-2\alpha - \frac{1}{1+\gamma}} d\xi \\
&\leq 4Ad_k(x)^{\frac{\gamma}{1+\gamma} - 2\alpha} + \int_{|\xi - \xi_0| > 2Ad_k(x)} (|z_k(\xi) - z_k(\xi_0)| - d_k(x))^{-2\alpha - \frac{1}{1+\gamma}} d\xi \\
&\leq 4A + \int_{|\xi - \xi_0| > 2Ad_k(x)} \left(\frac{1}{2} |z_k(\xi) - z_k(\xi_0)| \right)^{-2\alpha - \frac{1}{1+\gamma}} d\xi \\
&\leq 4A + (2A)^{2\alpha + \frac{1}{1+\gamma}} \int_{\mathbb{T}} |\xi - \xi_0|^{-2\alpha - \frac{1}{1+\gamma}} d\xi,
\end{aligned}$$

which is bounded by $C(\alpha, \gamma)A$ due to $2\alpha < \frac{\gamma}{1+\gamma}$ and $A \geq 1$. \square

The above results lead to the following lemma, from which Proposition 1.3 will follow.

Lemma 5.6. *Consider the setting of Proposition 1.3 and assume that ω is a $C^{1,\gamma}$ patch solution to (1.1)-(1.2) on $[0, T]$. Then $\Phi_t(x)$ is unique for each $(x, t) \in (\bar{D} \setminus \partial\Omega(0)) \times [0, T]$, and for each $T' \in (0, T)$, there is $B < \infty$ such that $d_t(x) := \text{dist}(x, \partial\Omega(t))$ satisfies*

$$d_t(\Phi_t(x)) \geq e^{-Bt} d_0(x) \quad \text{and} \quad (\Phi_t(x))_2 \geq e^{-Bt} x_2 \quad (5.7)$$

for each $(x, t) \in (\bar{D} \setminus \partial\Omega(0)) \times [0, T']$.

Proof. Let $A \geq 1$ be such that

$$A \geq \sup_{t \in [0, T']} \left[\max_k \|\Omega_k(t)\|_{1,\gamma} + \max_{k \neq i} \text{dist}(\partial\Omega_k(t), \partial\Omega_i(t))^{-1} \right],$$

and let $B := C(\alpha, \gamma)NA^{\frac{2+\gamma}{1+\gamma}}$, with $C(\alpha, \gamma)$ from Corollary 5.4 and Proposition 5.5. To satisfy the hypotheses of Proposition 5.3 on $[0, T']$, we will assume that $|\theta_k| \leq 1$ and that the lengths $2\pi L_k(t)$ of $\partial\Omega_k(t)$ satisfy $L_k(t) \geq 1$ for all k and $t \in [0, T']$. As we remarked before Lemma 5.1 and before Proposition 5.3, these two assumptions are not essential and can be removed by adjusting the constants involved in the bounds with extra factors of A and $\Theta := \sum_{k=1}^N |\theta_k|$. One can also see this by a scaling argument. Specifically, the scaling $\tilde{\theta}_k := \theta_k \Theta^{-1}$ and $\tilde{\Omega}_k(t) := \lambda \Omega_k(\lambda^{-2\alpha} \Theta^{-1} t)$ yields a patch solution $\tilde{\omega}$ on $[0, A^{2\alpha} \Theta T]$. Choosing $\lambda := A$ makes any constant speed parametrization $Z(t)$ of $\partial\tilde{\Omega}(t)$ satisfy $\inf_{t \in [0, A^{2\alpha} \Theta T']} |\tilde{z}_k(\pi, t) - \tilde{z}_k(0, t)| \geq \pi$ for each k because $F[Z(t)] \leq A$. Hence $|\partial\tilde{\Omega}_k(t)| \geq 2\pi$ for each k and $t \in [0, T']$, and of course $|\tilde{\theta}_k| \leq 1$. If the result holds for $\tilde{\theta}_k$

and $\tilde{\Omega}_k$ on $[0, A^{2\alpha}\Theta T']$, then it also holds for θ_k and Ω_k on $[0, T']$, but with B replaced by $A^{2\alpha}\Theta B$.

Fix now any $x \in \bar{D} \setminus \partial\Omega(0)$ and let $T' < T$ be any time such that $\Phi_t(x) \notin \partial\Omega(t)$ for all $t \leq T'$. To prove the lemma, it suffices to show (5.7) with this T' and the B from above.

