THE 2D MUSKAT PROBLEM I: LOCAL REGULARITY ON THE HALF-PLANE, PLANE, AND STRIPS

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Abstract. We prove local well-posedness for the Muskat problem on the half-plane, which models motion of an interface between two fluids of distinct densities (e.g., oil and water) in a porous medium (e.g., an aquifer) that sits atop an impermeable layer (e.g., bedrock). Our result allows for the interface to touch the bottom, and hence applies to the important scenario of the heavier fluid invading a region occupied by the lighter fluid along the impermeable layer. We use this result in the companion paper [24] to prove existence of finite time stable regime singularities in this model, including for arbitrarily small initial data. We do not require the interface and its derivatives to vanish at $\pm\infty$ or be periodic, and even allow it to be $O(|x|^{1-})$, which is an optimal bound on the power of growth. We also extend our results to the cases of the Muskat problem on the whole plane and on horizontal strips, where previous works did impose such limiting requirements.

1. Introduction and Main Results

The *Muskat problem* is a mathematical model for the motion of the interface between two incompressible immiscible fluids of different densities $\rho_1 > \rho_0$ (such as water and oil, or salt water and fresh water) inside a porous medium (such as a sand or sandstone aquifer) [1, 19]. It has also been used as a model for cell velocity in the growth of tumors [14, 20], and is related to the Hele-Shaw cell problem [18, 22]. If one instead considers a single fluid with continuously varying density, the corresponding model is the *incompressible porous medium (IPM) equation*. In both cases one studies the transport PDE

$$\partial_t \rho + u \cdot \nabla \rho = 0 \tag{1.1}$$

for fluid density ρ and velocity u, with the former of the form

$$\rho(\mathbf{x}, t) = \rho_1 - (\rho_1 - \rho_0)\chi_{\Omega_t}(\mathbf{x}) \tag{1.2}$$

in the case of the Muskat problem, where Ω_t is the (time-dependent) region of the lighter fluid. The velocity u is determined via *Darcy's law*, which after appropriate temporal scaling — by a factor that is the product of the gravitational constant, permeability of the medium, and the reciprocal of the fluid viscosity — becomes

$$u := -\nabla p - (0, \rho) \qquad \text{and} \qquad \nabla \cdot u = 0. \tag{1.3}$$

The term $-(0, \rho)$ models downward motion of denser regions of the fluid due to gravity and p is the fluid pressure, needed to obtain a divergence-free u. In the case of the Muskat problem, motion of the fluid interface $\partial\Omega_t$ determines the full dynamic, so only the value of u on this interface is relevant.

In two spatial dimensions, these models have been studied extensively on the whole plane $\mathbb{R}^2$, including in most of the references below. In particular, local regularity for the Muskat

problem was proved in the class of fluid interfaces that are graphs of functions from $H^3(\mathbb{R})$ [8] (as well as from $H^2(\mathbb{R})$ [5]; see also [23] for an earlier result for near-flat interfaces), as long as the lighter fluid lies above the heavier one (i.e., in the *stable regime*, when the Rayleigh-Taylor condition [21,22] is satisfied). This is the best result one can hope for in Sobolev spaces because the problem is ill-posed if the interface is in the unstable regime at any point, with the heavier fluid lying above the lighter one [8,23] (except in the space of analytic interfaces [4]). The question whether the latter situation can develop from a global lighter-above-heavier configuration, via interface *overturning*, was answered in the affirmative in [4] (overturning was studied in many other papers, both analytically and numerically, see e.g. [2,3,10,11,17]). This work therefore provided an example of finite time breakdown of local well-posedness for the Muskat problem on $\mathbb{R}^2$. Moreover, since it was shown in [4] that solutions with initial interfaces from $H^4(\mathbb{R})$ instantly become analytic and remain such until they turn over (if they do), it follows that overturning is the only mechanism for breakdown of local well-posedness on $\mathbb{R}^2$ (at least in $H^4(\mathbb{R})$, although instant smoothing is known to occur even for initial interfaces with corners [15,16]). We also note that other works studied the Muskat problem in settings with surface tension and for fluids with distinct viscosities (see, e.g., [7,13]).

However, since aquifers typically sit on top of (or in-between) impermeable rocky layers, of particular importance are the cases when the domain is the half-plane $\mathbb{R}\times\mathbb{R}^+$ or the strip $\mathbb{R}\times(0,l)$ for some $l>0$. On these domains the fluid velocity must also satisfy the *no-flow boundary condition* $u_2\equiv 0$ on the flat bottom $\mathbb{R}\times\{0\}$ (as well as on the top $\mathbb{R}\times\{l\}$ in the strip case). Moreover, these settings allow one to model the physically relevant and dynamically interesting scenario of *invasion* by one fluid of a region initially occupied by another fluid. In this case we have $\{a\le x_1\le b\}\subseteq\Omega_0$ for some interval $[a,b]$ and the heavier fluid may then flow into the region $\{a\le x_1\le b\}$ underneath the lighter fluid, along the impermeable bottom layer (and on the strip, the lighter fluid can similarly invade a region occupied by the heavier one along the top boundary).

While well-posedness for the Muskat problem on the half-plane or on the strip follows via straightforward adjustments of the whole plane proof when the fluid interface stays away from the domain boundary, all the relevant estimates blow up as this distance decreases. In particular, the invasion scenarios above cannot be modeled using currently available results. Our first motivation is to remedy this by showing local well-posedness for the Muskat problem on $\mathbb{R}\times\mathbb{R}^+$ and on $\mathbb{R}\times(0,l)$ for all sufficiently smooth initial interfaces, including those that touch the domain boundary. We do this in Theorems 1.1 and 1.3 below.

Our second motivation is to demonstrate existence of *(stable regime) singularity formation* for the fluid interface. Although the interface overturning examples in [4] involve breakdown of local well-posedness in Sobolev spaces, and the fluid interface acquires infinite slope in finite time, it *does not develop a singularity as a curve* at that time. In fact, as it is an analytic curve up to the time of overturning [4], it will remain such past this time if we continue it within the space of analytic interfaces because local well-posedness holds in that space even in the unstable regime [4]. Although it was proved in [3] that such interfaces can lose analyticity (and even C^4 regularity) in finite time if they remain in the unstable regime (see [11] for examples of interfaces that do not), the results from [4] show that interface curve singularities do not develop whithin the well-posedness theory in Sobolev spaces for

the Muskat problem on $\mathbb{R}^2$. And more importantly, stable regime singularities do not develop even for analytic interfaces on the plane.

In contrast, we show in the companion paper [24] (using the main results of this paper) that H^3 fluid interfaces on the half-plane can indeed develop stable regime finite time singularities, even for arbitrarily small initial data. The mechanism for this is invasion by the heavier fluid of a region occupied by the lighter fluid along the impermeable bottom layer, described above, specifically when the invasion proceeds from both directions. In this scenario, arguments in [24] suggest that the fluid interface develops a singularity at the meeting point of the two invading "fronts".

Finally, our third motivation is to generalize existing theory for the Muskat problem on $\mathbb{R}^2$ to fluid interfaces that need not be spatially periodic or vanishing at $\pm\infty$, which is the current state of the art. To better model real world situations, one should be able to include at least general bounded interfaces, with uniformly bounded Sobolev norms on unit spatial intervals. We do this here and go even further, allowing interfaces with $O(|x|^{1-})$ growth as $|x| \to \infty$. This is an optimal result (see Remark 2 after Theorem 1.1), and we establish it on the whole plane, half-plane, as well as on strips (of course, in the latter case all solutions are bounded). In addition, we obtain here maximum principles for such fluid interfaces on all these domains.

The treatment of such general classes of interfaces will require proper definitions of norms of the functions involved. We provide these next, after presenting the interface PDE for the Muskat problem on the half-plane, and then state our main results.

Basic setup. We will perform all of our analysis on the half-plane $\mathbb{R}\times\mathbb{R}^+$. The whole plane case, which is a greatly simplified version of the former, and the strip case, which will add some extra difficulties, will be treated in the last section. From (1.3) we see that the fluid *vorticity* $\nabla^\perp \cdot u$ equals ρ_{x_1}, where we use the convention $\nabla^\perp := (\partial_{x_2}, -\partial_{x_1})$ (which is the negative of the usual definition but will be more convenient here). It follows that

$$u(\mathbf{x},t) := \nabla^\perp \Delta^{-1}\rho_{x_1}(\mathbf{x},t) = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}^+}\left(\frac{(\mathbf{x}-\mathbf{y})^\perp}{|\mathbf{x}-\mathbf{y}|^2} - \frac{(\mathbf{x}-\bar{\mathbf{y}})^\perp}{|\mathbf{x}-\bar{\mathbf{y}}|^2}\right)\rho_{x_1}(\mathbf{y},t)\,d\mathbf{y}, \tag{1.4}$$

where Δ is the Dirichlet Laplacian on $\mathbb{R}\times\mathbb{R}^+$, $\bar{\mathbf{y}} := (y_1, -y_2)$, and $\mathbf{y}^\perp := (y_2, -y_1)$.

In the stable regime on the half-plane, the interface of the two fluids at time $t \geq 0$ is the graph of some (sufficiently regular) non-negative function $x_1 \mapsto f(x_1,t)$, with the lighter fluid lying above the heavier one (so $\Omega_t = \{x_2 > f(x_1,t)\}$). It is then not difficult to derive from (1.1)–(1.3) a PDE for the motion of this interface, which we do for the reader's convenience on both the half-plane and the plane in the next section (see Section 12 for the strip case). This then obviously fully determines the dynamic of the fluid as long as the interface remains the graph of a (regular-enough) non-negative function. We can assume without loss that $\rho_1 - \rho_0 = 2\pi$ (otherwise we scale time by a factor of $\frac{\rho_1-\rho_0}{2\pi}$), when the resulting PDE is

$$f_t(x,t) = PV\int_{\mathbb{R}}\left[\frac{y\,(f_x(x,t)-f_x(x-y,t))}{y^2+(f(x,t)-f(x-y,t))^2} + \frac{y\,(f_x(x,t)+f_x(x-y,t))}{y^2+(f(x,t)+f(x-y,t))^2}\right]dy \tag{1.5}$$

(we also replaced x_1 by x here). Dropping the second fraction similarly yields the corresponding PDE on the plane, which is (1.13) below, while the strip case results in (1.14).

In all three cases, the regularity level of $f(\cdot,t)$ considered here will be H^3, as in [8], but we will only require uniform boundedness of H^2-norms of $f_x(\cdot,t)$ on unit intervals and even allow $O(|x|^{1-})$ growth of f as $|x|\to\infty$. This motivates the following definitions. For $k=0,1,\dots$ define the norms

$$\|g\|_{\tilde L^2(\mathbb{R})}:=\sup_{x\in\mathbb{R}}\|g\|_{L^2([x-1,x+1])}\qquad\text{and}\qquad \|g\|_{\tilde H^k(\mathbb{R})}:=\sum_{j=0}^k\|g^{(j)}\|_{\tilde L^2(\mathbb{R})}$$

on the spaces of those $g\in L^2_{\rm loc}(\mathbb{R})$ for which these are finite. Then for $k\ge 1$ and any $\gamma\in[0,1]$ define the seminorms

$$\|g\|_{\ddot C^\gamma(\mathbb{R})}:=\sup_{|x-y|\ge 1}\frac{|g(x)-g(y)|}{|x-y|^\gamma}\qquad\text{and}\qquad \|g\|_{\tilde H^k_\gamma(\mathbb{R})}:=\|g'\|_{\tilde H^{k-1}(\mathbb{R})}+\|g\|_{\ddot C^{1-\gamma}(\mathbb{R})},$$

which vanish for constant functions. In particular, $\|g\|_{\tilde H^k_\gamma(\mathbb{R})}<\infty$ allows g to have $O(|x|^{1-\gamma})$ growth at $\pm\infty$. Our reason for using $1-\gamma$ here is that for all $g\in L^2_{\rm loc}(\mathbb{R})$ we now have

$$\|g\|_{\tilde H^k_{\gamma'}(\mathbb{R})}\le\|g\|_{\tilde H^k_\gamma(\mathbb{R})}+\|g'\|_{L^\infty(\mathbb{R})}\le 2\|g\|_{\tilde H^k_\gamma(\mathbb{R})}\tag{1.6}$$

when $k\ge 1$ and $0\le\gamma'\le\gamma\le 1$, which agrees with a similar inequality for the norms $\|\cdot\|_{C^{k,\gamma}}$ (and will also be convenient for the analogous continuous seminorms from Section 2).

Main results. We can now state our main results. We do this for the half-plane case in the first two theorems, and then extend them to the plane and strip cases in the third one. The first result includes local well-posedness for (1.5) as well as an important blow-up criterion.

Theorem 1.1. *If $\gamma\in(0,1]$ and $\|\psi\|_{\tilde H^3_\gamma(\mathbb{R})}<\infty$ for some $\psi\ge 0$, there is γ-independent*

$$T_\psi\ge C_\gamma\min\Big\{\|\psi\|^{-4}_{\tilde H^3_\gamma(\mathbb{R})},\,1+\Big|\ln\|\psi\|_{\tilde H^3_\gamma(\mathbb{R})}\Big|\Big\}\tag{1.7}$$

(with some $C_\gamma>0$ depending only on γ) such that the following hold.

(i) There is a classical solution $f\ge 0$ to (1.5) on $\mathbb{R}\times[0,T_\psi)$ with $f(\cdot,0)\equiv\psi$ such that

$$\sup_{t\in[0,T]}\|f(\cdot,t)\|_{\tilde H^3_\gamma(\mathbb{R})}<\infty\tag{1.8}$$

and

$$\sup_{t\in[0,T]}\|f_t(\cdot,t)\|_{W^{1,\infty}(\mathbb{R})}<\infty\tag{1.9}$$

for any $T\in[0,T_\psi)$. And if $T_\psi<\infty$, then for each $\gamma'\in(0,1]$ we have

$$\int_0^{T_\psi}\|f_x(\cdot,t)\|^4_{C^{1,\gamma'}(\mathbb{R})}\,dt=\infty.\tag{1.10}$$

(ii) For each $T\in[0,T_\psi)$, f from (i) is the unique non-negative classical solution to (1.5) on $\mathbb{R}\times[0,T]$ that satisfies (1.8). We also have

$$\sup_{t\in[0,T]}\|f(\cdot,t)-\psi\|_{\tilde H^3(\mathbb{R})}<\infty,\tag{1.11}$$

and if there are $\psi_{\pm\infty}\in\mathbb{R}$ such that $\psi-\psi_{\pm\infty}\in H^3(\mathbb{R}^\pm)$, then even

$$\sup_{t\in[0,T]}\|f(\cdot,t)-\psi\|_{H^3(\mathbb{R})}<\infty.\tag{1.12}$$

Remarks. 1. The last claim in (i) means that f develops a singularity at T_ψ if the latter is finite. This could be either overturning, when f_x blows up, or an interface curve singularity.

2. One may ask whether these results extend to the case $\gamma = 0$, when even linear growth of $f(\cdot,t)$ is allowed at $\pm\infty$. If, e.g., $\psi' = \chi_{[0,\infty)}$ outside of some bounded interval, it is easy to see that the PV integral in (1.5) is not defined at any $x \in \mathbb{R}$ with $\psi'(x) \neq 0$ when $t = 0$, while the one in (1.13) is not defined at x with $\psi'(x) \neq -1$. This shows that Theorem 1.1 is optimal with respect to the power of the allowable growth of $f(\cdot,t)$ at $\pm\infty$.

Our second main result contains several crucial quantitative results for (1.5). These include an L^∞ maximum principle for f (which was proved in [8] for $H^3(\mathbb{R})$ interfaces on the whole plane), the claim that the interface cannot "peel off" the bottom without a loss of regularity, and an estimate on the difference of two solutions that are initially close on a large interval. The latter result, which we will later use to obtain the maximum principle on strips, uses norms $\|\cdot\|_{C^{2,\gamma}_\gamma}$ defined in Section 2 as well as

$$\|g\|_{\tilde{L}^{2,\mu}(\mathbb{R})} := \left\| \sqrt{1+(\mu-|x|)_+}\, g(x) \right\|_{\tilde{L}^2(\mathbb{R})}.$$

Theorem 1.2. *Let γ, ψ, f be as in Theorem 1.1.*

(i) Then $\sup f(\cdot,t)$ is non-increasing and $\inf f(\cdot,t)$ is non-decreasing on $[0,T_\psi)$.

(ii) If $\inf \psi = 0$, then $\inf f(\cdot,t) = 0$ for all $t \in [0,T_\psi)$.

(iii) If $\|\tilde{\psi}\|_{\tilde{H}^3_\gamma(\mathbb{R})} < \infty$ and $\tilde{f} \geq 0$ is a solution to (1.5) with $\tilde{f}(\cdot,0) \equiv \tilde{\psi} \geq 0$, then

$$\left\| f(\cdot,t) - \tilde{f}(\cdot,t) \right\|_{\tilde{L}^{2,\mu}(\mathbb{R})} \leq C_\gamma \exp\left[C_\gamma \int_0^t \left(1 + \|f(\cdot,s)\|^2_{C^{2,\gamma}_\gamma} + \|\tilde{f}(\cdot,s)\|^2_{C^{2,\gamma}_\gamma} \right) ds \right] \left\| \psi - \tilde{\psi} \right\|_{\tilde{L}^{2,\mu}(\mathbb{R})}$$

holds for each $\mu \geq 0$ and $t \in [0, \min\{T_\psi, T_{\tilde{\psi}}\})$, where C_γ only depends on γ.

Remarks. 1. Since we have (1.8), part (iii) also yields a local L^∞ bound on $f - \tilde{f}$.

2. In [24] we also obtain an L^2 maximum principle for f and an L^∞ maximum principle for f_x, which were proved for $H^3(\mathbb{R})$ solutions on the whole plane in [6,9]. These will be the key ingredients in the proof of existence of finite time stable regime singularities for (1.5).

We finally extend the above results to the Muskat problem on the plane and on horizontal strips. The derivation of (1.5) in Section 2 also gives

$$f_t(x,t) = PV \int_{\mathbb{R}} \frac{y\,(f_x(x,t) - f_x(x-y,t))}{y^2 + (f(x,t) - f(x-y,t))^2} dy \tag{1.13}$$

in the former case (with (1.4) now involving the Laplacian on $\mathbb{R}^2$). Similarly, in the case of the strip $\mathbb{R} \times (0,l)$ we obtain

$$f_t(x,t) = \sum_{\pm} PV \int_{\mathbb{R}} (f_x(x,t) \pm f_x(x-y,t))\, \Theta_l\big(y, f(x,t) \pm f(x-y,t)\big)\, dy, \tag{1.14}$$

where

$$\Theta_l(y,r) := \frac{\pi}{2l} \frac{\sinh \frac{\pi y}{l}}{\cosh \frac{\pi y}{l} - \cos \frac{\pi r}{l}} \tag{1.15}$$

(see Section 12 below for the derivation of (1.14), which is from [12]). Our main results extend to both these PDE with the natural adjustments.

Theorem 1.3. *(i) Theorem 1.1 and Theorem 1.2(i,iii) hold for* (1.13) *in place of* (1.5)*, without requiring that* $f, \psi, \tilde{f}, \tilde{\psi} \geq 0$.

(ii) Theorems 1.1 and 1.2 hold for (1.14) *in place of* (1.5)*, with* $0 \leq f, \psi, \tilde{f}, \tilde{\psi} \leq l$*. In addition, if* $\sup \psi = l$*, then* $\sup f(\cdot, t) = l$ *for all* $t \in [0, T_\psi)$.

Remark. Note that Theorem 1.3(i) includes existing results from [8] for $\psi \in H^3(\mathbb{R})$ (see (1.12)) and for periodic $\psi \in H^3_{\text{loc}}(\mathbb{R})$, since in the latter case uniqueness of solutions implies periodicity of $f(\cdot, t)$ for each $t \in [0, T_\psi)$ (with the same period).

Discussion of the proof of Theorem 1.1 and organization of the paper. Most of the rest of this paper, namely Sections 2–10, forms the proof of Theorem 1.1. When one works in $H^3(\mathbb{R})$ and on the plane (so without a bottom and $f(\cdot, t)$ vanishes at $\pm\infty$), local well-posedness for the Muskat problem in the stable regime can be obtained via a simple adjustment of a proof from [8], where local well-posedness in $H^4(\mathbb{R}^2)$ was obtained for the Muskat problem on $\mathbb{R}^3$. The crux of that proof is an estimate on the rate of change of the L^2 norm of $f_{xxx}(\cdot, t)$.

In the case of Theorem 1.1, the crucial argument will be an analogous estimate, this time involving *local* L^2 norms of $f_{xxx}(\cdot, t)$ and including terms corresponding to the second fraction in (1.5). The former requires introduction of appropriate cut-off functions h, and then we split the formula for the rate of change of the resulting local norms into eight integrals $I_j^\pm$, four corresponding to each term in (1.5) (see (2.11) below).

In the proof in [8], which only involved the I_j^- integrals and no cut-off functions, the main challenge was to estimate the integral corresponding to our I_1^-. This is because it includes the most singular integrand (involving $\partial_x^4 f$), and we make use of the arguments from [8] when estimating I_1^- and the other integrals I_j^- originating from the first fraction in (1.5). Our inclusion of the cut-offs h, and potential growth of $f(\cdot, t)$ at $\pm\infty$, unsurprisingly introduce a number of non-trivial complications in these arguments as well as in the rest of our proofs. (We note that it is not possible to use the result from [8] to obtain Theorem 1.1 on the plane via some approximation scheme, even for just $\tilde{H}^3(\mathbb{R})$ interfaces with no growth as $x \to \pm\infty$.)

However, working on the half-plane increases the difficulty level much more dramatically. Note that if $\inf_{x\in\mathbb{R}} f(x, 0) > 0$, then the second fraction in (1.5) is sufficiently regular and the local well-posedness proof on $\mathbb{R}^2$ extends to this case, but the obtained constants *blow up* as $\inf_{x\in\mathbb{R}} f(x, 0)$ approaches 0. As a result, this approach fails when $\inf_{x\in\mathbb{R}} f(x, 0) = 0$, which is the most important case, as we explained above. In our proof, integrals I_j^+, which originate from the second fraction in (1.5), will be much more challenging to estimate in the region where f is small. In fact, estimates on I_3^+ and I_4^+ (whose counterparts I_3^- and I_4^- are easily bounded) form over two thirds of the proof of our main a priori bound (2.12) in Sections 3–6, even though these terms do not involve the highest derivatives of f!

The reason for this is the following. When f is a smooth function, the inside integrands in all the integrals I_j^- are bounded functions of y because their numerators vanish to at least the same degree as their denominators. Since $f_{xxx}(\cdot, t)$ is only L^2_{loc} and the integrands involve derivatives of order up to 4, obtaining estimates on these integrals in terms of only $f_{xxx}(\cdot, t)$ and lower order derivatives is far from trivial and requires various symmetrization arguments as well as leveraging of cancellations coming from adding the integrands for y and $-y$ (for $y > 0$). On the other hand, denominators of the inside integrands in I_j^+ only vanish when

$f(x,t)=0$, and in that case such cancellations still work because $f_x(x,t)=0$ (since $f\geq 0$). What makes these integrals much more difficult to estimate is the fact that their integrands become very large (even after the y vs. $-y$ cancellation) when $f(x,t)$ is *close to but not equal to 0* and y is small. And this is even more pronounced for larger j (when the numerators and denominators include more factors), due to all the $\pm$ signs being $+$ and hence most of the factors not vanishing at $y=0$, which is why estimates on I_3^+ and I_4^+ will be the most involved.

Related to (and an illustration of) this is also the following observation. In Section 11 we show that (1.5) is the same as

$$f_t(x,t)=\pi\chi_{(0,\infty)}(f(x,t))+\sum_{\pm}PV\int_{\mathbb{R}}\frac{y\,f_x(x,t)-(f(x,t)\pm f(x-y,t))}{y^2+(f(x,t)\pm f(x-y,t))^2}dy. \tag{1.16}$$

The derivation of (1.16) shows that the last integral with $\pm$ being $+$, which comes from the second fraction in (1.5), is *discontinuous* at all $x\in\partial\{f(\cdot,t)=0\}$, even when $f\in C_c^\infty(\mathbb{R})$. This also suggests that proving local well-posedness for (1.5) via (1.16) would likely be even more complicated than our proof below. Nevertheless, we do use (1.16) when proving a maximum principle for f_x in [24].

It follows from (5.8) and (2.9) below that the integrand in I_3^+ is $O\left(\frac{|y|f(x,t)}{\max\{|y|,f(x,t)\}^4}\right)$ when $f(x,t)\leq 1$, $f_{xx}(x,t)$ is not small, and $|y|\leq\sqrt{f(x,t)}$ (the integrand in I_4^+ is similar, albeit more complex), and hence it is far from being integrable uniformly in the size of $f(x,t)$. Even after adding the integrand at y and $-y$, it remains too large as we explain just before (5.18). However, we are able to identify here somewhat unexpected additional cancellations in these integrals. Namely, after isolating (appropriately defined) leading order terms in the integrands, we find these terms to be y-derivatives of certain special functions with large variations (see (5.18), (5.36), and (6.9) below). Integrating these allows us to bound both I_3^+ and I_4^+ in a way that ultimately yields the crucial a priori estimate (2.12).

With (2.12) in hand, we obtain analogous estimates for an approximating family of more regular ε-dependent models (7.1) in Section 7. In Sections 8 and 9 we then derive related bounds on the rate of change of local L^2 norms of differences of two solutions to either (1.5) or (7.1) (these rates for single solutions may not be bounded due to possible growth of f as $x\to\pm\infty$). This proves uniqueness for (1.5), and we then use all these estimates in Section 10 to obtain local-in-time $\tilde{H}^3_\gamma$ solutions to (1.5) as $\varepsilon\to 0$ limits of solutions to (7.1) (the latter are found via fairly standard methods), as well as finish the proof of Theorem 1.1.

Finally, the proofs of Theorems 1.2 and 1.3 appear in Sections 11 and 12, respectively. Having the proof of Theorem 1.1, these will involve mostly straightforward arguments, although the extension of Theorem 1.1 to horizontal strips will require some care.

2. Derivation of (1.5) and the Start of the Proof of Theorem 1.1

In this and all the following sections, C will be some universal constant that may change from line to line and always depends only on γ. We will also assume without loss that $\gamma\in(0,\frac{1}{2}]$, so that (2.2) below holds. We can do this because if we apply Theorem 1.1 with some $\gamma'<\gamma$ in place of γ (using (1.6) with $g:=\psi$), then (1.11) guarantees that (1.8) also holds.

Before starting the proof of Theorem 1.1, let us derive (1.5) and show that this derivation is valid in the setting considered here.

Derivation of (1.5) **and** (1.13)**.** From (1.2) we obtain

$$\rho_{x_1}(\cdot,t) = (\rho_1-\rho_0)\frac{f_x(P(\cdot),t)}{\sqrt{1+f_x(P(\cdot),t)^2}}\delta_{\Gamma_f(t)},$$

where $\Gamma_f(t) := \{(x,f(x,t))\,|\,x\in\mathbb{R}\}$ is the graph of $f(\cdot,t)$ and $P(x_1,x_2) := x_1$. This and assuming that $\rho_1-\rho_0 = 2\pi$ (which we do without loss) now turns (1.4) into

$$u(x,f(x,t),t) = PV\int_{\mathbb{R}}\sum_{\pm}\left[\mp\frac{(f(x,t)\pm f(y,t),y-x)}{(x-y)^2+(f(x,t)\pm f(y,t))^2}\right]f_x(y,t)\,dy \tag{2.1}$$

for points on the curve $\Gamma_f(t)$ at each time $t\geq 0$.

The evolution is now fully determined by the motion of this curve, and hence by (2.1). If we want to recast it in terms of f only, we can do that by adding a multiple of the tangent vector $(1,f_x(x,t))$ to $u(x,f(x,t),t)$, which does not change the motion of the curve as a set provided $f(\cdot,t)$ is continuously differentiable. We choose this multiple so that the resulting vector's first coordinate vanishes. That is, we replace u by

$$v(x,f(x,t),t) := u(x,f(x,t),t) + PV\int_{\mathbb{R}}\sum_{\pm}\frac{x-y}{(x-y)^2+(f(x,t)\pm f(y,t))^2}dy\,(1,f_x(x,t)),$$

whose first coordinate is

$$-\frac{1}{2}\,PV\int_{\mathbb{R}}\sum_{\pm}\frac{d}{dy}\log\left[(x-y)^2+(f(x,t)\pm f(y,t))^2\right]dy = 0.$$

Hence Γ_f for $f(\cdot,t)\geq 0$ moves with velocity u precisely when $f_t(x,t) = v_2(x,f(x,t),t)$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, which after the change of variables $y\leftrightarrow x-y$ becomes (1.5). Note also that this change of variables and (2.3) below show that the multiple of $(1,f_x(x,t))$ that we added above is in fact bounded when $\|f(\cdot,t)\|_{C^2_\gamma(\mathbb{R})}<\infty$ (see the definition below), which holds in the setting of Theorem 1.1. In the case of the whole plane, this argument similarly gives (1.13).

Continuous norms of f with power growth. For $k=0,1,\dots$ and $\gamma\in[0,1]$ let

$$\|g\|_{\dot C^\gamma(\mathbb{R})} := \sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|^\gamma} \qquad\text{and}\qquad \|g\|_{C^{k,\gamma}(\mathbb{R})} := \sum_{j=0}^{k}\|g^{(j)}\|_{L^\infty(\mathbb{R})} + \|g^{(k)}\|_{\dot C^\gamma(\mathbb{R})}$$

when $g\in L^2_{\rm loc}(\mathbb{R})$ (note that then $C^{k,0}(\mathbb{R}) = W^{k,\infty}(\mathbb{R})$), and for $k\geq 1$ let

$$\|g\|_{C^{k,\gamma}_\gamma(\mathbb{R})} := \|g'\|_{C^{k-1,\gamma}(\mathbb{R})} + \|g\|_{\ddot C^{1-\gamma}(\mathbb{R})} \qquad\text{and}\qquad \|g\|_{C^k_\gamma(\mathbb{R})} := \|g'\|_{C^{k-1,0}(\mathbb{R})} + \|g\|_{\ddot C^{1-\gamma}(\mathbb{R})}.$$

All but the second of these are seminorms that vanish on constant functions. Then

$$\|g\|_{C^{k,\gamma'}_{\gamma'}(\mathbb{R})} \leq \|g\|_{C^{k,\gamma}_\gamma(\mathbb{R})} + \|g'\|_{L^\infty(\mathbb{R})} + 2\|g^{(k)}\|_{L^\infty(\mathbb{R})} \leq 4\|g\|_{C^{k,\gamma}_\gamma(\mathbb{R})}$$

again holds when $k \geq 1$ and $0 \leq \gamma' \leq \gamma \leq 1$, and we have $\|g\|_{C_0^{k,0}(\mathbb{R})} < \infty$ iff $g' \in W^{k-1,\infty}(\mathbb{R})$. Finally, we note that when $\gamma \in (0, \frac{1}{2}]$ (which we assume here) and $k \geq 1$, then

$$\|g\|_{C_\gamma^{k-1,\gamma}(\mathbb{R})} \leq C_{k,\gamma} \|g\|_{\tilde{H}_\gamma^k(\mathbb{R})} \tag{2.2}$$

holds for all $g \in L^2_{\mathrm{loc}}(\mathbb{R})$, with some constant $C_{k,\gamma} < \infty$.

Basic bounds on f and (1.5). We start the proof of Theorem 1.1 by showing that the right-hand side of (1.5) remains bounded as long as $\|f(\cdot,t)\|_{\tilde{H}_\gamma^3(\mathbb{R})} < \infty$. Since most of our arguments apply at some fixed t, we will mostly drop t from the notation for the sake of simplicity. We then denote $g' := \partial_x g$, and all (semi)norms below will involve functions defined on $\mathbb{R}$ unless we specify otherwise.

We first claim that

$$\left| PV \int_{S \cup (-S)} \frac{y}{y^2 + (f(x) \pm f(x-y))^2} \, dy \right| \leq C \|f\|_{C_\gamma^2}. \tag{2.3}$$

for any $S \subseteq [0, \infty)$. The left-hand side is clearly bounded by

$$\int_S \frac{y|f(x+y) - f(x-y)| \, |f(x+y) + f(x-y) \pm 2f(x)|}{[y^2 + (f(x) \pm f(x+y))^2][y^2 + (f(x) \pm f(x-y))^2]} \, dy. \tag{2.4}$$

When $\pm$ is $-$, from $|f(x+y) - f(x-y)| \leq |f(x+y) - f(x)| + |f(x-y) - f(x)|$ we see that

$$y|f(x+y) - f(x-y)| \leq [y^2 + (f(x) - f(x+y))^2]^{1/2} [y^2 + (f(x) - f(x-y))^2]^{1/2},$$

so (2.4) is indeed no more than

$$2\|f''\|_{L^\infty} + 2\|f\|_{\ddot{C}^{1-\gamma}} \int_1^\infty y^{-1-\gamma} \, dy \leq C \|f\|_{C_\gamma^2}.$$

When $\pm$ is $+$, the same estimate holds for

$$\int_S \frac{y|f(x+y) - f(x-y)| \, |f(x+y) + f(x-y) - 2f(x)|}{[y^2 + (f(x) + f(x+y))^2][y^2 + (f(x) + f(x-y))^2]} \, dy,$$

while the remainder of (2.4) is bounded by

$$\|f'\|_{L^\infty} \int_S \frac{4|f(x)|}{y^2 + f(x)^2} \, dy \leq C \|f'\|_{L^\infty}$$

because $f > 0$. This also yields (2.3) in this case. We also have

$$\left| \int_{-1}^1 \frac{y \, (g'(x) - g'(x-y))}{y^2 + (f(x) \pm f(x-y))^2} \, dy \right| \leq C \|g'\|_{\dot{C}^\gamma},$$

and integration by parts in y yields

$$\begin{aligned} PV \int_{|y| \geq 1} \frac{y \, g'(x-y)}{y^2 + (f(x) \pm f(x-y))^2} \, dy &= \frac{g(x-1) - g(x)}{1 + (f(x) \pm f(x-1))^2} + \frac{g(x+1) - g(x)}{1 + (f(x) \pm f(x+1))^2} \\ &+ \int_{|y| \geq 1} (g(x-y) - g(x)) \frac{(f(x) \pm f(x-y))^2 \pm 2y f'(x-y)(f(x) \pm f(x-y)) - y^2}{[y^2 + (f(x) \pm f(x-y))^2]^2} \, dy. \end{aligned} \tag{2.5}$$

The last fraction is bounded by $(1+\|f'\|_{L^\infty})|y|^{-2}$, so

$$\left|PV\int_{|y|\ge 1}\frac{y\,g'(x-y)}{y^2+(f(x)\pm f(x-y))^2}\,dy\right|\le C(1+\|f'\|_{L^\infty})(\|g'\|_{L^\infty}+\|g\|_{\ddot{C}^{1-\gamma}}).$$

The above bounds thus imply

$$\left|PV\int_{\mathbb{R}}\frac{y\,g'(x-y)}{y^2+(f(x)\pm f(x-y))^2}\,dy\right|\le C(1+\|f'\|_{L^\infty})\|g\|_{C^{1,\gamma}_\gamma},$$

which together with (2.3) yields

$$\left|PV\int_{\mathbb{R}}\frac{y\,(g'(x)\pm g'(x-y))}{y^2+(f(x)\pm f(x-y))^2}\,dy\right|\le C(1+\|f'\|_{L^\infty})\|g\|_{C^{1,\gamma}_\gamma}+C\|f\|_{C^2_\gamma}\|g'\|_{L^\infty}. \tag{2.6}$$

In particular,

$$\left|PV\int_{\mathbb{R}}\frac{y\,(f'(x)\pm f'(x-y))}{y^2+(f(x)\pm f(x-y))^2}\,dy\right|\le C(1+\|f'\|_{L^\infty})\|f\|_{C^2_\gamma}. \tag{2.7}$$

Next we derive a pair of basic bounds on $f\ge 0$ in the region where it is small. Note that $|f'(z)|\ge\frac12|f'(x)|$ whenever $|z-x|\le\frac{|f'(x)|}{2\|f''\|_{L^\infty}}$, so $f\ge 0$ forces

$$\frac{|f'(x)|}{2\|f''\|_{L^\infty}}\frac{|f'(x)|}{2}\le|f(x)|.$$

Hence

$$|f'(x)|\le 2\|f''\|^{1/2}_{L^\infty}\sqrt{f(x)}\le(1+\|f''\|_{L^\infty})\sqrt{f(x)}, \tag{2.8}$$

and then for all $|y|\le\sqrt{f(x)}$ we have

$$|f'(x-y)|\le(1+2\|f'\|_{C^1})\min\left\{\sqrt{f(x)},1\right\}\qquad\text{and}\qquad f(x-y)\le 2(1+\|f'\|_{C^1})f(x).$$

When $|y|\ge\sqrt{f(x)}$, from (2.8) we obtain

$$|f'(x-y)|\le(1+2\|f'\|_{C^1})\min\{|y|,1\}\qquad\text{and}\qquad f(x-y)\le 2(1+\|f'\|_{C^1})y^2.$$

So we have

$$\begin{aligned}|f'(x-y)|&\le 2(1+\|f'\|_{C^1})\max\left\{y,\sqrt{f(x)}\right\},\\ f(x-y)&\le 2(1+\|f'\|_{C^1})\max\left\{y^2,f(x)\right\}\end{aligned} \tag{2.9}$$

for all $x,y\in\mathbb{R}$ when $f\ge 0$. We will use these two estimates extensively below.

A priori estimates for local norms of f_{xxx}. In this and the next four sections, we will assume that $f\ge 0$ and it is smooth on $\mathbb{R}\times[0,T]$ for some $T>0$. We will eventually apply the *a priori* arguments from these sections to solutions of a mollified version (7.1) of (1.5) in Section 7, and we will show in Section 10 that the latter exist and converge to a solution of (1.5) as the mollification parameter $\varepsilon\to 0$. We now fix some $h_0\in C^2(\mathbb{R})$ such that $\chi_{[-1,1]}\le h_0\le\chi_{[-4,4]}$ and $\max\{\|h_0'\|_{L^\infty},\|h_0''\|_{L^\infty}\}\le 1$. Then

$$\|f_{xxx}(\cdot,t)\|_{\tilde{L}^2(\mathbb{R})}\le\sup_{x_0\in\mathbb{R}}\left\|\sqrt{h_0(\cdot-x_0)}\,f_{xxx}(\cdot,t)\right\|_{L^2(\mathbb{R})} \tag{2.10}$$

clearly holds for all $t \in [0,T]$, so to estimate $\|f(\cdot,t)\|_{\tilde{H}^3_\gamma(\mathbb{R})}$, it will mainly be important to obtain appropriate uniform-in-x_0 bounds on the right-hand side of (2.10). We do this via the estimate (2.12) below. We note that an analogous estimate for the left-hand side of (2.10) seems to *fail*, which is why we have to introduce the more regular cut-off functions $h_0(\cdot - x_0)$. Let us also fix an arbitrary $x_0 \in \mathbb{R}$ and denote $h := h_0(\cdot - x_0)$ in the arguments below.

To estimate the right-hand side of (2.10), from (1.5) (and after dropping t) we obtain

$$\frac{d}{dt}\left\|\sqrt{h}\, f'''\right\|_{L^2}^2 = 2(I_1^+ + I_1^- + I_4^+ + I_4^-) + 6(I_2^+ + I_2^- + I_3^+ + I_3^-) \tag{2.11}$$

for smooth f, where with $A^\pm(x,y) := [y^2 + (f(x) \pm f(x-y))^2]^{-1}$ we have

$$\begin{aligned}
I_1^\pm &:= \int_{\mathbb{R}} h(x) f'''(x)\, PV \int_{\mathbb{R}} \frac{y\,(f''''(x) \pm f''''(x-y))}{y^2 + (f(x) \pm f(x-y))^2}\, dydx,\\
I_2^\pm &:= \int_{\mathbb{R}} h(x) f'''(x)\, PV \int_{\mathbb{R}} y\,(f'''(x) \pm f'''(x-y))\, \partial_x A^\pm(x,y)\, dydx,\\
I_3^\pm &:= \int_{\mathbb{R}} h(x) f'''(x)\, PV \int_{\mathbb{R}} y\,(f''(x) \pm f''(x-y))\, \partial_x^2 A^\pm(x,y)\, dydx,\\
I_4^\pm &:= \int_{\mathbb{R}} h(x) f'''(x)\, PV \int_{\mathbb{R}} y\,(f'(x) \pm f'(x-y))\, \partial_x^3 A^\pm(x,y)\, dydx.
\end{aligned}$$

Note that the PV integrals above all converge as $|y| \to \infty$ (and all but $I_1^\pm$ do so absolutely). As for convergence at $y = 0$, we see that the integrands are all bounded when $\pm$ is $-$ and f is smooth enough, and this is clearly also true for the ones with $+$ when also $f(x) > 0$. When $f(x) = 0$, then $f'(x) = 0$ because $f \geq 0$, so $f'(x-y) = O(|y|)$ and $f(x-y) = O(y^2)$. This shows that the inside integrand for I_2^+ is still bounded, while the ones for I_1^+, I_3^+, and I_4^+ are $\frac{C}{y} + O(1)$. Moreover, this C is non-zero only when $f''(x) \neq 0$ $(= f(x))$. Therefore "PV" can be removed from $I_2^\pm$, I_3^-, and I_4^-, and it is only needed in I_3^+, and I_4^+ at countably many points, which makes it irrelevant (in fact, we will eventually apply the arguments below to $f > 0$, so then "PV" will only be needed in $I_1^\pm$ as $|y| \to \infty$ anyway). As a result, there will be few PV integrals in our analysis of the terms $I_2^\pm$, $I_3^\pm$, and $I_4^\pm$. However, analysis of the integrands in I_3^+ and I_4^+ when $0 < f(x) \ll 1$ (e.g., when x is near a point where f vanishes) will turn out to be the most challenging task in our derivation of a priori estimates for (1.5).

We bound the above eight integrals in the next four sections, and obtain estimates (3.15), (4.2), (4.4), (5.11), and (6.6) that collectively yield the a priori estimate

$$\frac{d}{dt}\left\|\sqrt{h}\, f'''\right\|_{L^2}^2 \leq C(1 + \|f\|^4_{C^{2,\gamma}_\gamma})\,(\|f'\|^2_{\tilde{L}^2} + \|f''\|^2_{\tilde{L}^2} + \|f'''\|^2_{\tilde{L}^2}) \tag{2.12}$$

(which holds uniformly in x_0). Note that we do not prove (2.12) with the left-hand side multiplied by -1, due to (3.15) below. In fact, that inequality would hold in the unstable heavier-fluid-above scenario, but (2.12) then fails as stated and causes ill-posedness for (1.5) even in $H^3(\mathbb{R})$ in that setting [8]. With (2.12) in hand, we finish the proof of Theorem 1.1 in Sections 7–10 as described at the end of the introduction.

3. Estimates on $I_1^\pm$

Let us first estimate $I_1^\pm$, denoting $g := f'''$. For later reference, we will do this in terms of

$$\|g\|_{\tilde{L}^2_{x_0}(\mathbb{R})} := \left[\int_{\mathbb{R}} g(x)^2 \min\{1, |x-x_0|^{-2}\} dx\right]^{1/2} \qquad (\le 2\|g\|_{\tilde{L}^2(\mathbb{R})}) \tag{3.1}$$

rather than $\|g\|_{\tilde{L}^2}$ (we use the latter in the following sections). We have $I_1^\pm = J_1^\pm \pm J_2^\pm$, where

$$J_1^\pm := \int_{\mathbb{R}} h(x)g(x)\, PV \int_{\mathbb{R}} \frac{y\, g'(x)}{y^2 + (f(x) \pm f(x-y))^2}\, dydx,$$
$$J_2^\pm := \int_{\mathbb{R}} h(x)g(x)\, PV \int_{\mathbb{R}} \frac{(x-y)\, g'(y)}{(x-y)^2 + (f(x) \pm f(y))^2}\, dydx.$$

Integration by parts in x yields

$$\begin{aligned} J_1^\pm = &\int_{\mathbb{R}} h(x)g(x)^2\, PV \int_{\mathbb{R}} \frac{y\,(f(x) \pm f(x-y))(f'(x) \pm f'(x-y))}{[y^2 + (f(x) \pm f(x-y))^2]^2}\, dydx \\ &- \frac{1}{2}\int_{\mathbb{R}} h'(x)g(x)^2\, PV \int_{\mathbb{R}} \frac{y}{y^2 + (f(x) \pm f(x-y))^2}\, dydx =: K_1^\pm + K_2^\pm. \end{aligned} \tag{3.2}$$

From (2.3) we see that

$$|K_2^\pm| \le C\|f\|_{C^2_\gamma} \left\| \sqrt{|h'|}\, g \right\|^2_{L^2} \le C\|f\|_{C^2_\gamma} \|g\|^2_{\tilde{L}^2_{x_0}}. \tag{3.3}$$

As for $K_1^\pm$, we have

$$|K_1^\pm| \le C\left(\|f'\|_{L^\infty} + \|M^\pm\|_{L^\infty}\right) \|g\|^2_{\tilde{L}^2_{x_0}}, \tag{3.4}$$

where

$$M^\pm(x) := PV \int_{-1}^{1} \frac{y\,(f(x) \pm f(x-y))\,(f'(x) \pm f'(x-y))}{[y^2 + (f(x) \pm f(x-y))^2]^2}\, dy.$$

Here we estimated both $|y|$ and $|f(x) \pm f(x-y)|$ by $[y^2 + (f(x) \pm f(x-y))^2]^{1/2}$, and we will do so multiple times below (however, at other times we will estimate $|f(x) \pm f(x-y)|$ via (2.9), including via its first line when $\pm$ is $-$). To estimate M^-, we note that from

$$|f'(x+y) + f'(x-y) - 2f'(x)| \le 2\|f''\|_{\dot{C}^\gamma} |y|^{1+\gamma}$$

and

$$\frac{ab}{c} - \frac{a'b'}{c'} = \frac{(a-a')b}{c} + \frac{a'(b-b')}{c} + \frac{a'b'(c'-c)}{cc'} \tag{3.5}$$

we obtain

$$\begin{aligned} &\left| \frac{(f(x) - f(x-y))\,(f'(x) - f'(x-y))}{[y^2 + (f(x) - f(x-y))^2]^2} - \frac{(f(x+y) - f(x))\,(f'(x+y) - f'(x))}{[y^2 + (f(x) - f(x+y))^2]^2} \right| \\ &\le \frac{C\|f''\|^2_{L^\infty}|y|^3}{y^4} + \frac{C\|f'\|^2_{C^{1,\gamma}}|y|^{2+\gamma}}{y^4} + \frac{C\|f''\|^2_{L^\infty}|y|^3}{y^4} \le \frac{C\|f'\|^2_{C^{1,\gamma}}|y|^{2+\gamma}}{y^4}, \end{aligned}$$

for $|y| \le 1$ (we again used $|f(x+y) + f(x-y) - 2f(x)| \le \|f''\|_{L^\infty} y^2$). Hence

$$\|M^-\|_{L^\infty} \le \int_0^1 \frac{C\|f'\|^2_{C^{1,\gamma}} y^{3+\gamma}}{y^4}\, dy = C\|f'\|^2_{C^{1,\gamma}}. \tag{3.6}$$

We can bound M^+ via (3.5) and (2.9), which in particular imply

$$|f(x+y)-f(x-y)| \le 4(1+\|f'\|_{C^1})\sqrt{f(x)}\,|y|$$

for $|y| \le \sqrt{f(x)}$, and yield for these y

$$\left|\frac{(f(x)+f(x-y))\,(f'(x)+f'(x-y))}{[y^2+(f(x)+f(x-y))^2]^2} - \frac{(f(x)+f(x+y))\,(f'(x)+f'(x+y))}{[y^2+(f(x)+f(x+y))^2]^2}\right| \le \frac{C(1+\|f'\|_{C^1}^2)f(x)|y|}{\max\{|y|,f(x)\}^4} \le \frac{C(1+\|f'\|_{C^1}^2)f(x)}{\max\{|y|,f(x)\}^3}.$$