The second claim in (5.7) now follows directly from Proposition 5.5, so let us consider the first. Notice that the function $f(t) := d_t(\Phi_t(x))$ is Lipschitz on $[0, T']$ (this is proved via the argument from Lemma 4.10). Note that while f depends on x , we will suppress this in the notation. Hence f' exists almost everywhere and $f(t) - f(0) = \int_0^t f'(s)ds$ for $t \in [0, T']$. Gronwall's inequality now shows that if the first claim in (5.7) does not hold for some $t \in [0, T']$, then there must be $s \in [0, t]$ such that $f(s) > 0$ and $f'(s) < -Bf(s)$. Let $a := -\frac{1}{3}(Bf(s) + f'(s)) > 0$ and let $\delta > 0$ be such that

$$\left| [u(x'', s'') - u(x', s)] \cdot \frac{\Phi_s(x) - x'}{|\Phi_s(x) - x'|} \right| \leq Bf(s) + a \quad (5.8)$$

whenever $|x'' - \Phi_s(x)| \leq \delta$, $|s'' - s| \leq \delta$, and $|x' - P| \leq \delta$ for some $P \in \partial\Omega(s)$ such that $|\Phi_s(x) - P| = f(s)$. Existence of such δ follows from continuity of u (which holds by the last claim in Lemma 3.1) and Corollary 5.4, because (5.3) shows (5.8) with $(x'', s'', x') := (\Phi_s(x), s, P)$ and $Bf(s)$ instead of $Bf(s) + a$.

Let now $\delta' := \inf_{x' \in S} |\Phi_s(x) - x'| - f(s)$ (with $\delta' := \infty$ if $S = \emptyset$), where

$$S := \{x' \in \partial\Omega(s) : B(x', \delta) \cap \partial\Omega(s) \cap \overline{B(\Phi_s(x), f(s))} = \emptyset\},$$

and notice that $\delta' > 0$ since $\partial\Omega(s)$ is compact and so is S . Because of this, the distance of the points in S ($\subseteq \partial\Omega(s)$) from $\Phi_s(x)$ exceeds $f(s)$ by more than a positive constant, and thus their dynamics will not affect $f'(s)$. Also let $h := C^{-1} \min\{\delta', \delta\}$ ($\leq \delta$), where $C := 2\|u\|_{L^\infty} + a + 1$. Since $f'(s) = -(Bf(s) + 3a)$, there are $s' \in (s, s + h)$ with arbitrarily small $s' - s$ such that

$$f(s') < f(s) - (Bf(s) + 2a)(s' - s). \quad (5.9)$$

Pick such s' so that we also have $d_H(\partial\Omega(s'), X_{u(\cdot, s)}^{s'-s}[\partial\Omega(s)]) \leq a(s' - s)$ (which is possible by (1.6)), and let $Q \in \partial\Omega(s')$ be such that $|\Phi_{s'}(x) - Q| = f(s')$. There exists $\tilde{Q} \in \partial\Omega(s)$ such that

$$|\tilde{Q} + (s' - s)u(\tilde{Q}, s) - Q| \leq a(s' - s). \quad (5.10)$$

Therefore

$$|\Phi_s(x) - \tilde{Q}| \leq |\Phi_{s'}(x) - Q| + |\Phi_{s'}(x) - \Phi_s(x)| + |\tilde{Q} - Q| \leq |\Phi_{s'}(x) - Q| + (2\|u\|_{L^\infty} + a)h < f(s) + \delta',$$

which implies that $\tilde{Q} \in \partial\Omega(s) \setminus S$. Hence $|\tilde{Q} - P| \leq \delta$ for some $P \in \partial\Omega(s)$ with $|\Phi_s(x) - P| = f(s)$. Let us now write

$$\begin{aligned} \Phi_{s'}(x) - Q &= \Phi_s(x) - \tilde{Q} + \Phi_{s'}(x) - \Phi_s(x) + \tilde{Q} - Q \\ &= \Phi_s(x) - \tilde{Q} + \int_s^{s'} [u(\Phi_{s''}(x), s'') - u(\tilde{Q}, s)] ds'' + (s' - s)u(\tilde{Q}, s) + \tilde{Q} - Q. \end{aligned}$$