For $|y| \ge \sqrt{f(x)}$ we instead obtain

$$\left|\frac{(f(x)+f(x-y))\,(f'(x)+f'(x-y))}{[y^2+(f(x)+f(x-y))^2]^2}\right| \le \frac{C(1+\|f'\|_{C^1}^2)|y|^3}{|y|^4},$$

so it follows that

$$|M^+(x)| \le C(1+\|f'\|_{C^1}^2)\int_0^1 \left(\frac{yf(x)}{\max\{y,f(x)\}^3}+1\right)dy \le C(1+\|f'\|_{C^1}^2). \tag{3.7}$$

This, (3.3), (3.4), and (3.6) thus prove

$$|J_1^\pm| \le C(1+\|f\|_{C^{2,\gamma}_\gamma}^2)\|g\|_{\tilde{L}^2_{x_0}}^2. \tag{3.8}$$

Next we estimate $J_2^\pm$. Integration by parts in y yields

$$J_2^\pm = \int_{\mathbb{R}} h(x)g(x)\,PV\int_{\mathbb{R}} \frac{(x-y)\,\partial_y(g(y)-g(x))}{(x-y)^2+(f(x)\pm f(y))^2}\,dydx = K_3^\pm + K_4^\pm, \tag{3.9}$$

where

$$K_3^\pm := \int_{\mathbb{R}} h(x)g(x)\,PV\int_{\mathbb{R}} \frac{g(x)-g(y)}{(x-y)^2+(f(x)\pm f(y))^2}\,dydx,$$

$$K_4^\pm := -2\int_{\mathbb{R}} h(x)g(x)\int_{\mathbb{R}} \frac{(g(x)-g(y))(f(x)\pm f(y))\big[f(x)\pm f(y)\pm f'(y)(x-y)\big]}{[(x-y)^2+(f(x)\pm f(y))^2]^2}\,dydx$$

(note that the term obtained when ∂_y is applied to $x-y$ in the numerator is $-K_3^\pm$, so $K_4^\pm$ is in fact the term obtained when ∂_y is applied to the denominator minus $2K_3^\pm$). Symmetrization now shows that $K_3^\pm := \frac{1}{2}(L_1^\pm + L_2^\pm)$, where

$$L_1^\pm := \int_{\mathbb{R}^2} \frac{h(y)(g(x)-g(y))^2}{(x-y)^2+(f(x)\pm f(y))^2}\,dydx,$$

$$\begin{aligned} L_2^\pm &:= \int_{\mathbb{R}^2} \frac{(h(x)-h(y))g(x)(g(x)-g(y))}{(x-y)^2+(f(x)\pm f(y))^2}\,dydx \\ &= \int_{\mathbb{R}} g(x)^2\,PV\int_{\mathbb{R}} \frac{h(x)-h(y)}{(x-y)^2+(f(x)\pm f(y))^2}\,dydx \\ &= \int_{\mathbb{R}} g(x)^2\,PV\int_{\mathbb{R}} \frac{h(x)-h(x-y)}{y^2+(f(x)\pm f(x-y))^2}\,dydx \end{aligned}$$

and the third equality follows also by symmetrization. Note that symmetrization here uses

$$\int_{\mathbb{R}} PV \int_{\mathbb{R}} F(x,y)\,dxdy = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} F(x,y)\,dxdy = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \frac{F(x,y)+F(y,x)}{2}\,dxdy.$$

We then obviously have

$$L_1^+ - L_1^- \leq 0 \leq L_1^\pm. \tag{3.10}$$

On the other hand, for $|x-x_0|\leq 5$ we can estimate the inside integral in $L_2^\pm$ by $C(1+\|f\|_{C^2_\gamma})$ using $|h(x)-h(x-y)-h'(x)y|\leq y^2$ for $|y|\leq 1$ and then (2.3) with $S=[-1,1]$, while for $|x-x_0|\geq 5$ this integral is bounded by $\frac{8}{(|x-x_0|-4)^2}$ because $\operatorname{supp} h\subseteq [x_0-4,x_0+4]$. It follows that

$$|L_2^\pm| \leq C(1+\|f\|_{C^2_\gamma})\|g\|^2_{\tilde{L}^2_{x_0}},$$

and therefore

$$\begin{aligned} K_3^\pm &\geq -C(1+\|f\|_{C^2_\gamma})\|g\|^2_{\tilde{L}^2_{x_0}},\\ K_3^+ - K_3^- &\leq C(1+\|f\|_{C^2_\gamma})\|g\|^2_{\tilde{L}^2_{x_0}}. \end{aligned} \tag{3.11}$$

We now write $K_4^- = -L_3^- + L_4^- + L_5^-$, where

$$\begin{aligned} L_3^- &:= 2\int_{\mathbb{R}} h(x)g(x)^2\,PV\int_{\mathbb{R}} \frac{(f(x)-f(y))\big[f(x)-f(y)-f'(y)(x-y)\big]}{[(x-y)^2+(f(x)-f(y))^2]^2}\,dydx,\\ L_4^- &:= \frac{1}{2}\int_{\mathbb{R}} h(x)g(x)\,PV\int_{\mathbb{R}} \frac{g(y)(f(x)-f(y))(f''(x)+f''(y))\min\{(x-y)^2,1\}}{[(x-y)^2+(f(x)-f(y))^2]^2}\,dydx,\\ L_5^- &:= 2\int_{\mathbb{R}} h(x)g(x)\,PV\int_{\mathbb{R}} \frac{g(y)(f(x)-f(y))D(x,y)}{[(x-y)^2+(f(x)-f(y))^2]^2}\,dydx, \end{aligned}$$

where

$$D(x,y) := f(x)-f(y)-f'(y)(x-y)-\frac{f''(x)+f''(y)}{4}\min\{(x-y)^2,1\}.$$

After changing variables $y\leftrightarrow x-y$, we can treat L_3^- similarly to K_1^-, writing

$$(f(x)-f(x-y))[f(x)-f(x-y)-f'(x-y)y] = -(f(x+|y|)-f(x))[f(x)-f(x+|y|)+f'(x+|y|)|y|]$$

for $y\in(-1,0)$ and then also estimating

$$\big|[f(x)-f(x-y)-f'(x-y)y]-[f(x)-f(x+y)+f'(x+y)y]\big| = \frac{1}{2}|f''(z_1)-f''(z_2)|y^2 \leq \|f''\|_{\dot{C}^\gamma}y^{2+\gamma}$$

for $y\in(0,1)$, with some $z_1\in[x-y,x]$ and $z_2\in[x,x+y]$. We thus obtain

$$|L_3^-| \leq C(1+\|f\|^2_{C^{2,\gamma}_\gamma})\|g\|^2_{\tilde{L}^2_{x_0}}.$$

On the other hand, symmetrization and

$$|g(x)g(y)| \leq \frac{g(x)^2+g(y)^2}{2} \tag{3.12}$$

show that

$$|L_4^-| = \left|\frac{1}{4}\int_{\mathbb{R}} g(x)\int_{\mathbb{R}} \frac{g(y)(h(x)-h(y))(f(x)-f(y))(f''(x)+f''(y))\min\{(x-y)^2,1\}}{[(x-y)^2+(f(x)-f(y))^2]^2}\,dydx\right|,$$

$$\leq \frac{1}{4}\int_{\mathbb{R}} g(x)^2 \int_{\mathbb{R}} \frac{|h(x)-h(y)|\,|f(x)-f(y)|\,|f''(x)+f''(y)|\min\{(x-y)^2,1\}}{[(x-y)^2+(f(x)-f(y))^2]^2}\,dydx.$$

The last integral in y is bounded by $C\|f''\|_{L^\infty}$ when $|x-x_0|\leq 5$ and by $\frac{16\|f''\|_{L^\infty}}{(|x-x_0|-4)^3}$ when $|x-x_0|\geq 5$, so

$$|L_4^-|\leq C(1+\|f''\|_{L^\infty})\|g\|^2_{\tilde{L}^2_{x_0}}.$$

And since we clearly have

$$|4D(x,y)| = |2f''(z)-f''(x)-f''(y)|(x-y)^2 \leq 4\|f''\|_{C^\gamma}|x-y|^{2+\gamma}$$

with some z between x and y when $|x-y|\leq 1$, from (3.12) we again see that

$$|L_5^-|\leq C(1+\|f'\|_{C^{1,\gamma}})\|g\|^2_{\tilde{L}^2_{x_0}}.$$

Therefore we proved

$$|K_4^-|\leq C(1+\|f\|^2_{C^{2,\gamma}_\gamma})\|g\|^2_{\tilde{L}^2_{x_0}}. \tag{3.13}$$

Next write $K_4^+ = -L_3^+ - L_4^+ + L_5^+$, where

$$L_3^+ := 2\int_{\mathbb{R}} h(x)g(x)\int_{\mathbb{R}} \frac{(g(x)-g(y))(f(x)+f(y))^2}{[(x-y)^2+(f(x)+f(y))^2]^2}\,dydx,$$
$$L_4^+ := 2\int_{\mathbb{R}} h(x)g(x)^2\int_{\mathbb{R}} \frac{(f(x)+f(x-y))f'(x-y)y}{[y^2+(f(x)+f(x-y))^2]^2}\,dydx,$$
$$L_5^+ := 2\int_{\mathbb{R}} h(x)g(x)\int_{\mathbb{R}} \frac{g(y)(f(x)+f(y))f'(y)(x-y)}{[(x-y)^2+(f(x)+f(y))^2]^2}\,dydx,$$

and in L_4^+ we changed variables $y\leftrightarrow x-y$. We can now estimate L_3^+ in the same way as K_3^+, although this time a simpler argument can be used to estimate the term analogous to L_2^+. That is because after changing variables $y\leftrightarrow x-y$, this term becomes

$$\int_{\mathbb{R}} g(x)^2\int_{\mathbb{R}} \frac{(h(x)-h(x-y))(f(x)+f(x-y))^2}{[y^2+(f(x)+f(x-y))^2]^2}\,dydx, \tag{3.14}$$

and (2.9) shows that the absolute value of the inside integral is bounded by

$$C(1+\|f'\|^2_{C^1})\left(\int_{-1}^{1}\frac{\max\{y^2,f(x)\}^2|y|}{\max\{|y|,f(x)\}^4}\,dy+\int_{|y|\geq 1}\frac{dy}{y^2}\right)\leq C(1+\|f'\|^2_{C^1})$$

when $|x-x_0|\leq 5$. Hence similarly to (3.11) we obtain

$$L_3^+\geq -C(1+\|f'\|^2_{C^1})\|g\|^2_{\tilde{L}^2_{x_0}}$$

The argument bounding K_1^+ shows also that

$$|L_4^+|\leq C(1+\|f'\|^2_{C^1})\|g\|^2_{\tilde{L}^2_{x_0}}.$$

Finally, symmetrization again shows that

$$L_5^+ = \int_{\mathbb{R}} h(x)g(x)\int_{\mathbb{R}} \frac{g(y)(f(x)+f(y))(f'(y)-f'(x))(x-y)}{[(x-y)^2+(f(x)+f(y))^2]^2}\,dydx$$
$$+\int_{\mathbb{R}} g(x)\int_{\mathbb{R}} \frac{g(y)(h(x)-h(y))(f(x)+f(y))f'(x)(x-y)}{[(x-y)^2+(f(x)+f(y))^2]^2}\,dydx,$$

so (3.12) and a change of variables yield

$$\begin{aligned}|L_5^+| \le &\int_{x_0-5}^{x_0+5} g(x)^2 \int_{\mathbb{R}} \frac{|f(x)+f(x-y)|\,|f'(x)-f'(x-y)|\,|y|}{[y^2+(f(x)+f(x-y))^2]^2}\,dydx \\ &+ \int_{|x-x_0|\ge 5} g(x)^2 \int_{x-x_0-4}^{x-x_0+4} \frac{|f(x)+f(x-y)|\,|f'(x)-f'(x-y)|\,|y|}{[y^2+(f(x)+f(x-y))^2]^2}\,dydx \\ &+ \int_{\mathbb{R}} g(x)^2 \int_{\mathbb{R}} \frac{|h(x)-h(x-y)|(f(x)+f(x-y))(|f'(x)|+|f'(x-y)|)|y|}{[y^2+(f(x)+f(x-y))^2]^2}\,dydx.\end{aligned}$$

The integral in y in the second term is bounded by $\frac{16\|f'\|_{L^\infty}}{(|x-x_0|-4)^2}$. In the first term, the integral in y over $\mathbb{R}\setminus[-1,1]$ is bounded by $C\|f'\|_{L^\infty}$, and the same bound works for the integral over $[-1,1]$ when $f(x)\ge 1$. When $f(x)\le 1$, then (2.9) shows that the latter integral is bounded by

$$C(1+\|f'\|_{C^1}^2)\int_{-1}^{1} \frac{\max\{y^2, f(x)\}\,y^2}{\max\{|y|, f(x)\}^4}\,dy \le C(1+\|f'\|_{C^1}^2).$$

The third term in the bound on $|L_5^+|$ can be estimated in the same way, so the above bounds yield

$$K_4^+ \le C(1+\|f'\|_{C^1}^2)\|g\|_{\tilde{L}^2_{x_0}}^2.$$

This combines with (3.11) and (3.13) to yield

$$J_2^+ - J_2^- \le C(1+\|f\|_{C^{2,\gamma}_\gamma}^2)\|g\|_{\tilde{L}^2_{x_0}}^2,$$

and adding (3.8) to this finally concludes that

$$I_1^+ + I_1^- \le C(1+\|f\|_{C^{2,\gamma}_\gamma}^2)\|f'''\|_{\tilde{L}^2_{x_0}}^2. \tag{3.15}$$

4. Estimates on $I_2^\pm$

We note that (3.2) and a change of variables show that

$$I_2^\pm = \pm 2K_1^\pm \mp 2\int_{\mathbb{R}} h(x)f'''(x)\,PV\int_{\mathbb{R}} \frac{f'''(y)(x-y)(f(x)\pm f(y))(f'(x)\pm f'(y))}{[(x-y)^2+(f(x)\pm f(y))^2]^2}\,dydx.$$

Symmetrization shows that the second term equals

$$\int_{\mathbb{R}} f'''(x)\int_{\mathbb{R}} \frac{(h(x)-h(y))f'''(y)(x-y)(f(x)\pm f(y))(f'(x)\pm f'(y))}{[(x-y)^2+(f(x)\pm f(y))^2]^2}\,dydx.$$

The absolute value of this integral is bounded by

$$\int_{\mathbb{R}} f'''(x)^2\int_{\mathbb{R}} \frac{|h(x)-h(y)|\,|x-y|\,|f(x)\pm f(y)|\,|f'(x)\pm f'(y)|}{[(x-y)^2+(f(x)\pm f(y))^2]^2}\,dydx \tag{4.1}$$

When we replace $\pm$ by $-$, the inside integrand is bounded by $\|f'\|_{C^1}\min\{1, \frac{1}{(|x-x_0|-4)^2}\}$, so this, (3.4), and (3.6) yield

$$|I_2^-| \le C\|f'\|_{C^{1,\gamma}}^2\|f'''\|_{\tilde{L}^2}^2. \tag{4.2}$$

When we replace $\pm$ by $+$, then (4.1) becomes (after changing variables)

$$\int_{\mathbb{R}} f'''(x)^2\int_{\mathbb{R}} \frac{|h(x)-h(x-y)|\,|y|\,|f(x)+f(x-y)|\,|f'(x)+f'(x-y)|}{[y^2+(f(x)+f(x-y))^2]^2}\,dydx \tag{4.3}$$

The inside integral is bounded in the same way as (3.14), so from that bound, (3.4), and (3.7) we obtain

$$|I_2^+| \le C(1+\|f'\|_{C^1}^2)\|f'''\|_{\tilde{L}^2}^2. \tag{4.4}$$

5. Estimates on $I_3^\pm$

We can write $I_3^- = 2(J_3^- - J_4^-)$, where

$$J_3^- := \int_{\mathbb{R}} h(x)f'''(x)\int_{\mathbb{R}} \frac{y\,(f''(x)-f''(x-y))(f'(x)-f'(x-y))^2[3(f(x)-f(x-y))^2-y^2]}{[y^2+(f(x)-f(x-y))^2]^3}\,dydx,$$

$$J_4^- := \int_{\mathbb{R}} h(x)f'''(x)\int_{\mathbb{R}} \frac{y\,(f''(x)-f''(x-y))^2(f(x)-f(x-y))}{[y^2+(f(x)-f(x-y))^2]^2}\,dydx.$$

Since $f''(x)-f''(x-y) = y\int_0^1 f'''(x-sy)ds$ we obtain

$$\begin{aligned} |J_3^-| &\le C\|f'\|_{C^1}^2 \int_{\mathbb{R}^2\times[0,1]} h(x)|f'''(x)|\,|f'''(x-sy)|\min\{1,|y|^{-2}\}\,dxdyds \\ &\le C\|f'\|_{C^1}^2\|f'''\|_{\tilde{L}^2}^2 \int_{\mathbb{R}\times[0,1]} \min\{1,|y|^{-2}\}\,dyds \le C\|f'\|_{C^1}^2\|f'''\|_{\tilde{L}^2}^2. \end{aligned} \tag{5.1}$$

We use the same method to estimate J_4^-, bounding the second factor $f''(x)-f''(x-y)$ via $|f''(x)-f''(x-y)| \le \|f''\|_{\dot{C}^\gamma}|y|^\gamma$ for $|y|\le 1$. This yields

$$|J_4^-| \le C\|f\|_{C^{2,\gamma}_\gamma}^2\|f'''\|_{\tilde{L}^2}^2 \int_{\mathbb{R}\times[0,1]} \min\{|y|^{\gamma-1},|y|^{-1-\gamma}\}\,dyds \le C\|f\|_{C^{2,\gamma}_\gamma}^2\|f'''\|_{\tilde{L}^2}^2,$$

and it follows that

$$|I_3^-| \le C\|f\|_{C^{2,\gamma}_\gamma}^2\|f'''\|_{\tilde{L}^2}^2. \tag{5.2}$$

We now turn to the term I_3^+, whose treatment will be much more complicated. First we split $I_3^+ = I_3' + I_3'' + I_3'''$, where with $R_x := \mathbb{R}\setminus[-\sqrt{f(x)},\sqrt{f(x)}\,]$ we have

$$I_3' := \int_{\mathbb{R}} \chi_{(1,\infty)}(f(x))\,h(x)f'''(x)\int_{\mathbb{R}} y\,(f''(x)+f''(x-y))\,\partial_x^2 A^+(x,y)\,dydx,$$

$$I_3'' := \int_{\mathbb{R}} \chi_{[0,1]}(f(x))\,h(x)f'''(x)\,PV\int_{R_x} y\,(f''(x)+f''(x-y))\,\partial_x^2 A^+(x,y)\,dydx,$$

$$I_3''' := \int_{\mathbb{R}} \chi_{[0,1]}(f(x))\,h(x)f'''(x)\int_{-\sqrt{f(x)}}^{\sqrt{f(x)}} y\,(f''(x)+f''(x-y))\,\partial_x^2 A^+(x,y)\,dydx.$$

The inside integrand in all three integrals is twice the difference of inside integrands from J_3^- and J_4^- but with all $f^{(j)}(x)-f^{(j)}(x-y)$ replaced by $f^{(j)}(x)+f^{(j)}(x-y)$. It is clear that

$$\begin{aligned} |I_3'| &\le C\|f'\|_{C^1}^2 \int_{\mathbb{R}^2} h(x)|f'''(x)|\,(|f''(x)|+|f''(x-y)|)\min\{1,|y|^{-2}\}\,dxdyds \\ &\le C\|f'\|_{C^1}^2\|f''\|_{\tilde{L}^2}\|f'''\|_{\tilde{L}^2} \int_{\mathbb{R}} \min\{1,|y|^{-2}\}\,dyds \le C\|f'\|_{C^1}^2(\|f''\|_{\tilde{L}^2}^2+\|f'''\|_{\tilde{L}^2}^2). \end{aligned} \tag{5.3}$$

We leave the most challenging term I_3''' for later and deal with I_3'' next. We decompose it similarly to I_3^-, which gives us terms like J_3^- and J_4^- but with each $f^{(j)}(x)-f^{(j)}(x-y)$ replaced

by $f^{(j)}(x)+f^{(j)}(x-y)$. Then we decompose these terms further in two ways. First, we replace one factor $f''(x)+f''(x-y)$ in each integral by the sum of $2f''(x)$ and $f''(x-y)-f''(x)$, and then in the first "subterm" with $2f''(x)$ we also split $3(f(x)+f(x-y))^2-y^2$ into the sum of $3[y^2+(f(x)+f(x-y))^2]$ and $-4y^2$ in order to simplify the formulas. We thus obtain

$$I_3'' = 2(J_3^+ - J_4^+ + 6J_5^+ - 8J_6^+ - 2J_7^+),$$

where with $\tilde{h}(x) := \chi_{[0,1]}(f(x))\, h(x)$ we have

$$J_3^+ := \int_{\mathbb{R}} \tilde{h}(x) f'''(x) \int_{R_x} \frac{y\,(f''(x-y)-f''(x))(f'(x)+f'(x-y))^2[3(f(x)+f(x-y))^2-y^2]}{[y^2+(f(x)+f(x-y))^2]^3}\, dydx,$$

$$J_4^+ := \int_{\mathbb{R}} \tilde{h}(x) f'''(x) \int_{R_x} \frac{y\,(f''(x-y)-f''(x))(f''(x)+f''(x-y))(f(x)+f(x-y))}{[y^2+(f(x)+f(x-y))^2]^2}\, dydx,$$

$$J_5^+ := \int_{\mathbb{R}} \tilde{h}(x) f'''(x) f''(x)\, PV \int_{R_x} \frac{y\,(f'(x)+f'(x-y))^2}{[y^2+(f(x)+f(x-y))^2]^2}\, dydx,$$

$$J_6^+ := \int_{\mathbb{R}} \tilde{h}(x) f'''(x) f''(x)\, PV \int_{R_x} \frac{y^3\,(f'(x)+f'(x-y))^2}{[y^2+(f(x)+f(x-y))^2]^3}\, dydx,$$

$$J_7^+ := \int_{\mathbb{R}} \tilde{h}(x) f'''(x) f''(x)\, PV \int_{R_x} \frac{y\,(f''(x)+f''(x-y))(f(x)+f(x-y))}{[y^2+(f(x)+f(x-y))^2]^2}\, dydx.$$

We will estimate these integrals one by one.

First, the argument for J_3^- gives us also

$$|J_3^+| \le C(1+\|f'\|_{C^1}^2)\|f'''\|_{\tilde{L}^2}^2$$

because $|f'(x)+f'(x-y)| \le 4(1+\|f'\|_{C^1})|y|$ for $|y| \in [\sqrt{f(x)}, 1]$ by (2.9). From (2.9) we also have $|f(x-y)| \le 2(1+\|f'\|_{C^1})y^2$ for $|y| \in [\sqrt{f(x)}, 1]$, which can be used in the argument for J_4^- even though now we only have $|f''(x)+f''(x-y)| \le 2\|f''\|_{L^\infty}$. We thus obtain

$$|J_4^+| \le C(1+\|f'\|_{C^1}^2)\|f'''\|_{\tilde{L}^2}^2.$$

To estimate the last three integrals, we will use that

$$\int_{R_x} y\phi(y)dy = \int_{\sqrt{f(x)}}^{\infty} y\,[\phi(y)-\phi(-y)]\, dy \tag{5.4}$$

and assume below without loss that $x=0$ (and of course then also $f(0)\le 1$). For J_5^+, note that from (3.5) we obtain

$$\begin{aligned} y^5 \left| \frac{y(f'(0)+f'(-y))^2}{[y^2+(f(0)+f(-y))^2]^2} - \frac{y(f'(0)+f'(y))^2}{[y^2+(f(0)+f(y))^2]^2} \right| \\ \le y^2|2f'(0)+f'(y)+f'(-y)|\,|f'(y)-f'(-y)| \\ + 2y(f'(0)+f'(y))^2|f(y)-f(-y)|. \end{aligned}$$

The first term on the right-hand side can be estimated by

$$C(1+\|f'\|_{C^{1,\gamma}}^2)\, y^2 \min\left\{1, (\sqrt{f(0)}+y^{1+\gamma})y\right\}$$

for $y \geq \sqrt{f(0)}$ due to (2.9) and

$$\begin{aligned} |f'(y) - f'(-y)| &\leq 2\|f'\|_{C^1} \min\{1, y\}, \\ |2f'(0) + f'(y) + f'(-y)| &\leq 4\|f'\|_{L^\infty} && \text{for } y > 1, \\ |f'(y) + f'(-y) - 2f'(0)| &\leq 2\|f''\|_{\dot{C}^\gamma} |y|^{1+\gamma} && \text{for } y \in [0, 1]. \end{aligned} \tag{5.5}$$

The second term can be estimated by

$$C(1 + \|f'\|_{C^1}^3) y^2 \min\{1, y^3\}$$

for $y \geq \sqrt{f(0)}$ because (2.9) shows

$$|f'(z)| \leq 2(1 + \|f'\|_{C^1}) \min\{1, y\}$$

for $y \geq \sqrt{f(0)}$ and all $|z| \leq y$. Hence

$$\begin{aligned} |J_5^+| &\leq \int_{\mathbb{R}} h(x) |f'''(x)| \, |f''(x)| \int_{\sqrt{f(x)}}^\infty C(1 + \|f'\|_{C^{1,\gamma}}^3) \frac{\min\{1, (\sqrt{f(x)} + y^{1+\gamma}) y\}}{y^3} dy dx \\ &\leq C(1 + \|f'\|_{C^{1,\gamma}}^3) \|f''\|_{\tilde{L}^2} \|f'''\|_{\tilde{L}^2} \end{aligned}$$

because the inside integral is bounded by $C(1 + \|f'\|_{C^{1,\gamma}}^3)$. The same bound can be obtained for J_6^+ in exactly the same way.

Finally, J_7^+ can be estimated similarly via (5.4), using (3.5) to get

$$\begin{aligned} & y^5 \left| \frac{y\,(f''(0) + f''(-y))(f(0) + f(-y))}{[y^2 + (f(0) + f(-y))^2]^2} - \frac{y\,(f''(0) + f''(y))(f(0) + f(y))}{[y^2 + (f(0) + f(y))^2]^2} \right| \\ & \quad \leq y^2 |f''(y) - f''(-y)| \, (f(0) + f(-y)) + y^2 |f''(0) + f''(y)| \, |f(y) - f(-y)| \\ & \qquad + 2y |f''(0) + f''(y)| \, (f(0) + f(y)) |f(y) - f(-y)|. \end{aligned} \tag{5.6}$$

Then the above bounds together with (recall (2.9) and that $f(0) \leq 1$)

$$\begin{aligned} |f''(y) - f''(-y)| &\leq 2\|f''\|_{\dot{C}^\gamma} \min\{1, y^\gamma\}, \\ |f(\pm y)| &\leq 2(1 + \|f'\|_{C^1}) \min\{y, y^2\}, \\ |f(y) - f(-y)| &= \left| 2f'(0) y - \int_{-y}^y (z - y \operatorname{sgn} z) f''(z) dz \right| \leq 2|f'(0)| y + \|f''\|_{\dot{C}^\gamma} y^{2+\gamma} \\ &\leq 4(1 + \|f'\|_{C^{1,\gamma}})(\sqrt{f(0)} + y^{1+\gamma}) y \end{aligned}$$

for $y \geq \sqrt{f(0)}$, bound the right-hand side of (5.6) by

$$\begin{aligned} & C(1 + \|f'\|_{C^{1,\gamma}}^2) \left[y^3 \min\{1, y^{1+\gamma}\} + y^2 \min\left\{ y, (\sqrt{f(0)} + y^{1+\gamma}) y \right\} \right] \\ & \qquad + C(1 + \|f'\|_{C^{1,\gamma}}^3) \min\left\{ y^3, (\sqrt{f(0)} + y^{1+\gamma}) y^4 \right\} \end{aligned}$$

and we obtain

$$\begin{aligned} |J_7^+| &\leq \int_{\mathbb{R}} h(x) |f'''(x)| \, |f''(x)| \int_{\sqrt{f(x)}}^\infty C(1 + \|f'\|_{C^{1,\gamma}}^3) \frac{\min\{1, \sqrt{f(x)} + y^{1+\gamma}\}}{y^2} dy dx \\ &\leq C(1 + \|f'\|_{C^{1,\gamma}}^3) \|f''\|_{\tilde{L}^2} \|f'''\|_{\tilde{L}^2}. \end{aligned}$$

So the above arguments yield the bound

$$|I_3''| \le C(1+\|f'\|^3_{C^{1,\gamma}})(\|f''\|^2_{\tilde{L}^2}+\|f'''\|^2_{\tilde{L}^2}). \tag{5.7}$$

Let us now turn to I_3''' and consider its inside integrand, again for $x=0$. This is

$$y(f''(0)+f''(-y))\,\partial_x^2 A^+(0,y) = 6yP(-y) - 8y^3Q(-y) - 2yS(-y), \tag{5.8}$$

where

$$P(y) := \frac{(f''(0)+f''(y))(f'(0)+f'(y))^2}{[y^2+(f(0)+f(y))^2]^2},$$
$$Q(y) := \frac{(f''(0)+f''(y))(f'(0)+f'(y))^2}{[y^2+(f(0)+f(y))^2]^3},$$
$$S(y) := \frac{(f''(0)+f''(y))^2(f(0)+f(y))}{[y^2+(f(0)+f(y))^2]^2}.$$

Let us first deal with S above.

Lemma 5.1. *We have*

$$\left|\int_{-\sqrt{f(0)}}^{\sqrt{f(0)}} yS(y)dy\right| \le C(1+\|f'\|^4_{C^1})\,(|f''(0)|+M_f(0)) \tag{5.9}$$

when $f(0)\le 1$*, where*

$$M_f(x) := \sup_{y>0}\frac{1}{2y}\int_{-y}^{y}|f'''(x+z)|dz$$

is the Hardy-Littlewood maximal function for f'''*.*

Remark. This result and Lemma 5.2 below, which is its analog with $6yP(y)-8y^3Q(y)$ in place of $yS(y)$, now yield

$$|I_3'''| \le C(1+\|f'\|^4_{C^1})(\|f''\|^2_{\tilde{L}^2}+\|f'''\|^2_{\tilde{L}^2}). \tag{5.10}$$

This is because both results equally hold with any $x\in\mathbb{R}$ instead of just $x=0$, and it is well known that $\|M_f\|_{L^2}\le C\|f'''\|_{L^2}$ for $f\in L^2(\mathbb{R})$, which easily implies $\|M_f\|_{\tilde{L}^2}\le C\|f'''\|_{\tilde{L}^2}$. because for $y\ge 1$ we have

$$\frac{1}{2y}\int_{-y}^{y}|f'''(x+z)|dz \le \sup_{x'\in\mathbb{R}}\frac{1}{2}\int_{x'-1}^{x'+1}|f'''(z)|dz \le \|f'''\|_{\tilde{L}^2}.$$

Together with (5.2), (5.3), and (5.7), this yields

$$|I_3^{\pm}| \le C(1+\|f\|^4_{C^{2,\gamma}_\gamma})(\|f''\|^2_{\tilde{L}^2}+\|f'''\|^2_{\tilde{L}^2}). \tag{5.11}$$

We note that, as the proof of Lemma 5.2 shows, (5.9) does not hold individually for integrands $yP(y)$ and $y^3Q(y)$!

Proof. Let

$$a := f(0)\in[0,1], \qquad b := f'(0)\in\left[-2\|f''\|^{1/2}_{L^\infty}\sqrt{a}, 2\|f''\|^{1/2}_{L^\infty}\sqrt{a}\right], \qquad c := f''(0)$$

(see (2.8)). Then

$$f''(y) = c + g_0(y), \qquad f'(y) = b + cy + g_1(y), \qquad f(y) = a + by + \frac{c}{2}y^2 + g_2(y)$$

for all $y \in \mathbb{R}$, where

$$g_0(y) := \int_0^y f'''(z)dz, \qquad g_1(y) := \int_0^y (y-z)f'''(z)dz, \qquad g_2(y) := \int_0^y \frac{(y-z)^2}{2} f'''(z)dz.$$

Let also

$$G_j(y) := g_j(y) - g_j(-y) = \int_{-y}^y \frac{(y-z)^j}{j!} f'''(z)dz,$$
$$F_k(y) := (f(0) + f(y))^k + (f(0) + f(-y))^k.$$

By (3.5) we have

$$S(y) - S(-y) = S_1(y)G_0(y) + S_2(y) + S_3(y), \tag{5.12}$$

where

$$S_1(y) := \frac{(2f''(0) + f''(y) + f''(-y))\,(f(0) + f(y))}{[y^2 + (f(0) + f(y))^2]^2},$$
$$S_2(y) := \frac{(2c + g_0(-y))^2(2by + G_2(y))}{[y^2 + (f(0) + f(y))^2]^2},$$
$$S_3(y) := -\frac{(2c + g_0(-y))^2(f(0) + f(-y))\,[2y^2 + F_2(y)]\,F_1(y)\,(2by + G_2(y))}{[y^2 + (f(0) + f(y))^2]^2\,[y^2 + (f(0) + f(-y))^2]^2}.$$

First consider S_1 and let

$$\tilde{S}_1(y) := \frac{(2f''(0) + f''(y) + f''(-y))(f(0) + f(y))}{[y^2 + 4a^2]^2}.$$

From (2.9) we have

$$\begin{aligned} |f(0) + f(\pm y)| &\le 3(1 + \|f'\|_{C^1})a \\ |f(0) + f(\pm y) - 2a| = |f(\pm y) - f(0)| &\le 2(1 + \|f'\|_{C^1})\sqrt{a}\,y \end{aligned} \tag{5.13}$$

for all $y \in [0, \sqrt{a}]$, and so for these y we have

$$|(f(0) + f(\pm y))^2 - 4a^2| \le 20(1 + \|f'\|_{C^1}^2)a^{3/2}y$$

and then

$$\left| \frac{y^2 + 4a^2}{y^2 + (f(0) + f(\pm y))^2} - 1 \right| \le C(1 + \|f'\|_{C^1}^2)\frac{a^{3/2}}{\max\{y, a\}}. \tag{5.14}$$

This yields

$$|S_1(y) - \tilde{S}_1(y)| \le C(1 + \|f'\|_{C^1}^4)\frac{a^{5/2}}{\max\{y, a\}^5} \tag{5.15}$$

for all $y \in [0, \sqrt{a}]$. Similarly, (5.13) yields

$$|\tilde{S}_1(y)| \le C(1 + \|f'\|_{C^1}^2)\frac{a}{\max\{y, a\}^4} \tag{5.16}$$

for $y \in [0, \sqrt{a}]$, so

$$\int_0^{\sqrt{a}} y|\tilde{S}_1(y)G_0(y)|dy \le C(1+\|f'\|_{C^1}^2)\int_0^{\sqrt{a}} \frac{ay^2}{\max\{y,a\}^4}M_f(0)dy \le C(1+\|f'\|_{C^1}^2)M_f(0).$$

An analogous bound can be obtained using (5.15) in place of (5.16), and we thus have

$$\int_0^{\sqrt{a}} y|S_1(y)G_0(y)|dy \le C(1+\|f'\|_{C^1}^4)M_f(0), \tag{5.17}$$

which is dominated by the right-hand side of (5.9).

Let next

$$\tilde{S}_4(y) := \frac{8bc^2y}{[y^2+4a^2]^2} - \frac{128a^2bc^2y}{[y^2+4a^2]^3} = 8bc^2\frac{y^3-12a^2y}{[y^2+4a^2]^3},$$

which is $S_2(y)+S_3(y)$ after dropping all G_2 and g_0 terms and replacing all $f(0)+f(\pm y)$ by $2a$ (including those in $F_j(y)$); this is also the leading order term of $S_2(y)+S_3(y)$ when $bc \ne 0$ and $0 \le y \ll \sqrt{a}$. When $|b| \sim \sqrt{a}$ and $c \sim 1$, in the latter region we have $y|\tilde{S}_4(y)| \sim \sqrt{a}\,y^2\max\{y,a\}^{-4}$, which is too large to yield an upper bound on $\int_0^{\sqrt{a}} y|\tilde{S}_4(y)|dy$ that is independent of a when $a \ll 1$. However, we observe that

$$y\tilde{S}_4(y) = -\partial_y\,\frac{8bc^2y^3}{[y^2+4a^2]^2}, \tag{5.18}$$

so this and $|b| \le 2\|f''\|_{L^\infty}^{1/2}\sqrt{a}$ yield

$$\left|\int_0^{\sqrt{a}} y\tilde{S}_4(y)dy\right| \le C\|f''\|_{L^\infty}^{1/2}c^2, \tag{5.19}$$

which is dominated by the right-hand side of (5.9).

Now let

$$S_4(y) := \frac{(2c+g_0(-y))^2(2by+G_2(y))}{[y^2+4a^2]^2} - \frac{16a^2(2c+g_0(-y))^2(2by+G_2(y))}{[y^2+4a^2]^3},$$

which is obtained from $S_2(y)+S_3(y)$ in the same way as $\tilde{S}_4(y)$, but without dropping the G_2 and g_0 terms. From (5.13),

$$\begin{aligned} 2c+g_0(-y) &= f''(0)+f''(-y),\\ 2by+G_2(y) &= f(y)-f(-y), \end{aligned} \tag{5.20}$$

(2.9), (5.14), and $|g_0(-y)| \le 2yM_f(0)$ for $y \in [0,1]$ we obtain

$$\begin{aligned} &|S_2(y)+S_3(y)-S_4(y)|\\ &\quad\le C\|f''\|_{L^\infty}(|c|+M_f(0)\sqrt{a})(1+\|f'\|_{C^1})\sqrt{a}\,y\left(\frac{(1+\|f'\|_{C^1}^2)a^{3/2}}{\max\{y,a\}^5}+\frac{(1+\|f'\|_{C^1}^2)a^{3/2}y}{\max\{y,a\}^6}\right)\\ &\quad\le C(1+\|f'\|_{C^1}^4)(|c|+M_f(0))\frac{a^2}{\max\{y,a\}^4} \end{aligned} \tag{5.21}$$

for $y \in [0, \sqrt{a}]$, similarly to (5.15). This bound follows after we first replace $f(0) + f(-y)$ by $2a$ and $F_1(y)$ by $4a$ in S_3, and then all the $(f(0) + f(\pm y))^2$ by $4a^2$ in both S_2 and S_3. Thus

$$\int_0^{\sqrt{a}} y|S_2(y) + S_3(y) - S_4(y)|dy \leq C(1 + \|f'\|_{C^1}^4)(|c| + M_f(0)), \tag{5.22}$$

which is again dominated by the right-hand side of (5.9). For $y \in [0, \sqrt{a}]$ we similarly have

$$|S_4(y) - \tilde{S}_4(y)| \leq C(1 + \|f'\|_{C^1}^2) \frac{\sqrt{a}\, y + y^2}{\max\{y, a\}^4} \int_{-y}^{y} |f'''(z)|dz \leq C(1 + \|f'\|_{C^1}^2) \frac{M_f(0)a}{\max\{y, a\}^3}, \tag{5.23}$$

so we obtain

$$\int_0^{\sqrt{a}} y|S_4(y) - \tilde{S}_4(y)|dy \leq C(1 + \|f'\|_{C^1}^2) M_f(0). \tag{5.24}$$

The result therefore follows from (5.12), (5.17), (5.19), (5.22), and (5.24). □

Next we consider the terms P and Q.

Lemma 5.2. *When $f(0) \leq 1$, we have*

$$\left| \int_{-\sqrt{f(0)}}^{\sqrt{f(0)}} \left[3yP(y) - 4y^3 Q(y) \right] dy \right| \leq C(1 + \|f'\|_{C^1}^4) \left(|f''(0)| + M_f(0) \right) \tag{5.25}$$

Proof. This proof will follow along the same lines as that of Lemma 5.1 so we will skip some details. Let $a, b, c, g_j, G_j^{\pm}$ be as above, and let us start with P. By (3.5) we have

$$P(y) - P(-y) = P_1(y)G_0(y) + P_2(y) + P_3(y), \tag{5.26}$$

where

$$P_1(y) := \frac{(f'(0) + f'(y))^2}{[y^2 + (f(0) + f(y))^2]^2},$$
$$P_2(y) := \frac{(2c + g_0(-y))\,(2f'(0) + f'(y) + f'(-y))\,(2cy + G_1(y))}{[y^2 + (f(0) + f(y))^2]^2},$$
$$P_3(y) := -\frac{(2c + g_0(-y))(f'(0) + f'(-y))^2\,[2y^2 + F_2(y)]\,F_1(y)\,(2by + G_2(y))}{[y^2 + (f(0) + f(y))^2]^2\,[y^2 + (f(0) + f(-y))^2]^2}.$$

Let

$$\tilde{P}_1(y) := \frac{(f'(0) + f'(y))^2}{[y^2 + 4a^2]^2}.$$

Then similarly to (5.15) and (5.16) we obtain

$$|P_1(y) - \tilde{P}_1(y)| \leq C(1 + \|f'\|_{C^1}^4) \frac{a^{5/2}}{\max\{y, a\}^5},$$
$$|\tilde{P}_1(y)| \leq C(1 + \|f'\|_{C^1}^2) \frac{a}{\max\{y, a\}^4}$$

for $y \in [0, \sqrt{a}]$, where we also used the first bound from (2.9). Just as (5.17), we now get

$$\int_0^{\sqrt{a}} y|P_1(y)G_0(y)|dy \leq C(1 + \|f'\|_{C^1}^4) M_f(0). \tag{5.27}$$

Let now

$$\tilde{P}_4(y) := \frac{16bc^2y}{[y^2+4a^2]^2} - \frac{128ab^3cy}{[y^2+4a^2]^3} = 16bc^2\frac{y^3-12a^2y}{[y^2+4a^2]^3} + 128abc\frac{(2ac-b^2)y}{[y^2+4a^2]^3},$$

which is $P_2(y)+P_3(y)$ after dropping all G_j and g_0 terms, replacing all $f(0)+f(\pm y)$ by $2a$ (including those in $F_j(y)$), and replacing all $f'(0)+f'(\pm y)$ terms by $2b$. Then

$$y\tilde{P}_4(y) = 128abc\frac{(2ac-b^2)y^2}{[y^2+4a^2]^3} - \partial_y\,\frac{16bc^2y^3}{[y^2+4a^2]^2}, \tag{5.28}$$

but this time the first term means that

$$\left|\int_0^{\sqrt{a}} y\tilde{P}_4(y)dy\right| \sim a^{-1/2} \tag{5.29}$$

for $a \ll 1$ whenever $a \neq 0$, $|b| \not\ll \sqrt{a}$, $|c| \not\ll 1$, and $|2ac-b^2| \not\ll a$. We will therefore leave $\tilde{P}_4$ for now and return to it at the end of this proof.

Next let

$$P_4(y) := \frac{4b(2c+g_0(-y))\,(2cy+G_1(y))}{[y^2+4a^2]^2} - \frac{32ab^2(2c+g_0(-y))\,(2by+G_2(y))}{[y^2+4a^2]^3}$$

which is obtained from $P_2(y)+P_3(y)$ in the same way as $\tilde{P}_4(y)$, but without dropping the G_2 and g_0 terms. Similarly to (5.21), we now obtain

$$\begin{aligned}
&|P_2(y)+P_3(y)-P_4(y)| \\
&\quad\le C\|f''\|_{L^\infty}^2\frac{M_f(0)y^3}{\max\{y,a\}^4} + C(|c|+M_f(0)\sqrt{a})(1+\|f'\|_{C^1}^4)\frac{a^2y^2}{\max\{y,a\}^6} \\
&\quad+ C(|c|+M_f(0)\sqrt{a})(1+\|f'\|_{C^1}^2)\sqrt{a}\,y\left(\frac{(1+\|f'\|_{C^1}^2)a^{3/2}}{\max\{y,a\}^5} + \frac{(1+\|f'\|_{C^1})a^{3/2}y}{\max\{y,a\}^6}\right) \\
&\quad\le C(1+\|f'\|_{C^1}^4)(|c|+M_f(0))\frac{a^2+y^3}{\max\{y,a\}^4}
\end{aligned}$$

for $y \in [0,\sqrt{a}]$, where we also used $|b| \le 2\|f''\|_{L^\infty}^{1/2}\sqrt{a}$ and

$$\begin{aligned}
2cy+G_1(y) &= f'(y)-f'(-y), \\
|(f'(0)+f'(\pm y))^n-(2b)^n| &\le C_n(1+\|f'\|_{C^1}^n)a^{(n-1)/2}y, \\
|2f'(0)+f'(y)+f'(-y)-4b| &= |g_1(y)+g_1(-y)| \le 2y^2M_f(0)
\end{aligned} \tag{5.30}$$

for $y \in [0,\sqrt{a}]$ and $n \ge 1$ (here we only need $n=2$ but other n will be useful in Lemma 6.1 below). The above estimate is obtained if we first replace all the terms $f'(0)+f'(\pm y)$ by $2b$, where the errors are the first two terms after the first inequality above (in the first of them we use the first line in (5.20) and the first and third line in (5.30); in the second we use the second line in (5.20), (2.9), the second line in (5.30), and (5.13)). Then during the following replacements we use the bound $|b| \le 2\|f''\|_{L^\infty}^{1/2}\sqrt{a}$, obtaining essentially the same errors as in Lemma 5.1 (these are collected in the third term after the first inequality above; here we

used the second line in (5.20), the first line in (5.30), the second line in (5.13), and (5.14)). From this we see that

$$\int_0^{\sqrt{a}} y|P_2(y)+P_3(y)-P_4(y)|dy \le C(1+\|f'\|_{C^1}^4)(|c|+M_f(0)). \tag{5.31}$$

By (5.20), the first line in (5.30), and (2.9) we again have

$$|P_4(y)-\tilde{P}_4(y)| \le C(1+\|f'\|_{C^1}^2)\frac{a}{\max\{y,a\}^4}\int_{-y}^{y}|f'''(z)|dz \le C(1+\|f'\|_{C^1}^2)\frac{M_f(0)a}{\max\{y,a\}^3}$$

for $y \in [0,\sqrt{a}]$, so we obtain

$$\int_0^{\sqrt{a}} y|P_4(y)-\tilde{P}_4(y)|dy \le C(1+\|f'\|_{C^1}^2)M_f(0).$$

Finally we can conclude from this and (5.31) that

$$\int_0^{\sqrt{a}} y|P_2(y)+P_3(y)-\tilde{P}_4(y)|dy \le C(1+\|f'\|_{C^1}^4)(|c|+M_f(0)). \tag{5.32}$$

The treatment of Q is the same as P, with two adjustments due to the power 3 in the denominator of Q. First, all the pointwise estimates on Q_j have an extra factor $\max\{y,a\}^{-2}$, but this is cancelled in all the integrals by the extra factor y^2 in front of Q in (5.8) (relative to P). Second, the term Q_3, corresponding to P_3, has an extra factor $\frac{3}{2}$ relative to P_3 (besides $\max\{y,a\}^{-2}$). This means that we still get

$$\int_0^{\sqrt{a}} y^3|Q_1(y)G_0(y)|dy \le C(1+\|f'\|_{C^1}^4)M_f(0). \tag{5.33}$$

and

$$\int_0^{\sqrt{a}} y^3|Q_2(y)+Q_3(y)-\tilde{Q}_4(y)|dy \le C(1+\|f'\|_{C^1}^4)(|c|+M_f(0)) \tag{5.34}$$

via the same arguments as (5.27) and (5.32), but now the critical term (analogous to $\tilde{P}_4$) is

$$\tilde{Q}_4(y) := \frac{16bc^2y}{[y^2+4a^2]^3} - \frac{192ab^3cy}{[y^2+4a^2]^4} = 16bc^2\frac{y^3-20a^2y}{[y^2+4a^2]^4} + 192abc\frac{(2ac-b^2)y}{[y^2+4a^2]^4}.$$