Multiplying this equality by $\frac{\Phi_s(x) - \tilde{Q}}{|\Phi_s(x) - \tilde{Q}|}$, and using (5.10), we obtain

$$\begin{aligned} |\Phi_{s'}(x) - Q| &\geq |\Phi_s(x) - \tilde{Q}| - \int_s^{s'} \left| [u(\Phi_{s''}(x), s'') - u(\tilde{Q}, s)] \cdot \frac{\Phi_s(x) - \tilde{Q}}{|\Phi_s(x) - \tilde{Q}|} \right| ds'' - a(s' - s) \\ &\geq f(s) - (Bf(s) + 2a)(s' - s), \end{aligned}$$

where in the last step we used that $|\tilde{Q} - P| \leq \delta$, and that from $\|u\|_{L^\infty h} \leq \delta$ we have $|\Phi_{s''}(x) - \Phi_s(x)| \leq \delta$ for any $s'' \in [s, s']$, so (5.8) applies. The obtained inequality contradicts (5.9), and the proof is finished. \square

Remark. We note that the argument above can be extended in a straightforward manner to show that for $(x, t) \in (\bar{D} \setminus \partial\Omega(0)) \times [0, T]$ we in fact have

$$\frac{d^+}{dt} d_t(\Phi_t(x)) = \inf_{P \in \partial\Omega(t) \cap \partial B(\Phi_t(x), d_t(\Phi_t(x)))} \left\{ [u(\Phi_t(x), t) - u(P, t)] \cdot \frac{\Phi_t(x) - P}{|\Phi_t(x) - P|} \right\}, \quad (5.11)$$

where $\frac{d^+}{dt}$ is the right derivative. The left derivative has sup in place of inf.

Proof of Proposition 1.3. (a) This follows from Lemma 5.6 and smoothness of u away from $\partial\Omega$ (see Lemma 3.2). Indeed, these show that $\Phi_t : [\bar{D} \setminus \partial\Omega(0)] \rightarrow [\bar{D} \setminus \partial\Omega(t)]$ is injective, and it is surjective by solving the ODE in (1.4) backwards in time, with any given terminal condition $\Phi_t(x) = y \in \bar{D} \setminus \partial\Omega(t)$. Note that all estimates of Lemma 5.6 still apply in this case.

(b) First note that $\Phi_t : [\bar{D} \setminus \partial\Omega(0)] \rightarrow [\bar{D} \setminus \partial\Omega(t)]$ is measure preserving because it is such when restricted to any closed subset of $\bar{D} \setminus \partial\Omega(0)$ (due to $\nabla \cdot u \equiv 0$, compactness of $\partial\Omega(t)$, and its continuity in time). Continuity of $\Phi_t(x)$ and $\partial\Omega(t)$ in time also shows that Φ_t must preserve connected components of $\bar{D} \setminus \partial\Omega$.

In addition, since the ODE in (1.4) has unique backwards-in-time solutions with terminal conditions $\Phi_t(x) = y \in \bar{D} \setminus \partial\Omega(t)$, and they satisfy $x \in \bar{D} \setminus \partial\Omega(0)$ (due to $\Phi_t : [\bar{D} \setminus \partial\Omega(0)] \rightarrow [\bar{D} \setminus \partial\Omega(t)]$ being a bijection), any $\Phi_t(x)$ for $(x, t) \in \partial\Omega_k(0) \times [0, T]$ must be in $\partial\Omega(t)$ (and hence in $\partial\Omega_k(t)$ by continuity). We then also have that for each

$t \in [0, T)$ and $y \in \partial\Omega_k(t)$ there is a solution of (1.4) such that $\Phi_t(x) = y$ (obtained by solving (1.4) backwards), and Φ_t being a bijection shows that we must have $x \in \partial\Omega_k(0)$.

Finally, (1.8) together with uniform continuity of u on a neighborhood of the compact set $\partial\Omega(t) \times \{t\}$ shows that (1.6) holds for each $t \in (0, T)$. Hence ω is a patch solution to (1.1)-(1.2) on $[0, T)$. \square

Proof of Theorem 1.5. This is identical to the proofs of Proposition 1.3 and Theorem 1.4 in the case $D = \mathbb{R} \times \mathbb{R}^+$, but with Theorem 2.8 being valid for all $\alpha \in (0, \frac{1}{2})$ when $D = \mathbb{R}^2$ [13], so that Proposition 4.5, Corollary 4.7, and Theorem 4.12 then also hold with $\alpha \in (0, \frac{1}{2})$. \square

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