Hence

$$y^3\tilde{Q}_4(y) = 192abc\frac{(2ac-b^2)y^4}{[y^2+4a^2]^4} - \partial_y\frac{16bc^2y^5}{[y^2+4a^2]^3}, \tag{5.35}$$

which together with (5.28) yields

$$\begin{aligned}\frac{3y\tilde{P}_4(y)-4y^3\tilde{Q}_4(y)}{16} &= -24abc(2ac-b^2)\frac{y^4-4a^2y^2}{[y^2+4a^2]^4} - \partial_y\frac{3bc^2y^3}{[y^2+4a^2]^2} + \partial_y\frac{4bc^2y^5}{[y^2+4a^2]^3}\\ &= \partial_y\frac{8abc(2ac-b^2)y^3}{[y^2+4a^2]^3} - \partial_y\frac{3bc^2y^3}{[y^2+4a^2]^2} + \partial_y\frac{4bc^2y^5}{[y^2+4a^2]^3}.\end{aligned} \tag{5.36}$$

This way we essentially cancelled the problematic term in (5.28) by adding to it its equally problematic counterpart from (5.35), and using $|b| \le 2\|f''\|_{L^\infty}^{1/2}\sqrt{a}$ we obtain

$$\left|\int_0^{\sqrt{a}} \left[3y\tilde{P}_4(y) - 4y^3\tilde{Q}_4(y)\right] dy\right| \le C(\|f''\|_{L^\infty}^{1/2}c^2 + \|f''\|_{L^\infty}^{3/2}|c|) \le C\|f''\|_{L^\infty}^{3/2}|c|. \tag{5.37}$$

In view of (5.26), (5.27), (5.32), (5.33), and (5.34), this finishes the proof. □

6. Estimates on $I_4^\pm$

Finally we will treat $I_4^\pm$. Note that $I_4^- = 6J_3^- - 24J_5^- + I_2^-$, where

$$J_5^- := \int_{\mathbb{R}} h(x)f'''(x)\int_{\mathbb{R}} \frac{y\,(f'(x)-f'(x-y))^4(f(x)-f(x-y))[(f(x)-f(x-y))^2-y^2]}{[y^2+(f(x)-f(x-y))^2]^4}\,dydx.$$

Similarly to (5.1) we now obtain

$$\begin{aligned}|J_5^-| &\le C\|f'\|_{C^1}^4 \int_{\mathbb{R}^2\times[0,1]} h(x)|f'''(x)|\,|f''(x-sy)|\min\{1,|y|^{-4}\}\,dxdyds\\ &\le C\|f'\|_{C^1}^4\|f''\|_{\tilde{L}^2}\|f'''\|_{\tilde{L}^2}\int_{\mathbb{R}\times[0,1]}\min\{1,|y|^{-4}\}\,dyds \le C\|f'\|_{C^1}^4(\|f''\|_{\tilde{L}^2}^2+\|f'''\|_{\tilde{L}^2}^2).\end{aligned}$$

This, (4.2), and (5.1) now yield

$$|I_4^-| \le C(1+\|f'\|_{C^{1,\gamma}}^4)(\|f''\|_{\tilde{L}^2}^2+\|f'''\|_{\tilde{L}^2}^2). \tag{6.1}$$

Term I_4^+ will again be the challenging one. As with I_3^+, split $I_4^+ = I_4' + I_4'' + I_4''' + I_4''''$, where with $R_x := \mathbb{R}\setminus[-\sqrt{f(x)},\sqrt{f(x)}\,]$ and $\tilde{h}(x) := \chi_{[0,1]}(f(x))\,h(x)$ we have

$$\begin{aligned}
I_4' &:= \int_{\mathbb{R}} \chi_{(1,\infty)}(f(x))\,h(x)f'''(x)\int_{\mathbb{R}}\left[y\,(f'(x)+f'(x-y))\,\partial_x^3A^+(x,y)+2yW(x,y)\right]dydx,\\
I_4'' &:= \int_{\mathbb{R}} \tilde{h}(x)f'''(x)\,PV\int_{R_x}\left[y\,(f'(x)+f'(x-y))\,\partial_x^3A^+(x,y)+2yW(x,y)\right]dydx,\\
I_4''' &:= \int_{\mathbb{R}} \tilde{h}(x)f'''(x)\int_{-\sqrt{f(x)}}^{\sqrt{f(x)}}\left[y\,(f'(x)+f'(x-y))\,\partial_x^3A^+(x,y)+2yW(x,y)\right]dydx,\\
I_4'''' &:= -\int_{\mathbb{R}} h(x)f'''(x)\int_{\mathbb{R}} 2yW(x,y)\,dydx,
\end{aligned}$$

and

$$W(x,y) := \frac{(f'''(x)+f'''(x-y))(f(x)+f(x-y))(f'(x)+f'(x-y))}{[y^2+(f(x)+f(x-y))^2]^2}.$$

Note that $I_4'''' = I_2^+$, so (4.4) yields

$$|I_4''''| \le C(1+\|f'\|_{C^1}^2)\|f'''\|_{\tilde{L}^2}^2. \tag{6.2}$$

Next, the inside integrand in I_4' is (6.5) below with $(x, x-y)$ in place of $(0,-y)$, so

$$\begin{aligned}|I_4'| &\le C\|f'\|_{C^1}^3\int_{\mathbb{R}^2} h(x)|f'''(x)|\sum_{j=1}^2\left(|f^{(j)}(x)|+|f^{(j)}(x-y)|\right)\min\{1,|y|^{-3}\}\,dxdyds\\ &\le C\|f'\|_{C^1}^3(\|f'\|_{\tilde{L}^2}^2+\|f''\|_{\tilde{L}^2}^2+\|f'''\|_{\tilde{L}^2}^2).\end{aligned} \tag{6.3}$$

We also have $I_4'' = 6(J_3^+ + 6J_5^+ - 8J_6^+) - 24J_8^+$, where

$$J_8^+ := \int_{\mathbb{R}} \tilde{h}(x) f'''(x) \int_{R_x} \frac{y\,(f'(x)+f'(x-y))^4 (f(x)+f(x-y))[(f(x)+f(x-y))^2 - y^2]}{[y^2+(f(x)+f(x-y))^2]^4}\,dydx.$$

Estimates from the last section show that

$$|J_3^+| + |J_5^+| + |J_6^+| \le C(1+\|f'\|_{C^{1,\gamma}}^3)(\|f''\|_{\tilde{L}^2}^2 + \|f'''\|_{\tilde{L}^2}^2),$$

while we obtain

$$\begin{aligned} |J_8^+| &\le C(1+\|f'\|_{C^1}^4) \int_{\mathbb{R}^2\times[0,1]} h(x)|f'''(x)|(|f'(x)| + |f''(x-sy)|)\min\{1,|y|^{-4}\}\,dxdyds \\ &\le C(1+\|f'\|_{C^1}^4)(\|f'\|_{\tilde{L}^2}^2 + \|f''\|_{\tilde{L}^2}^2 + \|f'''\|_{\tilde{L}^2}^2) \end{aligned}$$

if inside the integral we split one factor $f'(x)+f'(x-y)$ into $2f'(x)$ and $f'(x-y)-f'(x)$, and then use (2.9) to estimate both integrals (the latter in the same way as J_3^-). Therefore

$$|I_4''| \le C(1+\|f'\|_{C^{1,\gamma}}^4)(\|f'\|_{\tilde{L}^2}^2 + \|f''\|_{\tilde{L}^2}^2 + \|f'''\|_{\tilde{L}^2}^2) \tag{6.4}$$

It remains to estimate I_4'''. Consider again $x=0$, and then we have

$$y(f'(0)+f'(-y))\,\partial_x^3 A^+(0,y) + 2yW(0,y) = 18yP(-y) - 24y^3Q(-y) - 24yU(-y) + 48y^3V(-y), \tag{6.5}$$

where

$$\begin{aligned} U(y) &:= \frac{(f'(0)+f'(y))^4(f(0)+f(y))}{[y^2+(f(0)+f(y))^2]^3}, \\ V(y) &:= \frac{(f'(0)+f'(y))^4(f(0)+f(y))}{[y^2+(f(0)+f(y))^2]^4}. \end{aligned}$$

We can use Lemma 5.2 to estimate the first two terms on the right-hand side of (6.5). Since the next lemma will deal with the other two terms, similarly to (5.10) it follows that

$$|I_4'''| \le C(1+\|f'\|_{C^1}^4)(\|f'\|_{\tilde{L}^2}^2 + \|f''\|_{\tilde{L}^2}^2 + \|f'''\|_{\tilde{L}^2}^2)$$

by using these results for all $x \in \mathbb{R}$ instead of just $x=0$. This, (6.1), (6.2), (6.3), and (6.4) then yield

$$|I_4^\pm| \le C(1+\|f'\|_{C^{1,\gamma}}^4)(\|f'\|_{\tilde{L}^2}^2 + \|f''\|_{\tilde{L}^2}^2 + \|f'''\|_{\tilde{L}^2}^2). \tag{6.6}$$

Lemma 6.1. *When $f(0) \le 1$, we have*

$$\left| \int_{-\sqrt{f(0)}}^{\sqrt{f(0)}} \left[2y^3V(y) - yU(y)\right] dy \right| \le C(1+\|f'\|_{C^1}^4)\left(|f'(0)| + |f''(0)| + M_f(0)\right). \tag{6.7}$$

Proof. By (3.5) we again have

$$\begin{aligned} U(y) - U(-y) &= U_1(y) + U_2(y) + U_3(y), \\ V(y) - V(-y) &= V_1(y) + V_2(y) + V_3(y), \end{aligned} \tag{6.8}$$

where

$$U_1(y) := \frac{(f(0)+f(y))(2cy + G_1(y)) \sum_{n=0}^3 (f'(0)+f'(y))^n (f'(0)+f'(-y))^{3-n}}{[y^2+(f(0)+f(y))^2]^3},$$

$$U_2(y) := \frac{(f'(0)+f'(-y))^4(2by+G_2(y))}{[y^2+(f(0)+f(y))^2]^3},$$
$$U_3(y) := -\frac{(f'(0)+f'(-y))^4(f(0)+f(-y))\,H_2(y)\,F_1(y)\,(2by+G_2(y))}{[y^2+(f(0)+f(y))^2]^3\,[y^2+(f(0)+f(-y))^2]^3}$$

and

$$V_1(y) := \frac{(f(0)+f(y))(2cy+G_1(y))\sum_{n=0}^3(f'(0)+f'(y))^n(f'(0)+f'(-y))^{3-n}}{[y^2+(f(0)+f(y))^2]^4},$$
$$V_2(y) := \frac{(f'(0)+f'(-y))^4(2by+G_2(y))}{[y^2+(f(0)+f(y))^2]^4},$$
$$V_3(y) := -\frac{(f'(0)+f'(-y))^4(f(0)+f(-y))\,H_3(y)\,F_1(y)\,(2by+G_2(y))}{[y^2+(f(0)+f(y))^2]^4\,[y^2+(f(0)+f(-y))^2]^4},$$

with

$$H_j(y) := \sum_{n=0}^{j}[y^2+(f(0)+f(y))^2]^n\,[y^2+(f(0)+f(-y))^2]^{j-n}$$

(so in particular, $H_1(y) = 2y^2 + F_2(y)$). None of these terms is analogous to S_1 and P_1 because U and V do not have the $f''(0)+f''(y)$ term. Instead $U_1+U_2+U_3$ and $V_1+V_2+V_3$ are treated the same way as S_2+S_3, P_2+P_3, and Q_2+Q_3 in Lemmas 5.1 and 5.2. We leave some of the details to the reader, but provide here the main estimates, discuss how to deal with an extra complication, and show cancellation of the critical leading terms

$$\begin{aligned}\tilde{U}_4(y) &:= \frac{128ab^3cy}{[y^2+4a^2]^3} + \frac{32b^5y}{[y^2+4a^2]^3} - \frac{768a^2b^5y}{[y^2+4a^2]^4}\\ &= 32b^3(4ac+b^2)\frac{y^3-4a^2y}{[y^2+4a^2]^4} + 512a^2b^3\frac{(2ac-b^2)y}{[y^2+4a^2]^4},\\ \tilde{V}_4(y) &:= \frac{128ab^3cy}{[y^2+4a^2]^4} + \frac{32b^5y}{[y^2+4a^2]^4} - \frac{1024a^2b^5y}{[y^2+4a^2]^5}\\ &= 32b^3(4ac+b^2)\frac{3y^3-20a^2y}{3[y^2+4a^2]^5} + 2048a^2b^3\frac{(2ac-b^2)y}{3[y^2+4a^2]^5}.\end{aligned}$$

These were obtained from $U_1+U_2+U_3$ and $V_1+V_2+V_3$ by dropping all the G_j terms, and replacing all the $f(0)+f(\pm y)$ and $f'(0)+f'(\pm y)$ terms by $2a$ and $2b$, respectively.

Doing the latter but keeping the G_j terms defines terms U_4 and V_4, and we first obtain an estimate on $U_1+U_2+U_3-U_4$. We can do this similarly to the analogous estimates for $S_2+S_3-S_4$ and $P_2+P_3-P_4$, using (2.9), (5.13), (5.14), (5.20), and (5.30). But if we simply estimate each $f'(0)+f'(\pm y)$ by $4(1+\|f'\|_{C^1})\sqrt{a}$ and each b by $2\|f''\|_{L^\infty}^{1/2}\sqrt{a}$, our final estimate will lack the term $|c|+M_f(0)$ from (5.21) because U_2, U_3, V_2, V_3 do not contain the factor $2c+g_0(-y)$ or $2cy+G_1(y)$, which can be estimated by $2(|c|+M_f(0)\sqrt{a})$ and $2(|c|+M_f(0)\sqrt{a})y$, respectively. (This would add 1 to the last parenthesis in (6.7), which is not an issue when we consider solutions with $f''' \in \tilde{L}^2(\mathbb{R})$ but would become one for those with $f''' \in L^2(\mathbb{R})$, relevant to (1.12).) Instead, when we first replace $(f'(0)+f'(-y))^4$ by

$(2b)^4$ in either of these terms and estimate the difference, we need to use

$$|f'(-y) - f'(0)| \leq 2(|c| + M_f(0)\sqrt{a})y \leq 2(|c| + M_f(0))y,$$

which replaces the second line in (5.30) by

$$|(f'(0) + f'(\pm y))^n - (2b)^n| \leq C_n(1 + \|f'\|_{C^1}^{n-1})(|c| + M_f(0))a^{(n-1)/2}y,$$

In the case of U_2 we then also need to estimate

$$\frac{16b^4(2by + G_2(y))}{[y^2 + (f(0) + f(y))^2]^3} - \frac{16b^4(2by + G_2(y))}{[y^2 + 4a^2]^3},$$

but doing this by only using (5.14) and $|b| \leq 2\|f''\|_{L^\infty}^{1/2}\sqrt{a}$ is again insufficient. However, we do obtain the desired estimate, with $|b|+|c|+M_f(0)$ in place of $|c|+M_f(0)$, if we also employ the bound

$$b^2 \leq 4(|b| + |c| + M_f(0))a$$

once. This clearly holds when $|b| \leq 4a$, while if $|b| > 4a$, then $f(\pm\frac{2a}{b}) \geq 0$ forces

$$\frac{b}{2} \leq \sup_{|y|\leq 2a/|b|} |f'(y) - f'(0)| \leq (|c| + M_f(0))2ab^{-1}.$$

With a similar adjustment when estimating U_3, we eventually obtain

$$|U_1(y) + U_2(y) + U_3(y) - U_4(y)| \leq C(1 + \|f'\|_{C^1}^4)(|b| + |c| + M_f(0))\frac{a^2}{\max\{y, a\}^4},$$

$$|V_1(y) + V_2(y) + V_3(y) - V_4(y)| \leq C(1 + \|f'\|_{C^1}^4)(|b| + |c| + M_f(0))\frac{a^2}{\max\{y, a\}^6}$$

for $y \in [0, \sqrt{a}]$, so that

$$\int_0^{\sqrt{a}} y|U_1(y) + U_2(y) + U_3(y) - U_4(y)|dy \leq C(1 + \|f'\|_{C^1}^4)(|b| + |c| + M_f(0))$$

and

$$\int_0^{\sqrt{a}} y^3|V_1(y) + V_2(y) + V_3(y) - V_4(y)|dy \leq C(1 + \|f'\|_{C^1}^4)(|b| + |c| + M_f(0)).$$

We easily also get

$$|U_4(y) - \tilde{U}_4(y)| \leq C(1 + \|f'\|_{C^1}^2)\frac{a^3}{\max\{y, a\}^6}\int_{-y}^{y} |f'''(z)|dz \leq C(1 + \|f'\|_{C^1}^2)\frac{M_f(0)a}{\max\{y, a\}^3},$$

$$|V_4(y) - \tilde{V}_4(y)| \leq C(1 + \|f'\|_{C^1}^2)\frac{a^3}{\max\{y, a\}^8}\int_{-y}^{y} |f'''(z)|dz \leq C(1 + \|f'\|_{C^1}^2)\frac{M_f(0)a}{\max\{y, a\}^5}$$

for $y \in [0, \sqrt{a}]$, so

$$\int_0^{\sqrt{a}} \Big(y|U_4(y) - \tilde{U}_4(y)| + y^3|V_4(y) - \tilde{V}_4(y)|\Big)\, dy \leq C(1 + \|f'\|_{C^1}^2)M_f(0).$$

Finally, we now have the crucial identity

$$\frac{2y^3\tilde{V}_4(y) - y\tilde{U}_4(y)}{32} = -\partial_y\frac{2b^3(4ac + b^2)y^5}{3[y^2 + 4a^2]^4} + \partial_y\frac{b^3(4ac + b^2)y^3}{3[y^2 + 4a^2]^3} + 16a^2b^3(2ac - b^2)\frac{5y^4 - 12a^2y^2}{3[y^2 + 4a^2]^5}$$

$$= -\partial_y \frac{2b^3(4ac+b^2)y^5}{3[y^2+4a^2]^4} + \partial_y \frac{b^3(4ac+b^2)y^3}{3[y^2+4a^2]^3} - \partial_y \frac{16a^2b^3(2ac-b^2)y^3}{3[y^2+4a^2]^4}, \tag{6.9}$$

hence

$$\left| \int_0^{\sqrt{a}} \left[2y^3 \tilde{V}_4(y) - y\tilde{U}_4(y) \right] dy \right| \le Cb^2 \|f''\|_{L^\infty}^{3/2}(1+a) \le C\|f''\|_{L^\infty}^2 |b|.$$

The result now follows from the above bounds and (6.8). □

7. A Priori Estimates for an Approximating Family of Equations

We will find solutions to (1.5) as $\varepsilon \to 0$ limits of solutions to the mollified problems

$$\partial_t f_\varepsilon(x,t) = \phi_\varepsilon * PV \int_{\mathbb{R}} \sum_{\pm} \frac{y\,(\partial_x(\phi_\varepsilon * f_\varepsilon)(x,t) \pm \partial_x(\phi_\varepsilon * f_\varepsilon)(x-y,t))}{y^2 + ((\phi_\varepsilon * f_\varepsilon)(x,t) \pm (\phi_\varepsilon * f_\varepsilon)(x-y,t))^2} dy, \tag{7.1}$$

where $\phi_\varepsilon(x) := \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ for $\varepsilon \in (0,1)$ and $\phi \in C^\infty(\mathbb{R})$ vanishes on $\mathbb{R}\setminus[-1,1]$, is even, decreasing on $[0,1]$ with $\phi(0) = 1$ and $\|\phi'\|_{L^\infty} \le 2$, and $\int_{-1}^1 \phi(x)dx = 1$. Dropping again t in the arguments, denoting $g' := \partial_x g$, and letting $F_\varepsilon := \phi_\varepsilon * f_\varepsilon$ for $f_\varepsilon \ge 0$, we obtain

$$\frac{d}{dt} \left\| \sqrt{h}\, f_\varepsilon''' \right\|_{L^2}^2 = 2(I_{1'}^+ + I_{1'}^- + I_{4'}^+ + I_{4'}^-) + 6(I_{2'}^+ + I_{2'}^- + I_{3'}^+ + I_{3'}^-),$$

where with $A_\varepsilon^\pm(x,y) := [y^2 + (F_\varepsilon(x) \pm F_\varepsilon(x-y))^2]^{-1}$ we have

$$I_{1'}^\pm := \int_{\mathbb{R}} h(x) f_\varepsilon'''(x)\, \phi_\varepsilon * PV \int_{\mathbb{R}} \frac{y\,(F_\varepsilon''''(x) \pm F_\varepsilon''''(x-y))}{y^2 + (F_\varepsilon(x) \pm F_\varepsilon(x-y))^2}\, dydx,$$

$$I_{2'}^\pm := \int_{\mathbb{R}} h(x) f_\varepsilon'''(x)\, \phi_\varepsilon * \int_{\mathbb{R}} y\,(F_\varepsilon'''(x) \pm F_\varepsilon'''(x-y))\, \partial_x A_\varepsilon^\pm(x,y)\, dydx,$$

$$I_{3'}^\pm := \int_{\mathbb{R}} h(x) f_\varepsilon'''(x)\, \phi_\varepsilon * \int_{\mathbb{R}} y\,(F_\varepsilon''(x) \pm F_\varepsilon''(x-y))\, \partial_x^2 A_\varepsilon^\pm(x,y)\, dydx,$$

$$I_{4'}^\pm := \int_{\mathbb{R}} h(x) f_\varepsilon'''(x)\, \phi_\varepsilon * \int_{\mathbb{R}} y\,(F_\varepsilon'(x) \pm F_\varepsilon'(x-y))\, \partial_x^3 A_\varepsilon^\pm(x,y)\, dydx.$$

As explained at the end of Section 2, we can drop PV in the last three integrals, and below we will also drop it in $I_{1'}^\pm$ (where it is only needed as $|y| \to \infty$). Note that now F_ε does have the degree of regularity required for the arguments in the previous sections (it is in fact smooth), which carry over with the following adjustments.

Let us first consider $I_{1'}^\pm$. From $\int_{\mathbb{R}} F(\phi * G)dx = \int_{\mathbb{R}} (\phi * F)G dx$ for even ϕ we have

$$I_{1'}^\pm = J_{1'}^\pm + J_{0'}^\pm := \int_{\mathbb{R}} h(x) F_\varepsilon'''(x) \int_{\mathbb{R}} \frac{y\,(F_\varepsilon''''(x) \pm F_\varepsilon''''(x-y))}{y^2 + (F_\varepsilon(x) \pm F_\varepsilon(x-y))^2}\, dydx$$
$$+ \int_{\mathbb{R}^2} (h(z) - h(x))\phi_\varepsilon(x-z) f_\varepsilon'''(z) \int_{\mathbb{R}} \frac{y\,(F_\varepsilon''''(x) \pm F_\varepsilon''''(x-y))}{y^2 + (F_\varepsilon(x) \pm F_\varepsilon(x-y))^2}\, dydxdz.$$

We can estimate $J_{1'}^\pm$ in the same way as $I_1^\pm$, and similarly to (3.15) we obtain

$$J_{1'}^+ + J_{1'}^- \le C(1 + \|f_\varepsilon\|_{C_\gamma^{2,\gamma}}^2)\|f_\varepsilon'''\|_{\tilde{L}^2}^2 \tag{7.2}$$

with some ε-independent C, where we also used

$$\|\phi_\varepsilon * g\|_{\tilde{L}^2}^2 \le 2\|g\|_{\tilde{L}^2}^2.$$

On the other hand, we have

$$J_{0'}^{\pm} = \int_{\mathbb{R}^2} f_\varepsilon'''(z) f_\varepsilon'''(v) \int_{\mathbb{R}} (h(z)-h(x))\phi_\varepsilon(x-z) \int_{\mathbb{R}} \frac{y\left(\phi_\varepsilon'(x-v) \pm \phi_\varepsilon'(x-y-v)\right)}{y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2}\, dydxdzdv. \tag{7.3}$$

From (2.3) we obtain

$$\left| \int_{\mathbb{R}} \frac{y}{y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2}\, dy \right| \le C\|F_\varepsilon\|_{C^2_\gamma} \le C\|f_\varepsilon\|_{C^2_\gamma},$$

so

$$\left| \int_{\mathbb{R}} \frac{y\phi_\varepsilon'(x-v)}{y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2}\, dy \right| \le \frac{C\|f_\varepsilon\|_{C^2_\gamma}}{\varepsilon^2}\chi_{[0,\varepsilon]}(|x-v|) \tag{7.4}$$

because $\phi_\varepsilon(x-z)=0$ when $|x-z|\ge\varepsilon$. When $|x-v|\ge 2\varepsilon$, then

$$\left| \frac{d}{dy} \frac{y}{y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2} \right| \le \frac{C(1+\|f_\varepsilon'\|_{L^\infty})}{|x-v|^2}$$

for all $y\in[x-v-\varepsilon, x-v+\varepsilon]$, so from

$$\int_{x-v-\varepsilon}^{x-v+\varepsilon} \phi_\varepsilon'(x-y-v) = 0 \qquad \text{and} \qquad \int_{x-v-\varepsilon}^{x-v+\varepsilon} |\phi_\varepsilon'(x-y-v)| = 2\varepsilon^{-1}$$

we get

$$\left| \int_{\mathbb{R}} \frac{y\phi_\varepsilon'(x-y-v)}{y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2}\, dy \right| \le \frac{C(1+\|f_\varepsilon'\|_{L^\infty})}{|x-v|^2}. \tag{7.5}$$

When $|x-v|\le 2\varepsilon$, then from $|\phi_\varepsilon'(x-y-v)-\phi_\varepsilon'(x-v)|\le C\varepsilon^{-3}|y|$ and the fact that (7.4) also holds with integration over $[-y',y']$ for any $y'\ge 0$ instead of over $\mathbb{R}$,

$$\left| \int_{\mathbb{R}} \frac{y\phi_\varepsilon'(x-y-v)}{y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2}\, dy \right| = \left| \int_{-3\varepsilon}^{3\varepsilon} \frac{y\phi_\varepsilon'(x-y-v)}{y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2}\, dy \right| \le \frac{C(1+\|f_\varepsilon\|_{C^2_\gamma})}{\varepsilon^2}.$$

From this, (7.3), (7.4), (7.5), and the properties of h it now follows that

$$|J_{0'}^{\pm}| \le \int_{x_0-4-\varepsilon}^{x_0+4+\varepsilon} |f_\varepsilon'''(z)| \int_{\mathbb{R}} |f_\varepsilon'''(v)| \frac{C\varepsilon(1+\|f_\varepsilon\|_{C^2_\gamma})}{\max\{|z-v|,\varepsilon\}^2} dvdz \le C(1+\|f_\varepsilon\|_{C^2_\gamma})\|f_\varepsilon'''\|^2_{\tilde{L}^2}$$

because

$$\int_{\mathbb{R}} \frac{\varepsilon|f_\varepsilon'''(v)|}{\max\{|z-v|,\varepsilon\}^2}\, dv \le \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{2(n-1)}\varepsilon^2} \int_{-2^n\varepsilon}^{2^n\varepsilon} |f_\varepsilon'''(z+w)|\, dw \le 4M_{f_\varepsilon}(z).$$

Hence this and (7.2) show that

$$I_{1'}^{+} + I_{1'}^{-} \le C(1+\|f_\varepsilon\|^2_{C^{2,\gamma}_\gamma})\|f_\varepsilon'''\|^2_{\tilde{L}^2} \tag{7.6}$$

As for $I_{2'}^{\pm}$, similarly to (4.2) and (4.4) we will obtain

$$|I_{2'}^{\pm}| \le C(1+\|f_\varepsilon'\|^2_{C^{1,\gamma}})\|f_\varepsilon'''\|^2_{\tilde{L}^2} \tag{7.7}$$

if we can prove the same bound for

$$J_{2'}^{\pm} := \int_{\mathbb{R}^2} (h(z)-h(x))\phi_\varepsilon(x-z) f_\varepsilon'''(z) \int_{\mathbb{R}} y\left(F_\varepsilon'''(x)\pm F_\varepsilon'''(x-y)\right) \partial_x A_\varepsilon^{\pm}(x,y)\, dydxdz$$

$$= 2\int_{\mathbb{R}^2} f_\varepsilon'''(z) f_\varepsilon'''(v) \int_{\mathbb{R}} (h(z)-h(x))\phi_\varepsilon(x-z)$$
$$\int_{\mathbb{R}} \frac{y\,(\phi_\varepsilon(x-v)\pm\phi_\varepsilon(x-y-v))(F_\varepsilon(x)\pm F_\varepsilon(x-y))(F_\varepsilon'(x)\pm F_\varepsilon'(x-y))}{[y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2]^2}\,dydxdzdv.$$

Similarly to (7.3), this will follow once we bound the inside integral by

$$\frac{C(1+\|f_\varepsilon'\|^2_{C^{1,\gamma}})}{\max\{|x-v|,\varepsilon\}^2},$$

and we do this analogously. First, the estimation of $K_1^\pm$ from (3.2) yields

$$\left|\int_{S\cup(-S)} \frac{y(F_\varepsilon(x)\pm F_\varepsilon(x-y))(F_\varepsilon'(x)\pm F_\varepsilon'(x-y))}{[y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2]^2}\,dy\right| \le C(1+\|F_\varepsilon'\|^2_{C^{1,\gamma}}) \le C(1+\|f_\varepsilon'\|^2_{C^{1,\gamma}}) \tag{7.8}$$

for any $S\subseteq[0,\infty)$, so

$$\left|\int_{\mathbb{R}} \frac{y\phi_\varepsilon(x-v)(F_\varepsilon(x)\pm F_\varepsilon(x-y))(F_\varepsilon'(x)\pm F_\varepsilon'(x-y))}{[y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2]^2}\,dy\right| \le \frac{C(1+\|f_\varepsilon'\|^2_{C^{1,\gamma}})}{\varepsilon}\chi_{[0,\varepsilon]}(|x-v|). \tag{7.9}$$

When $|x-v|\ge 2\varepsilon$, then

$$\left|\frac{y(F_\varepsilon(x)\pm F_\varepsilon(x-y))(F_\varepsilon'(x)\pm F_\varepsilon'(x-y))}{[y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2]^2}\right| \le \left|\frac{F_\varepsilon'(x)\pm F_\varepsilon'(x-y)}{y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2}\right| \le \frac{C\|f_\varepsilon'\|_{L^\infty}}{|x-v|^2}$$

for all $y\in[x-v-\varepsilon,x-v+\varepsilon]$, so

$$\left|\int_{\mathbb{R}} \frac{y\phi_\varepsilon(x-y-v)(F_\varepsilon(x)\pm F_\varepsilon(x-y))(F_\varepsilon'(x)\pm F_\varepsilon'(x-y))}{[y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2]^2}\,dy\right| \le \frac{C\|f_\varepsilon'\|_{L^\infty}}{|x-v|^2}. \tag{7.10}$$

When $|x-v|\le 2\varepsilon$, then from $|\phi_\varepsilon(x-y-v)-\phi_\varepsilon(x-v)|\le C\varepsilon^{-2}|y|$ and (7.9) we obtain

$$\left|\int_{\mathbb{R}} \frac{y\phi_\varepsilon(x-y-v)(F_\varepsilon(x)\pm F_\varepsilon(x-y))(F_\varepsilon'(x)\pm F_\varepsilon'(x-y))}{[y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2]^2}\,dy\right| \tag{7.11}$$
$$= \left|\int_{-3\varepsilon}^{3\varepsilon} \frac{y\phi_\varepsilon(x-y-v)(F_\varepsilon(x)\pm F_\varepsilon(x-y))(F_\varepsilon'(x)\pm F_\varepsilon'(x-y))}{[y^2+(F_\varepsilon(x)\pm F_\varepsilon(x-y))^2]^2}\,dy\right| \le \frac{C(1+\|f_\varepsilon'\|^2_{C^{1,\gamma}})}{\varepsilon^{3/2}}.$$

This is immediate when $\pm$ is $-$, while in the other case we use (2.9) to bound the last integral with $\phi_\varepsilon(x-y-v)-\phi_\varepsilon(x-v)$ in place of $\phi_\varepsilon(x-y-v)$ by

$$\frac{C(1+\|F_\varepsilon'\|_{C^1})}{\varepsilon^2}\int_0^{3\varepsilon}\frac{\max\{y,\sqrt{F_\varepsilon(x)}\}}{\max\{y,F_\varepsilon(x)\}}dy \le \frac{C(1+\|f_\varepsilon'\|_{C^1})}{\varepsilon^{3/2}}.$$

The required bound on $J_{2'}^\pm$ now follows from the above estimates.

Integral $I_{3'}^\pm$ is estimated in the same way as $I_3^\pm$ because we did not use integration by parts in x or symmetrization in the process; instead, estimates on all the relevant integrals immediately followed after we obtained appropriate bounds on their inside integrals or integrands. The first inequality in (5.1) becomes

$$|J_{3'}^-| \le C\|F_\varepsilon'\|^2_{C^1}\int_{\mathbb{R}^2\times[0,1]} h(x)|f_\varepsilon'''(x)|\,(\phi_\varepsilon * |F_\varepsilon'''|)(x-sy)\min\{1,|y|^{-2}\}\,dxdyds,$$

which yields the same estimate on $J_{3'}^-$ as we obtained for J_3^-, with f_ε in place of f. Bounds on the terms analogous to J_3^+ and $J_4^\pm$ are adjusted in the same way. The term analogous to J_5^+ will be bounded by

$$C(1+\|F_\varepsilon'\|_{C^{1,\gamma}}^3)\int_{\mathbb{R}} h(x)|f_\varepsilon'''(x)|(\phi_\varepsilon * |F_\varepsilon''|)(x)\,dx \le C(1+\|f_\varepsilon'\|_{C^{1,\gamma}}^3)\|f_\varepsilon''\|_{\tilde{L}^2}\|f_\varepsilon'''\|_{\tilde{L}^2},$$

those corresponding to J_6^+, J_7^+, and I_3' are treated the same way, and (5.10) becomes

$$\begin{aligned}|I_{3'}'''| &\le C(1+\|F_\varepsilon'\|_{C^1}^4)\int_{\mathbb{R}} h(x)|f_\varepsilon'''(x)|(\phi_\varepsilon * (|F_\varepsilon''|+M_{F_\varepsilon}))(x)\,dx\\ &\le C(1+\|f_\varepsilon'\|_{C^1}^4)(\|f_\varepsilon''\|_{\tilde{L}^2}^2+\|f_\varepsilon'''\|_{\tilde{L}^2}^2).\end{aligned}$$

However, in all those terms that constitute $I_{3'}^+$, the estimates on the inside integrals/integrands carry over only when $|f(x+z)-f(x)|\le\frac{1}{2}$ for all $|z|\le\varepsilon$ because in the relevant bounds in Section 5 we assumed that $f(x)$ is away from either 0 or ∞. We therefore need to assume here that $\varepsilon\le\frac{1}{2}\|f_\varepsilon'\|_{L^\infty}^{-1}$.

The same adjustments apply to the terms constituting $I_{4'}^\pm$ (except for $I_{2'}^\pm$, which is bounded by (7.7)) in a decomposition analogous to that of $I_4^\pm$ in Section 6. These estimates, together with (7.6) and (7.7), finally yield

$$\frac{d}{dt}\left\|\sqrt{h}\,f_\varepsilon'''\right\|_{L^2}^2 \le C(1+\|f_\varepsilon\|_{C_\gamma^{2,\gamma}}^4)\,(\|f_\varepsilon'\|_{\tilde{L}^2}^2+\|f_\varepsilon''\|_{\tilde{L}^2}^2+\|f_\varepsilon'''\|_{\tilde{L}^2}^2) \tag{7.12}$$

in place of (2.12), provided $\varepsilon\le\frac{1}{2}\|f_\varepsilon'\|_{L^\infty}^{-1}$.

8. Uniqueness of Solutions

Assume that $f_1, f_2\ge 0$ are classical solutions to (1.5) on $\mathbb{R}\times[0,T]$ that both satisfy (1.8). Then the second claim in Theorem 1.1(ii) holds for f_j $(j=1,2)$ because (2.7) yields

$$\|f_j(\cdot,\tau)-f_j(\cdot,0)\|_{L^\infty}\le C\int_0^\tau s_j(t)(1+s_j(t))\,dt \tag{8.1}$$

for all $\tau\in[0,T]$, where $s_j(t):=\|f_j(\cdot,t)\|_{\tilde{H}_\gamma^3(\mathbb{R})}^2$. Therefore $f:=f_1-f_2\in L^\infty([0,T];\tilde{H}^3(\mathbb{R}))$ whenever $f_1(\cdot,0)-f_2(\cdot,0)\in\tilde{H}^3(\mathbb{R})$, and for all $t\in[0,T)$ we have

$$\frac{d}{dt}\left\|\sqrt{h}\,f(\cdot,t)\right\|_{L^2}^2 = 2(I_5^+ + I_5^- + I_6^+ + I_6^-)$$

with $h:=h_0(\cdot-x_0)$ as above, where (we again drop t in the notation and let $g':=\partial_x g$)

$$\begin{aligned}I_5^\pm &:= \int_{\mathbb{R}} h(x)f(x)\,PV\int_{\mathbb{R}}\frac{y\,(f'(x)\pm f'(x-y))}{y^2+(f_1(x)\pm f_1(x-y))^2}\,dydx,\\ I_6^\pm &:= \int_{\mathbb{R}} h(x)f(x)\int_{\mathbb{R}} yB^\pm(x,x-y)\,dydx = \int_{\mathbb{R}} h(x)f(x)\int_{\mathbb{R}}(x-y)B^\pm(x,y)\,dydx,\end{aligned}$$

and with $f_3:=f_1+f_2$ we have

$$B^\pm(x,y):=\frac{f_2'(x)\pm f_2'(y)}{(x-y)^2+(f_1(x)\pm f_1(y))^2}-\frac{f_2'(x)\pm f_2'(y)}{(x-y)^2+(f_2(x)\pm f_2(y))^2}$$

$$= -\frac{(f_2'(x) \pm f_2'(y))\,(f_3(x) \pm f_3(y))\,(f(x) \pm f(y))}{[(x-y)^2 + (f_1(x) \pm f_1(y))^2]\,[(x-y)^2 + (f_2(x) \pm f_2(y))^2]} =: B_0^{\pm}(x,y)\,(f(x) \pm f(y)).$$

The arguments in Section 3 estimating $I_1^{\pm}$ identically apply to $I_5^{\pm}$ with (f, f_1) in place of (g, f), and just as (3.15) they now prove

$$I_5^+ + I_5^- \le C(1 + \|f_1\|^2_{C^{2,\gamma}_\gamma})\|f\|^2_{\tilde{L}^2_{x_0}}. \tag{8.2}$$

Symmetrization shows that

$$\int_{\mathbb{R}} h(x)f(x)\,PV\int_{\mathbb{R}}(x-y)B_0^{\pm}(x,y)f(y)\,dydx = \frac{1}{2}\int_{\mathbb{R}^2}(h(x)-h(y))(x-y)B_0^{\pm}(x,y)f(x)f(y)\,dydx,$$

so with $E := [x_0 - 4, x_0 + 4]$ we have

$$\left|\int_{\mathbb{R}} h(x)f(x)\,PV\int_{\mathbb{R}}(x-y)B_0^{-}(x,y)f(y)\,dydx\right| \le C\|f_2'\|_{C^1}\int_{\mathbb{R}^2}\frac{\chi_E(x)+\chi_E(y)}{\max\{1,|x-y|^2\}}|f(x)|\,|f(y)|\,dydx$$
$$\le C\|f_2'\|_{C^1}\int_{\mathbb{R}^2}\frac{\chi_E(x)+\chi_E(y)}{\max\{1,|x-y|^2\}}f(x)^2\,dydx \le C\|f_2'\|_{C^1}\|f\|^2_{\tilde{L}^2_{x_0}}.$$

On the other hand, letting $\hat{y} := \min\{|y|, 1\}$, $a_j := f_j(x)$ $(j = 1, 2)$, $a_3 := \max\{a_1, a_2\}$, and $\lambda := 1 + \|f_1'\|_{C^1} + \|f_2'\|_{C^1}$, changing variables back via $y \leftrightarrow x - y$, and using also (2.9) yields

$$\left|\int_{\mathbb{R}} h(x)f(x)\,PV\int_{\mathbb{R}}(x-y)B_0^{+}(x,y)f(y)\,dydx\right|$$
$$\le C\lambda^2\int_{\mathbb{R}^2}\frac{(\chi_E(x)+\chi_E(x-y))\hat{y}|y|\max\{\hat{y},\sqrt{\hat{a}_2}\}\max\{\hat{y}|y|,a_3\}}{\max\{|y|,a_1\}^2\max\{|y|,a_2\}^2}|f(x)|\,|f(x-y)|\,dydx,$$

where we used $|h(x) - h(x-y)| \le \hat{y}(\chi_E(x) + \chi_E(x-y))$ and

$$f_j(x-y) \le C(1 + \|f_j'\|_{C^1})(a_j + \min\{y^2, |y|\}) \le C(1 + \|f_j'\|_{C^1})\max\{\hat{y}|y|, a_j\} \tag{8.3}$$

for $j = 1, 2$. If we employ $|f(x)|\,|f(x-y)| \le f(x)^2 + f(x-y)^2$, split the integral in two accordingly, and change variables $x \leftrightarrow x - y$ in the second, we find that

$$\left|\int_{\mathbb{R}} h(x)f(x)\,PV\int_{\mathbb{R}}(x-y)B_0^{+}(x,y)f(y)\,dydx\right|$$
$$\le C\lambda^2\int_{\mathbb{R}}f(x)^2\int_{\mathbb{R}}\frac{(\chi_E(x)+\chi_E(x-y))\hat{y}|y|\max\{\hat{y},\sqrt{\hat{a}_2}\}\max\{\hat{y}|y|,a_3\}}{\max\{|y|,a_1\}^2\max\{|y|,a_2\}^2}\,dydx,$$

so we need to estimate the inside integral. It is clearly bounded by

$$4\int_0^1\frac{\max\{y,\sqrt{a_3}\}^3}{\max\{y,a_3\}^2}dy + 4\int_1^\infty\frac{1}{y^2}dy \le C$$

for any $x \in \mathbb{R}$, while for $|x - x_0| \ge 5$ it is bounded by

$$\int_{|x-x_0|-4}^{|x-x_0|+4}\frac{1}{y^2}\,dy \le \frac{C}{|x-x_0|^2}.$$

From the above estimates it now follows that

$$\left|\int_{\mathbb{R}} h(x)f(x)\,PV\int_{\mathbb{R}}(x-y)B_0^{\pm}(x,y)f(y)\,dydx\right| \le C\lambda^2\|f\|^2_{\tilde{L}^2_{x_0}}. \tag{8.4}$$

Finally, we will show that

$$\left| \int_{\mathbb{R}} h(x) f(x)^2 \, PV \int_{\mathbb{R}} (x-y) B_0^{\pm}(x,y) \, dy dx \right| \leq C \lambda_\gamma^2 \|f\|^2_{\tilde{L}^2_{x_0}}, \tag{8.5}$$

with $\lambda_\gamma := 1 + \|f_1\|_{C^{2,\gamma}_\gamma} + \|f_2\|_{C^{2,\gamma}_\gamma}$ This and (8.4) then yield

$$|I_6^{\pm}| \leq C(1 + \|f_1\|^2_{C^{2,\gamma}_\gamma} + \|f_2\|^2_{C^{2,\gamma}_\gamma}) \|f\|^2_{\tilde{L}^2_{x_0}}, \tag{8.6}$$

which together with (8.2) implies

$$\frac{d}{dt} \left\| \sqrt{h} \, f(\cdot,t) \right\|^2_{L^2} \leq C \left(1 + \|f_1(\cdot,t)\|^2_{C^{2,\gamma}_\gamma} + \|f_2(\cdot,t)\|^2_{C^{2,\gamma}_\gamma} \right) \|f(\cdot,t)\|^2_{\tilde{L}^2_{x_0}}. \tag{8.7}$$

Uniformity of (8.7) in $x_0 \in \mathbb{R}$ (recall that $h = h_0(\cdot - x_0) =: h_{x_0}$) and

$$\|f(\cdot,t)\|_{\tilde{L}^2} \leq C \sup_{x_0 \in \mathbb{R}} \left\| \sqrt{h_{x_0}} \, f(\cdot,t) \right\|_{L^2}$$

yield

$$\frac{d}{dt} \sup_{x_0 \in \mathbb{R}} \left\| \sqrt{h_{x_0}} \, f(\cdot,t) \right\|_{L^2} \leq C \left(1 + \|f_1(\cdot,t)\|^2_{C^{2,\gamma}_\gamma} + \|f_2(\cdot,t)\|^2_{C^{2,\gamma}_\gamma} \right) \sup_{x_0 \in \mathbb{R}} \left\| \sqrt{h_{x_0}} \, f(\cdot,t) \right\|_{L^2} \tag{8.8}$$

for all $t \in [0,T]$. So Grönwall's inequality shows that $f_1 \equiv f_2$ whenever $f_1(\cdot,0) \equiv f_2(\cdot,0)$, proving the first claim in Theorem 1.1(ii). From $\|\psi\|_{\tilde{H}^3_\gamma} < \infty$ and (2.7) we also obtain (1.11).

To prove (8.5), it clearly suffices to show

$$\left| PV \int_{S \cup (-S)} y B_0^{\pm}(x, x-y) \, dy \right| \leq C \lambda_\gamma^2 \tag{8.9}$$

for any $S \subseteq [0,\infty)$. Using a version of (3.5) with an extra term in the denominator yields

$$y |B_0^+(x,x+y) - B_0^+(x,x-y)| \leq C\lambda B_1(x,y) + C\lambda^2 \sum_{n=2}^{4} B_n(x,y) \tag{8.10}$$

for $y \geq 0$, where (recall (2.9), $a_j := f_j(x)$ $(j = 1,2)$, and $a_3 := \max\{a_1, a_2\}$)

$$B_1(x,y) := \frac{y^2 \left(f_3(x) + f_3(x-y) \right)}{\max\{y,a_1\}^2 \max\{y,a_2\}^2},$$

$$B_2(x,y) := \frac{y^2 \, \max\left\{\hat{y}, \hat{a}_2^{1/2}\right\} \, \max\left\{\hat{y}, \hat{a}_3^{1/2}\right\}}{\max\{y,a_1\}^2 \max\{y,a_2\}^2},$$

$$B_3(x,y) := \frac{y^2 \, \max\left\{\hat{y}, \hat{a}_2^{1/2}\right\} \left(f_3(x) + f_3(x+y) \right) \left(2f_1(x) + f_1(x+y) + f_1(x-y) \right) \max\left\{\hat{y}, \hat{a}_1^{1/2}\right\}}{\left[y^2 + (f_1(x) + f_1(x-y))^2\right] \left[y^2 + (f_1(x) + f_1(x+y))^2\right] \left[y^2 + (f_2(x) + f_2(x-y))^2\right]},$$

$$B_4(x,y) := \frac{y^2 \, \max\left\{\hat{y}, \hat{a}_2^{1/2}\right\} \left(f_3(x) + f_3(x+y) \right) \left(2f_2(x) + f_2(x+y) + f_2(x-y) \right) \max\left\{\hat{y}, \hat{a}_2^{1/2}\right\}}{\left[y^2 + (f_1(x) + f_1(x+y))^2\right] \left[y^2 + (f_2(x) + f_2(x-y))^2\right] \left[y^2 + (f_2(x) + f_2(x+y))^2\right]}.$$

We have

$$\int_0^\infty B_1(x,y) dy \leq C\lambda \int_0^1 \frac{\max\{y^2, a_3\}}{\max\{y,a_3\}^2} \, dy, + \int_1^\infty \frac{2a_3 + C\lambda_\gamma y^{1-\gamma}}{\max\{y,a_3\}^2} \, dy \leq C\lambda_\gamma$$

$$\int_0^\infty B_2(x,y)dy \le \int_0^\infty \frac{\max\{\hat{y}^2,\hat{a}_3\}}{\max\{y,a_3\}^2}\,dy \le C,$$
$$\int_0^\infty B_4(x,y)dy \le \int_0^\infty \frac{\max\{\hat{y}^2,\hat{a}_2\}}{\max\{y,a_2\}^2}\,dy \le C,$$

where in the last line we used $\max\{\hat{y}y,a_j\}\le\max\{y,a_j\}$. When $a_1\ge a_2$, we also obtain

$$\int_0^\infty B_3(x,y)dy \le \int_0^\infty \frac{\max\{\hat{y}^2,\hat{a}_1\}}{\max\{y,a_1\}^2}\,dy \le C,$$

while in the case $a_2\ge a_1$ we have

$$\int_0^\infty B_3(x,y)dy \le \int_0^\infty \frac{\max\{\hat{y},\hat{a}_1^{1/2}\}\max\{\hat{y},\hat{a}_2^{1/2}\}}{\max\{y,a_1\}\max\{y,a_2\}}\,dy \le C$$

because in the last inequality we can assume without loss that $a_2\le 1$, and then use

$$\int_0^{\sqrt{a_1}} \frac{a_1^{1/2}a_2^{1/2}}{\max\{y,a_1\}\max\{y,a_2\}}\,dy \le C,$$
$$\int_{\sqrt{a_1}}^1 \frac{\max\{y,a_2^{1/2}\}}{\max\{y,a_2\}}\,dy \le \int_0^1 \frac{\max\{y,a_2^{1/2}\}}{\max\{y,a_2\}}\,dy \le C.$$

These bounds and (8.10) yield (8.9) when $\pm$ is $+$. To prove (8.9) with $-$, write the numerator of $B_0^-(x,x-y)$ as

$$[f_2'(x)-f_2'(x-y)]\,[f_3(x)-f_3(x-y)-f_3'(x)y]+yf_3'(x)\,[f_2'(x)-f_2'(x-y)]\,.$$

The first term is bounded by $C\lambda^2\hat{y}^2y$, yielding the desired estimate for the fraction involving it directly. The integral involving the second fraction is evaluated analogously to

$$\int_S y|B_0^-(x,x+y)-B_0^-(x,x-y)|dy,$$

and then we use a version of (3.5) with one term in the numerator and two in the denominator, together with the bounds

$$|2f_2'(x)-f_2'(x+y)-f_2'(x-y)|\le 2\|f_2'\|_{C^{1,\gamma}}|\hat{y}|^{1+\gamma},$$
$$|f_2'(x)-f_2'(x-y)|\,|2f_j(x)-f_j(x+y)-f_j(x-y)|\le C\lambda^2\hat{y}^2y.$$

It remains to prove the last claim in Theorem 1.1(ii), which we do at the end of Section 10.

9. Estimates on the Difference of Approximating Solutions

We will construct solutions to (1.5) as $\varepsilon\to 0$ limits of appropriate solutions to (7.1), so we first need to extend the estimates from Section 8 to involve solutions f_1 and f_2 to (7.1) with $\varepsilon\in(0,\frac12)$ and $\varepsilon'\in(0,\frac12)$, respectively. Let again $f:=f_1-f_2$ and drop t in the notation. Let

$$H_f^\pm(x,y):=\frac{y\,(f'(x)\pm f'(x-y))}{y^2+(f(x)\pm f(x-y))^2}, \tag{9.1}$$

so that

$$\partial_t f(x)=\phi_\varepsilon*PV\int_{\mathbb{R}}(H^-_{\phi_\varepsilon*f_1}(x,y)+H^+_{\phi_\varepsilon*f_1}(x,y))dy-\phi_{\varepsilon'}*PV\int_{\mathbb{R}}(H^-_{\phi_{\varepsilon'}*f_2}(x,y)+H^+_{\phi_{\varepsilon'}*f_2}(x,y))dy$$

(we drop PV below). Then

$$\frac{d}{dt}\left\|\sqrt{h}\,f\right\|_{L^2}^2 = 2(I_{5'}^+ + I_{5'}^- + I_{6'}^+ + I_{6'}^- + I_{7'}^+ + I_{7'}^-)$$

where $F_j := \phi_\varepsilon * f_j$ $(j=1,2)$, $F := \phi_\varepsilon * f = F_1 - F_2$,

$$\begin{aligned}
I_{5'}^\pm &:= \int_{\mathbb{R}} h(x)f(x)\,\phi_\varepsilon * \int_{\mathbb{R}} \frac{y\,(F'(x)\pm F'(x-y))}{y^2+(F_1(x)\pm F_1(x-y))^2}\,dydx,\\
I_{6'}^\pm &:= \int_{\mathbb{R}} h(x)f(x)\,\phi_\varepsilon * \int_{\mathbb{R}} yB_5^\pm(x,x-y)\,dydx,\\
I_{7'}^\pm &:= \int_{\mathbb{R}} h(x)f(x)\left(\phi_\varepsilon * \int_{\mathbb{R}} yB_7^\pm(x,y)\,dydx - \phi_{\varepsilon'} * \int_{\mathbb{R}} yB_8^\pm(x,y)\,dydx\right),
\end{aligned}$$

and with $F_3 := F_1 + F_2$ and $F_{2'} := \phi_{\varepsilon'} * f_2$ we let

$$\begin{aligned}
B_5^\pm(x,y) &:= \frac{F_2'(x)\pm F_2'(y)}{(x-y)^2+(F_1(x)\pm F_1(y))^2} - \frac{F_2'(x)\pm F_2'(y)}{(x-y)^2+(F_2(x)\pm F_2(y))^2}\\
&= -\frac{(F_2'(x)\pm F_2'(y))\,(F_3(x)\pm F_3(y))\,(F(x)\pm F(y))}{[(x-y)^2+(F_1(x)\pm F_1(y))^2]\,[(x-y)^2+(F_2(x)\pm F_2(y))^2]} =: B_6^\pm(x,y)\,(F(x)\pm F(y)),\\
B_7^\pm(x,y) &:= \frac{F_2'(x)\pm F_2'(x-y)}{y^2+(F_2(x)\pm F_2(x-y))^2},\\
B_8^\pm(x,y) &:= \frac{F_{2'}'(x)\pm F_{2'}'(x-y)}{y^2+(F_{2'}(x)\pm F_{2'}(x-y))^2}.
\end{aligned}$$

Similarly to (8.2), we obtain

$$I_{5'}^+ + I_{5'}^- \le C(1+\|f_1\|_{C_\gamma^{2,\gamma}}^2)\|f\|_{\tilde{L}^2}^2 \tag{9.2}$$

by applying the argument yielding (7.6) to (f, F_1) in place of $(f_\varepsilon''', \phi_\varepsilon * f_\varepsilon)$.

Integrals $I_{6'}^\pm$ can also be estimated in the same way as $I_6^\pm$ in Section 8 and we obtain

$$|I_{6'}^\pm| \le C(1+\|f_1\|_{C_\gamma^{2,\gamma}}^2 + \|f_2\|_{C_\gamma^{2,\gamma}}^2)\|f\|_{\tilde{L}^2}^2, \tag{9.3}$$

provided we prove the same bound for the integrals

$$\begin{aligned}
J_9^\pm &:= \int_{\mathbb{R}^2} (h(z)-h(x))\phi_\varepsilon(x-z)f(z)\int_{\mathbb{R}} (x-y)\,B_6^\pm(x,y)F(y)\,dydxdz\\
&= \int_{\mathbb{R}^2} f(z)f(v)\int_{\mathbb{R}} (h(z)-h(x))\phi_\varepsilon(x-z)\int_{\mathbb{R}} y\,\phi_\varepsilon(x-y-v)B_6^\pm(x,x-y)\,dydxdzdv
\end{aligned}$$

that arise when symmetrizing the integrals at the start of the argument. We obtain

$$|J_9^\pm| \le C(1+\|F_1'\|_{C^{1,\gamma}}^2 + \|F_2'\|_{C^{1,\gamma}}^2)\|f\|_{\tilde{L}^2}^2 \le C(1+\|f_1'\|_{C^{1,\gamma}}^2 + \|f_2'\|_{C^{1,\gamma}}^2)\|f\|_{\tilde{L}^2}^2$$

via the argument bounding $J_{2'}^\pm$ in Section 7. This involves first bounding the last $dydx$ integral above with $\phi_\varepsilon(x-v)$ in place of $\phi_\varepsilon(x-y-v)$ (see (7.9), but now use (8.9) instead of (7.8)), then bounding the original integral when $|x-v| \ge 2\varepsilon$ (see (7.10)), and finally bounding the integral with $\phi_\varepsilon(x-y-v) - \phi_\varepsilon(x-v)$ when $|x-v| \le 2\varepsilon$ (see (7.11), and again use (8.9) instead of (7.8)).

We finally claim that

$$|I_{7'}^{\pm}| \le C(1+\|f_2\|^2_{C^{2,\gamma}_\gamma})\sqrt{\varepsilon+\varepsilon'}\,\|f\|_{\tilde{L}^2} \tag{9.4}$$

(note that $\|f\|_{\tilde{L}^2}$ is not squared here). Assume that $\varepsilon \ge \varepsilon'$, as both cases are identical. Since $\|\phi_\varepsilon * F - \phi_{\varepsilon'} * G\|_{L^\infty} \le \|(\phi_\varepsilon - \phi_{\varepsilon'}) * F\|_{L^\infty} + \|\phi_\varepsilon * (F-G)\|_{L^\infty} \le \varepsilon\|F'\|_{L^\infty} + \|F-G\|_{L^\infty}$, to prove (9.4), it suffices to show

$$\left|\int_{\mathbb{R}} y\partial_x B_7^{\pm}(x,y)\,dy\right| \le C(1+\|f_2\|^2_{C^{2,\gamma}_\gamma}), \tag{9.5}$$

$$\left|\int_{\mathbb{R}} y(B_7^{\pm}(x,y) - B_9^{\pm}(x,y))\,dy\right| \le C(1+\|f_2\|^2_{C^2_\gamma})\,\sqrt{\varepsilon}, \tag{9.6}$$

$$\left|\int_{\mathbb{R}} y(B_8^{\pm}(x,y) - B_9^{\pm}(x,y))\,dy\right| \le C(1+\|f_2'\|^2_{C^1})\,\sqrt{\varepsilon}, \tag{9.7}$$

where

$$B_9^{\pm}(x,y) := \frac{F_{2'}'(x) \pm F_{2'}'(x-y)}{y^2 + (F_2(x) \pm F_2(x-y))^2}.$$

We have

$$y\partial_x B_7^{\pm}(x,y) = \frac{y(F_2''(x) \pm F_2''(x-y))}{y^2 + (F_2(x) \pm F_2(x-y))^2} - 2\frac{y(F_2'(x) \pm F_2'(x-y))^2(F_2(x) \pm F_2(x-y))}{[y^2 + (F_2(x) \pm F_2(x-y))^2]^2},$$

and from (2.6) we see that

$$\left|\int_{\mathbb{R}} \frac{y\,(F_2''(x) \pm F_2''(x-y))}{y^2 + (F_2(x) \pm F_2(x-y))^2}\,dy\right| \le C(1+\|F_2\|^2_{C^{2,\gamma}_\gamma}).$$

Hence (9.5) will follow once we show

$$\left|\int_{\mathbb{R}} \frac{y(F_2'(x) \pm F_2'(x-y))^2(F_2(x) \pm F_2(x-y))}{[y^2 + (F_2(x) \pm F_2(x-y))^2]^2}\,dy\right| \le C(1+\|F_2'\|^2_{C^1}).$$

This is immediate when $\pm$ is $-$, while (2.9) shows that in the other case the left-hand side is bounded by

$$C(1+\|F_2'\|^2_{C^1})\int_0^\infty \frac{\max\{\hat{y}^2, F_2(x)\}}{\max\{y, F_2(x)\}^2}\,dy \le C(1+\|F_2'\|^2_{C^1}).$$

To prove (9.6), we use

$$|(F_2'(x) \pm F_2'(x-y)) - (F_{2'}'(x) \pm F_{2'}'(x-y))| \le C\|f_2''\|_{L^\infty}\,\varepsilon$$

for $|y| \ge \varepsilon$ to see that

$$\int_{-1}^1 |y|\,|B_7^-(x,y) - B_9^-(x,y)|\,dy \le C\|f_2''\|_{L^\infty}\int_0^1 \frac{y\min\{y,\varepsilon\}}{y^2}\,dy \le C\|f_2''\|_{L^\infty}\,\varepsilon\,|\ln\varepsilon|,$$

$$\int_{\varepsilon\le|y|\le 1} |y|\,|B_7^+(x,y) - B_9^+(x,y)|\,dy \le C\|f_2''\|_{L^\infty}\,\varepsilon\,|\ln\varepsilon|, \tag{9.8}$$

while (2.9) shows that

$$\int_0^\varepsilon y|B_j^+(x,y) - B_j^+(x,-y)|\,dy \le C(1+\|f_2'\|^2_{C^1})\int_0^\varepsilon \frac{y\max\left\{y, \sqrt{F_2(x)}\right\}}{\max\{y, F_2(x)\}^2}\,dy \le C(1+\|f_2'\|^2_{C^1})\,\sqrt{\varepsilon}$$

holds for $j = 7, 9$. We note that we gave up $\sqrt{\varepsilon}$ here by only estimating $|F_2'(x) + F_2'(x-y)| \leq 2\|f_2'\|_{L^\infty}$ and $|F_{2'}'(x) + F_{2'}'(x-y)| \leq 2\|f_2'\|_{L^\infty}$ in one of the terms in the integrand, but this will not cause a problem. Finally, we have

$$\left| \int_{|y|\geq 1} \frac{y(F_2'(x) - F_{2'}'(x))}{y^2 + (F_2(x) \pm F_2(x-y))^2}\, dy \right| \leq C\|f_2\|_{C^2_\gamma} \|F_2' - F_{2'}'\|_{L^\infty} \leq C\|f_2\|^2_{C^2_\gamma}\, \varepsilon$$

by (2.3), while integration by parts as in (2.5) yields

$$\left| \int_{|y|\geq 1} \frac{y(F_2'(x-y) - F_{2'}'(x-y))}{y^2 + (F_2(x) \pm F_2(x-y))^2}\, dy \right| \leq C\|F_2 - F_{2'}\|_{L^\infty}(1 + \|F_2'\|_{L^\infty}) \leq C(1 + \|f_2'\|^2_{L^\infty})\, \varepsilon.$$

These estimates collectively yield (9.6).

Inequality (9.7) is proved similarly, using

$$|(F_2(x) \pm F_2(x-y)) - (F_{2'}(x) \pm F_{2'}(x-y))| \leq C\|f_2'\|_{L^\infty}\, \varepsilon$$

and with no need to treat integrals over $\{|y| \geq 1\}$ separately. But there is an extra complication in the bound corresponding to (9.8). Namely, with $G_2(x,y) := F_2(x) + F_2(x-y)$ and $G_{2'}(x,y) := F_{2'}(x) + F_{2'}(x-y)$ we have

$$B_8^+(x,y) - B_9^+(x,y) = \frac{\partial_x G_{2'}(x,y)(G_2(x,y) + G_{2'}(x,y))\,(G_2(x,y) - G_{2'}(x,y))}{[y^2 + G_2(x,y)^2]\,[y^2 + G_{2'}(x,y)^2]}$$

and $|G_2(x,y) - G_{2'}(x,y)| \leq C\|f_2'\|_{L^\infty}\varepsilon$. Using (2.9), it follows that

$$\int_{|y|\geq\varepsilon} |y|\,|B_8^+(x,y) - B_9^+(x,y)|\, dy \leq C(1 + \|f_2'\|^2_{C^1})\, \varepsilon \int_\varepsilon^\infty \frac{\max\{\hat{y}, \sqrt{F_{2'}(x)}\}}{\max\{y, F_2(x)\}\max\{y, F_{2'}(x)\}}\, dy.$$

The last integral is bounded by

$$\int_\varepsilon^\infty \frac{\max\{\hat{y}, \sqrt{F_{2'}(x)}\}}{y\max\{y, F_{2'}(x)\}}\, dy \leq \frac{C}{\sqrt{\varepsilon}},$$

which yields the desired bound.

We can now conclude from (9.2), (9.3), and (9.4) that

$$\frac{d}{dt}\left\|\sqrt{h}\, f\right\|_{L^2} \leq C\left(1 + \|f_1\|^2_{C^{2,\gamma}_\gamma} + \|f_2\|^2_{C^{2,\gamma}_\gamma}\right)\left(\sqrt{\varepsilon + \varepsilon'} + \|f\|_{\tilde{L}^2}\right)$$

when f_1 and f_2 solve (7.1) with $\varepsilon \in (0, \frac{1}{2})$ and $\varepsilon' \in (0, \frac{1}{2})$, respectively, and $f := f_1 - f_2$. Similarly to (8.8), it follows that

$$\frac{d}{dt}\sup_{x_0\in\mathbb{R}}\left\|\sqrt{h_{x_0}}\, f(\cdot,t)\right\|_{L^2} \leq C\left(1 + \sum_{j=1}^2 \|f_j(\cdot,t)\|^2_{C^{2,\gamma}_\gamma}\right)\left(\sqrt{\varepsilon + \varepsilon'} + \sup_{x_0\in\mathbb{R}}\left\|\sqrt{h_{x_0}}\, f(\cdot,t)\right\|_{L^2}\right). \tag{9.9}$$

10. Existence of Solutions

First we claim that the operator in (9.1) satisfies

$$\sup_{x\in\mathbb{R}}\left|PV\int_{\mathbb{R}}(H^-_{F_1}(x,y)+H^+_{F_1}(x,y))\,dy-PV\int_{\mathbb{R}}(H^-_{F_2}(x,y)+H^+_{F_2}(x,y))\,dy\right| \\ \le C(1+\|F_1\|^2_{C^2_\gamma}+\|F_2\|^2_{C^2_\gamma})\|F_1-F_2\|_{C^{1,\gamma}}. \tag{10.1}$$

The inside of the absolute value is

$$\sum_{\pm}PV\int_{\mathbb{R}}\frac{y\,(F'(x)\pm F'(x-y))}{y^2+(F_1(x)\pm F_1(x-y))^2}\,dy+\sum_{\pm}PV\int_{\mathbb{R}}yB^{\pm}_5(x,x-y)\,dy,$$

where $F:=F_1-F_2$, $F_3:=F_1+F_2$, and $B^{\pm}_5$ is from Section 9. The two integrals in the first sum are bounded by $C(1+\|F_1\|_{C^2_\gamma}+\|F_2\|_{C^2_\gamma})\|F\|_{C^{1,\gamma}}$ due to (2.6). The same bound easily holds for the integral with B^-_5, as well as for the one with B^+_5 over $|y|\ge 1$. We write the last integral as $\int_0^1 y|B^+_5(x,x-y)-B^+_5(x,x+y)|\,dy$, and again use (2.9) and a version of (3.5) with five terms. There we always obtain y^2 in the numerator, and this cancels the smallest of $y^2+(F_1(x)+F_1(x\pm y))^2$ and $y^2+(F_2(x)+F_2(x\pm y))^2$ that appear in the denominator, so the integral is bounded by $C(1+\|F_1'\|^2_{C^1}+\|F_2'\|^2_{C^1})\|F\|_{C^1}$ times

$$\int_0^1\frac{\max\{y^2,F_1(x)\}}{\max\{y,F_1(x)\}^2}\,dy+\int_0^1\frac{\max\{y^2,F_2(x)\}}{\max\{y,F_2(x)\}^2}\,dy+\int_0^1\frac{\max\{y,\sqrt{F_1(x)}\}\max\{y,\sqrt{F_2(x)}\}}{\max\{y,F_1(x)\}\max\{y,F_2(x)\}}\,dy\le C$$

(see an estimate at the end of Section 8 for the last integral). This proves (10.1).

Then for each $\varepsilon>0$ and $n\in\mathbb{N}$ we have

$$\left\|\phi_\varepsilon * PV\int_{\mathbb{R}}(H^-_{\phi_\varepsilon*f_1}(x,y)+H^+_{\phi_\varepsilon*f_1}(x,y))\,dy-\phi_\varepsilon * PV\int_{\mathbb{R}}(H^-_{\phi_\varepsilon*f_2}(x,y)+H^+_{\phi_\varepsilon*f_2}(x,y))\,dy\right\|_{C^n} \\ \le C_n(1+\|f_1\|^2_{C^2_\gamma}+\|f_2\|^2_{C^2_\gamma})\,\varepsilon^{-n-2}\|f_1-f_2\|_{L^\infty}.$$

for some (n,γ)-dependent constant C_n. This means that we can apply Picard's existence theorem to (7.1) to obtain its local-in-time solutions f for initial data from the metric space $X_\psi:=\{g\in\phi*\psi+C^2(\mathbb{R})\,\big|\,\inf g>0\}$, where $\psi\ge 0$ with $\|\psi\|_{\tilde H^3_\gamma}<\infty$ is the initial datum from Theorem 1.1. From (2.7) and (7.1) we see that these are C^∞ if so are the initial data, and they can be extended in time as long as they remain uniformly positive and $f(\cdot,t)-\phi*\psi$ remains bounded in $C^2(\mathbb{R})$. This is because

$$\|g\|_{C^2_\gamma}=\|g'\|_{C^1}+\|g\|_{\ddot C^{1-\gamma}}\le\|(g-\phi*\psi)'\|_{C^1}+\|(\phi*\psi)'\|_{C^1}+\|\phi*\psi\|_{\ddot C^{1-\gamma}}+2\|g-\phi*\psi\|_{L^\infty}$$

and $\|\psi\|_{C^2_\gamma}<\infty$. For any $\varepsilon>0$, let f_ε be such C^∞ solution with

$$f_\varepsilon(\cdot,0)=\phi_\varepsilon*\psi+2\varepsilon\in X_\psi. \tag{10.2}$$

Let $T_\varepsilon\in(0,\infty]$ be its time of existence and let $F_\varepsilon:=\phi_\varepsilon*f_\varepsilon$.

Recall that $h_{x_0}:=h_0(\cdot-x_0)$ for $x_0\in\mathbb{R}$ and let

$$\|g\|_{\tilde H_\gamma}:=\|g\|_{\ddot C^\gamma}+\sup_{x_0\in\mathbb{R}}\left\|\sqrt{h_{x_0}}\,g'''\right\|_{L^2}.$$

Then

$$\max\left\{\|g\|_{\tilde H^3_\gamma},\|g\|_{C^{2,\gamma}_\gamma}\right\}\le C\|g\|_{\tilde H_\gamma}\le C\|g\|_{\tilde H^3_\gamma} \tag{10.3}$$

because we assume that $\gamma \in (0, \frac{1}{2}]$ (recall (2.2)), so from (2.7) and (7.12) we see that

$$\frac{d}{dt}\|f_\varepsilon(\cdot,t)\|_{\tilde{H}_\gamma} \le C\left(1+\|f_\varepsilon(\cdot,t)\|^4_{C^{2,\gamma}_\gamma}\right)\|f_\varepsilon(\cdot,t)\|_{\tilde{H}_\gamma} \le C\left(1+\|f_\varepsilon(\cdot,t)\|^4_{\tilde{H}_\gamma}\right)\|f_\varepsilon(\cdot,t)\|_{\tilde{H}_\gamma} \quad (10.4)$$

as long as $t < T_\varepsilon$ and $\varepsilon \le \frac{1}{2}\|f'_\varepsilon(\cdot,t)\|^{-1}_{L^\infty}$. Let $T > 0$ be such that the solution to $z' = C(1+z^4)z$ with $z(0) = \|\psi\|_{\tilde{H}_\gamma} + 1$ satisfies $z(T) = M := \|\psi\|_{\tilde{H}_\gamma} + 2$, and let $T'_\varepsilon := \min\{T_\varepsilon, T\} > 0$. Then (10.4) implies that for all $\varepsilon \in (0, (CM)^{-1})$ we have

$$\sup_{t\in[0,T'_\varepsilon)} \|f_\varepsilon(\cdot,t)\|_{\tilde{H}_\gamma} \le M. \quad (10.5)$$

It remains to check that f_ε stays uniformly positive for some uniform time. Fix any small enough $\varepsilon > 0$ as above and $t \in [0, T'_\varepsilon)$. Then for any $x \in \mathbb{R}$ we have (again dropping t)

$$PV\int_{\mathbb{R}} (H^-_{F_\varepsilon}(x,y) + H^+_{F_\varepsilon}(x,y))\,dy = I^+_8(x) + I^-_8(x) - 4I_9(x), \quad (10.6)$$

where

$$I^\pm_8(x) := PV\int_{\mathbb{R}} \frac{y\,F'_\varepsilon(x)}{y^2 + (F_\varepsilon(x) \pm F_\varepsilon(x-y))^2}dy,$$
$$I_9(x) := \int_{\mathbb{R}} \frac{y\,F'_\varepsilon(x-y)F_\varepsilon(x)F_\varepsilon(x-y)}{[y^2 + (F_\varepsilon(x) + F_\varepsilon(x-y))^2]\,[y^2 + (F_\varepsilon(x) - F_\varepsilon(x-y))^2]}dy.$$

From (2.3) we see that

$$|I^\pm_8(x)| \le C\|F_\varepsilon\|_{C^2_\gamma}|F'_\varepsilon(x)|. \quad (10.7)$$

Writing $I_9(x)$ as $\int_0^\infty y(B(x,x-y) - B(x,x+y))\,dy$ and again using (2.9) and a version of (3.5) with two terms in the denominator, we see that

$$|I_9(x)| \le C(1+\|F'_\varepsilon\|^5_{C^1})|F_\varepsilon(x)|. \quad (10.8)$$

The two "worst" terms in the latter estimate are (with the factor $F_\varepsilon(x)$ removed)

$$\left|\int_0^\infty \frac{y\,F'_\varepsilon(x+y)F_\varepsilon(x+y)\,(2F_\varepsilon(x) + F_\varepsilon(x+y) + F_\varepsilon(x-y))\,(F_\varepsilon(x+y) - F_\varepsilon(x-y))}{[y^2 + (F_\varepsilon(x) + F_\varepsilon(x-y))^2]\,[y^2 + (F_\varepsilon(x) + F_\varepsilon(x+y))^2]\,[y^2 + (F_\varepsilon(x) - F_\varepsilon(x-y))^2]}\,dy\right|$$
$$\le C(1+\|F'_\varepsilon\|^2_{C^1})\int_0^\infty \frac{\max\left\{\hat{y}^2, F_\varepsilon(x)\right\}}{\max\{y, F_\varepsilon(x)\}^2}\,dy \le C(1+\|F'_\varepsilon\|^2_{C^1}),$$

which holds because

$$F_\varepsilon(x+y)\,(2F_\varepsilon(x) + F_\varepsilon(x+y) + F_\varepsilon(x-y)) \le 2(F_\varepsilon(x) + F_\varepsilon(x+y))^2 + (F_\varepsilon(x) + F_\varepsilon(x-y))^2,$$

and

$$\left|\int_0^\infty \frac{y\,F'_\varepsilon(x+y)F_\varepsilon(x+y)\,(2F_\varepsilon(x) - F_\varepsilon(x+y) - F_\varepsilon(x-y))\,(F_\varepsilon(x+y) - F_\varepsilon(x-y))}{[y^2 + (F_\varepsilon(x) + F_\varepsilon(x+y))^2]\,[y^2 + (F_\varepsilon(x) - F_\varepsilon(x-y))^2]\,[y^2 + (F_\varepsilon(x) - F_\varepsilon(x+y))^2]}\,dy\right|$$
$$\le C(1+\|F'_\varepsilon\|^5_{C^1})\left[\int_0^1 \frac{\max\left\{y^2, F_\varepsilon(x)\right\}^2}{\max\{y, F_\varepsilon(x)\}^2}\,dy + \int_1^\infty \frac{1}{y^2}\,dy\right] \le C(1+\|F'_\varepsilon\|^5_{C^1}).$$

From (10.6), (10.7), and (10.8) we thus obtained

$$\left|PV\int_{\mathbb{R}} (H^-_{F_\varepsilon}(x,y) + H^+_{F_\varepsilon}(x,y))\,dy\right| \le C(1+\|F_\varepsilon\|_{C^2_\gamma})\,(\|F_\varepsilon\|^4_{C^2_\gamma}F_\varepsilon(x) + |F'_\varepsilon(x)|). \quad (10.9)$$

Let now $m(t) := \inf f_\varepsilon(\cdot,t) > 0$ and consider any $x' \in \mathbb{R}$ such that $f_\varepsilon(x',t) \le m(t) + \varepsilon^2$. Then (2.9) applied to $f_\varepsilon(\cdot,t) - m(t) \ge 0$ shows that

$$\sup_{|x-x'|\le 2\varepsilon} |\partial_x f_\varepsilon(x,t)| \le C(1 + \|\partial_x f_\varepsilon(\cdot,t)\|_{C^1})\,\varepsilon,$$

so (10.9) and (7.1) yield

$$|\partial_t f_\varepsilon(x',t)| \le C(1 + \|f_\varepsilon(\cdot,t)\|^5_{C^2_\gamma})\,(m(t)+\varepsilon). \tag{10.10}$$

It follows that

$$m'(t) \ge -C(1 + \|f_\varepsilon(\cdot,t)\|^5_{C^2_\gamma})\,(m(t)+\varepsilon),$$

which together with $m(0) \ge 2\varepsilon$ (recall (10.2)) yields $m(t) \ge \varepsilon$ for all $t \le \min\{T_\varepsilon, T'\}$, where $T' := \min\{T, \frac{1}{3C(1+M^5)}\} > 0$. This, (10.5), and (10.3) show that f_ε can be continued within X_ψ past time T_ε if $T_\varepsilon \le T'$, hence $T_\varepsilon > T'$ for all small $\varepsilon > 0$.

From (10.5) and (10.9) we see that $\tilde f_\varepsilon := f_\varepsilon - \phi * \psi \in L^\infty([0,T'];\tilde H^3(\mathbb{R}))$. Then (9.9) and $f_\varepsilon \ge 0$ show that $\tilde f_\varepsilon$ converges to some $\tilde f \ge -\phi*\psi$ in $L^\infty([0,T'];\tilde L^2(\mathbb{R}))$ as $\varepsilon \to 0$ because we also have $\tilde f_\varepsilon(\cdot,0) \to \psi - \phi*\psi$ in $L^\infty(\mathbb{R}) \subseteq \tilde L^2(\mathbb{R})$ (this yields $\tilde f(\cdot,0) = \psi - \phi*\psi$ as well). Then (10.5) shows that $f := \tilde f + \phi*\psi \ge 0$ satisfies

$$\sup_{t\in[0,T']} \|f(\cdot,t)\|_{\tilde H_\gamma} \le M, \tag{10.11}$$

and $\tilde f_\varepsilon \to \tilde f$ holds also in $L^\infty([0,T'];C^2(\mathbb{R}))$. Then $F_\varepsilon - \phi*\psi \to \tilde f$ in this space, so (10.1) shows that

$$\lim_{\varepsilon\to 0} \|\partial_t f_\varepsilon - G\|_{L^\infty(\mathbb{R}\times[0,T'])} = 0, \tag{10.12}$$

where G is the right-hand side of (1.5). Thus $G \in C(\mathbb{R}\times[0,T'])$ because all $\partial_t f_\varepsilon$ are smooth. But then (10.12) and pointwise convergence $f_\varepsilon \to f$ shows that $\partial_t f$ exists and equals G, which means that $f \ge 0$ is a classical solution to (1.5) that satsfies (10.11). We can now continue it on some time interval $[0,T_\psi)$ with $T_\psi \in (0,\infty]$ maximal such that

$$\sup_{t\in[0,T]} \|f(\cdot,t)\|_{\tilde H_\gamma} < \infty \tag{10.13}$$

for all $T \in (0,T_\psi)$. This and (10.3) yield (1.8). And since (10.4) implies

$$\frac{d}{dt}\|f(\cdot,t)\|_{\tilde H_\gamma} \le C\left(1 + \|f(\cdot,t)\|^4_{C^{2,\gamma}_\gamma}\right)\|f(\cdot,t)\|_{\tilde H_\gamma} \tag{10.14}$$

at $t=0$, this then holds for any $t \in [0,T_\psi)$ because we can apply the above arguments with initial time t and initial condition $f(\cdot,t)$ (the newly obtained $\tilde f_\varepsilon$ converge to the same $\tilde f$ in $L^\infty([t,T'];C^2(\mathbb{R}))$ by the uniqueness claim in Theorem 1.1(ii), proved in Section 8).

Next, for any $\gamma' \in (0,\gamma]$ we have $\|\cdot\|_{\tilde H_{\gamma'}} \le \|\cdot\|_{\tilde H_\gamma}$, so f is also the unique solution to (1.5) on the time interval $[0,T_\psi)$ when γ is replaced by γ' (this is still the T_ψ corresponding to γ). Then (10.14) holds also for γ' and any $t \in [0,T_\psi)$. Assume now that $T_\psi < \infty$ but (1.10) fails. Since (2.7) shows that for $\gamma'' \in \{\gamma,\gamma'\}$ we have

$$\frac{d}{dt}\|f(\cdot,t)\|_{\ddot C^{\gamma''}} \le C\left(1 + \|f_x(\cdot,t)\|^2_{C^1}\right)(1 + \|f(\cdot,t)\|_{\ddot C^{\gamma''}}),$$

it follows that $\sup_{t\in[0,T_\psi)}\|f(\cdot,t)\|_{\ddot{C}^{\gamma''}}<\infty$. This with $\gamma''=\gamma$, together with the definition of T_ψ and (10.14) yield

$$\lim_{t\to T_\psi}\|f_{xxx}(\cdot,t)\|_{\tilde{L}^2}=\infty.$$

But then also

$$\lim_{t\to T_\psi}\|f(\cdot,t)\|_{\tilde{H}_{\gamma'}}=\infty,$$

so (10.14) with γ' in place of γ forces (1.10) to hold, a contradiction. This now also proves (1.10) for all $\gamma'\in(0,1)$, and shows that T_ψ is independent of γ.

Since f can clearly be continued at least as long as the solution z above that defined T exists, (10.3) also yields (1.7). Finally, (10.1) with $F_1:=f(\cdot+z,t)$ $(z\neq 0)$ and $F_2:=f(\cdot,t)$, shows that for any $t\in[0,T_\psi)$ we have

$$\|f_t(\cdot,t)\|_{W^{1,\infty}}\le\|f_t(\cdot,t)\|_{L^\infty}+C\left(1+\|f(\cdot,t)\|^2_{C^2_\gamma}\right)\|f_x(\cdot,t)\|_{C^{1,\gamma}}.$$

Then (2.7), (1.8), and (10.3) yield (1.9), which concludes the proof of Theorem 1.1(i).

Proof of the last claim in Theorem 1.1(ii). All the arguments in Sections 3–7 trivially extend to the case $h\equiv 1$, with all $\|\cdot\|_{\tilde{L}^2}$ and $\|\cdot\|_{\tilde{L}^2_{x_0}}$ replaced by $\|\cdot\|_{L^2}$, because then we always have $h(x)-h(y)\equiv 0\equiv h'(x)$ and so terms with these factors vanish. Taking $\varepsilon\to 0$ in the corresponding version of (7.12) yields (dropping t again)

$$\frac{d}{dt}\|f'''\|^2_{L^2}\le C(1+\|f\|^4_{C^{2,\gamma}_\gamma})\left(\|f'\|^2_{L^2}+\|f''\|^2_{L^2}+\|f'''\|^2_{L^2}\right)\tag{10.15}$$

on the time interval $[0,T_\psi)$, similarly to (10.14).

Let next $\psi_0:=\phi*(\psi_{-\infty}\chi_{(-\infty,0)}+\psi_\infty\chi_{[0,\infty)})$ and $f_0(x,t):=f(x,t)-\psi_0(x)$. Then

$$\begin{aligned}\frac{d}{dt}\|f_0\|^2_{L^2}&=2\sum_{\pm}\int_{\mathbb{R}}f_0(x)\,PV\int_{\mathbb{R}}\frac{y\left(f_0'(x)\pm f_0'(x-y)\right)}{y^2+(f(x)\pm f(x-y))^2}\,dydx\\&\quad+2\sum_{\pm}\int_{\mathbb{R}}f_0(x)\,PV\int_{\mathbb{R}}\frac{y\left(\psi_0'(x)\pm\psi_0'(x-y)\right)}{y^2+(f(x)\pm f(x-y))^2}\,dydx.\end{aligned}$$

The arguments in Section 3 estimating $I_1^\pm$, with $h\equiv 1$, show that the first sum on the right-hand side is bounded above by $C(1+\|f\|^2_{C^{2,\gamma}_\gamma})\|f_0\|^2_{L^2}$. From $\|\psi_0\|_{C^2}\le C|\psi_\infty-\psi_{-\infty}|$ and (2.6) we see that the inside integrals in the second sum are bounded by $C(1+\|f\|_{C^2_\gamma})|\psi_\infty-\psi_{-\infty}|$ for all $x\in\mathbb{R}$. But they are also bounded by $C|\psi_\infty-\psi_{-\infty}||x|^{-1}$ when $|x|\ge 2$ because $\psi_0'\equiv 0$ on $\mathbb{R}\setminus(-1,1)$. Therefore

$$\frac{d}{dt}\|f_0\|^2_{L^2}\le C(1+\|f\|^2_{C^{2,\gamma}_\gamma})\|f_0\|_{L^2}(\|f_0\|_{L^2}+|\psi_\infty-\psi_{-\infty}|),$$

which together with (10.15) yields

$$\frac{d}{dt}(\|f'''\|_{L^2}+\|f_0\|_{L^2})\le C(1+\|f\|^4_{C^{2,\gamma}_\gamma})\left(\|f'''\|_{L^2}+\|f_0\|_{L^2}+|\psi_\infty-\psi_{-\infty}|\right).\tag{10.16}$$

From this, (1.8), and $\psi_0\in H^3(\mathbb{R})$ we get (1.12) with ψ_0 in place of ψ, and then (1.12) holds as well.

11. Proof of Theorem 1.2

Proof of (ii). Note that (10.10), $\tilde f_\varepsilon(\cdot,t)\to\tilde f(\cdot,t)$ in $C^2(\mathbb{R})$, $\partial_t\tilde f_\varepsilon(\cdot,t)\to\tilde f_t(\cdot,t)$ in $L^\infty(\mathbb{R})$, and (2.7) show that for all $t\in[0,T_\psi)$ we have

$$\left|\frac{d}{dt}\inf f(\cdot,t)\right|\le C\min\left\{(1+\|f(\cdot,t)\|^5_{C^2_\gamma})\inf f(\cdot,t),\ (1+\|f_x(\cdot,t)\|_{L^\infty})\|f(\cdot,t)\|_{C^2_\gamma}\right\}. \tag{11.1}$$

This, (10.3), and (10.13) imply that $\inf f(\cdot,t)=0$ for all $t\in[0,T_\psi)$ if $\inf\psi=0$.

Proof of (i). Since

$$\frac{f(x)\pm f(x-y)\pm y\,f'(x-y)}{y^2+(f(x)\pm f(x-y))^2}=-\frac{d}{dy}\arctan\frac{f(x)\pm f(x-y)}{y},$$

we see that

$$\int_{\mathbb{R}}\frac{f(x)\pm f(x-y)\pm y\,f'(x-y)}{y^2+(f(x)\pm f(x-y))^2}\,dy=\begin{cases}\pi & \pm\text{ is }+\text{ and }f(x)>0\\ 0 & \pm\text{ is }-\text{ or }f(x)=0\end{cases}$$

for $f\ge0$ with $\|f\|_{C^2_\gamma(\mathbb{R})}<\infty$. Hence (1.5) can be equivalently written as (1.16).

Assume that $M(t):=\sup f(\cdot,t)>0$ for some $t\in[0,T_\psi)$ (if $M(t)=0$, then $f(\cdot,t')\equiv0$ for all $t'\in[t,T_\psi)$). Let $\delta\in(0,\sqrt{M(t)})$ and consider any $x\in\mathbb{R}$ such that $f(x,t)\ge M(t)-\delta^2$. Then (2.3) and (2.9) for the function $M(t)-f(\cdot,t)$ show that (we again drop t)

$$\left|PV\int_{\mathbb{R}}\frac{yf'(x)}{y^2+(f(x)\pm f(x-y))^2}dy\right|\le C(1+\|f\|^2_{C^2_\gamma})\delta, \tag{11.2}$$

and we clearly have

$$-\int_{|y|\ge\delta}\frac{f(x)-f(x-y)}{y^2+(f(x)-f(x-y))^2}dy+\int_{|y|\ge\delta}\frac{f(x)-f(x-y)}{y^2+(f(x)+f(x-y))^2}dy\le\int_{|y|\ge\delta}\frac{\delta^2}{y^2}=2\delta. \tag{11.3}$$

Writing

$$f(x)-f(x-y)=[f(x)-f(x-y)-f'(x)y]+f'(x)y$$

and then using (2.3) and (2.9) yields

$$\left|\int_{-\delta}^{\delta}\frac{f(x)-f(x-y)}{y^2+(f(x)\pm f(x-y))^2}dy\right|\le C(1+\|f\|^2_{C^2_\gamma})\delta. \tag{11.4}$$

Finally,

$$-\int_{\mathbb{R}}\frac{2f(x)}{y^2+(f(x)+f(x-y))^2}dy\le-\int_{\mathbb{R}}\frac{2f(x)}{y^2+(2f(x)+\delta^2)^2}dy=-\pi\,\frac{2f(x,t)}{2f(x,t)+\delta^2}, \tag{11.5}$$

so from (11.2)–(11.5) and (1.16) we obtain

$$f_t(x)\le C(1+\|f\|^2_{C^2_\gamma}+\delta M(t)^{-1})\delta.$$

This holds uniformly in $t\in[0,T]$ for any $T<T_\psi$, so taking $\delta\to0$ shows that

$$\limsup_{t'\to t^+}\frac{M(t')-M(t)}{t'-t}\le0$$

for each $t\in[0,T_\psi)$. Hence M is non-increasing on $[0,T_\psi)$.

Assume now that $m(t) := \inf f(\cdot,t) > 0$ for some $t \in [0,T_\psi)$ (we already proved that if $m(t)=0$, then $m(t')=0$ for all $t' \in [t,T_\psi)$). Let $\delta \in (0,\sqrt{m(t)})$ and consider any $x \in \mathbb{R}$ such that $f(x,t) \le m(t)+\delta^2$. Then (dropping t) we again have (11.2) and (11.4), while (11.3) becomes

$$-\int_{|y|\ge\delta} \frac{f(x)-f(x-y)}{y^2+(f(x)-f(x-y))^2}dy + \int_{|y|\ge\delta} \frac{f(x)-f(x-y)}{y^2+(f(x)+f(x-y))^2}dy \ge -\int_{|y|\ge\delta}\frac{\delta^2}{y^2} = -2\delta$$

and (11.5) becomes

$$-\int_{\mathbb{R}} \frac{2f(x)}{y^2+(f(x)+f(x-y))^2}dy \ge -\int_{\mathbb{R}} \frac{2f(x)}{y^2+(2f(x)-\delta^2)^2}dy = -\pi\,\frac{2f(x,t)}{2f(x,t)-\delta^2}.$$

So we now obtain

$$f_t(x) \ge -C(1+\|f\|^2_{C^2_\gamma}+\delta m(t)^{-1})\delta,$$

and conclude as above that m is non-decreasing on $[0,T_\psi)$.

Proof of (iii). Let $f_0 := f-\tilde f$ and

$$A(t) := \sup_{x_0\in\mathbb{R}} (1+(\mu-|x_0|)_+)\left\|\sqrt{h_{x_0}}\,f_0(\cdot,t)\right\|^2_{L^2}$$

for $t \in [0,\min\{T_\psi,T_{\tilde\psi}\})$. Then

$$\|f_0(\cdot,t)\|^2_{\tilde L^2_{x_0}} \le \sum_{n=1}^{\lfloor\frac12(\mu-|x_0|)_+\rfloor} \frac{CA(t)n^{-2}}{1+(\mu-|x_0|)_+} + \sum_{n\ge\lfloor\frac12(\mu-|x_0|)_+\rfloor} CA(t)n^{-2} \le \frac{CA(t)}{1+(\mu-|x_0|)_+}$$

for all $x_0\in\mathbb{R}$. From this and (8.7) we see that

$$A'(t) \le C\left(1+\|f(\cdot,t)\|^2_{C^{2,\gamma}_\gamma}+\|\tilde f(\cdot,t)\|^2_{C^{2,\gamma}_\gamma}\right)A(t),$$

and the claim follows.

12. The Proof of Theorem 1.3

Proof of (i). Derivation of (1.13) and the proof of Theorem 1.3 are already contained in Sections 2–11, if we simply ignore all the arguments involving integrals resulting from the second term in (1.5) (which simplifies the proof considerably). This includes estimates on all integrals with superscript + and the argument in Section 10 showing that f_ε remains positive for a uniformly positive time. Note that these were the only places where we used (2.9), other than the proof of Theorem 1.2(i) (where it applies even for (1.13), and where we also ignore all the integrals involving $f(x)+f(x-y)$).

Proof of (ii). For the Muskat Problem on the strip $\mathbb{R}\times(0,l)$, we consider (1.4) with the Laplacian on $\mathbb{R}\times(0,l)$, which means that with $\mathbf{y}_n := (y_1, y_2-2ln)$ we now have

$$u(\mathbf{x},t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}\times(0,l)}\left(\frac{(\mathbf{x}-\mathbf{y}_n)^\perp}{|\mathbf{x}-\mathbf{y}_n|^2}-\frac{(\mathbf{x}-\bar{\mathbf{y}}_n)^\perp}{|\mathbf{x}-\bar{\mathbf{y}}_n|^2}\right)\rho_{x_1}(\mathbf{y},t)\,d\mathbf{y}. \tag{12.1}$$

The derivation of (1.5) in Section 2, together with assuming that $\rho_1 - \rho_0 = 2\pi$, now yields (1.14) with Θ_l from (1.15) when we also use that

$$\sum_{n\in\mathbb{Z}} \frac{y}{y^2+(r-2ln)^2} = \frac{\pi}{l}\sum_{n\in\mathbb{Z}} \frac{\frac{\pi y}{l}}{(\frac{\pi y}{l})^2+(\frac{\pi r}{l}-2\pi n)^2} = \frac{\pi}{l}\Theta_\pi(\tfrac{\pi y}{l},\tfrac{\pi r}{l}) = \Theta_l(y,r) \tag{12.2}$$

(this derivation, including the proof of the middle equality in (12.2), appears in [12]). Let us assume for simplicity that $l=\pi$, as the general case is identical, and define

$$\begin{aligned}
\Theta_0^-(y,r) &:= \frac{y}{y^2+r^2}\\
\Theta_0^+(y,r) &:= \frac{y}{y^2+r^2} + \frac{y}{y^2+(2\pi-r)^2},\\
\Theta^-(y,r) &:= \Theta_\pi(y,r) - \Theta_0^-(y,r) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{y}{y^2+(r-2\pi n)^2},\\
\Theta^+(y,r) &:= \Theta_\pi(y,r) - \Theta_0^+(y,r) = \sum_{n\in\mathbb{Z}\setminus\{0,1\}} \frac{y}{y^2+(r-2\pi n)^2}.
\end{aligned} \tag{12.3}$$

The reason for this is that when $r = f(x)-f(x-y)$ (we again drop t from the notation), then denominators of all the terms in the first sum in (12.2) are no less than π^2, except for the one with $n=0$. And when $r=f(x)+f(x-y)$, then the same is true for all $n\neq 0,1$.

We now separate the right-hand side of (1.14) accordingly when replicating Sections 3-6 for (1.14). The integrals $I_j^\pm$ $(j=1,2,3,4)$, arising from differentiation of

$$PV\int_{\mathbb{R}} (f_x(x)\pm f_x(x-y))\,\Theta_0^\pm(y, f(x)\pm f(x-y))\,dy$$

in x are the same as in Section 2, but those with $\pm$ being $+$ each have a second term that comes from $\frac{y}{y^2+(\pi-f(x)+\pi-f(x-y))^2}$. All the estimates in Sections 3–6 carry over to this term, with $\pi-f$ in place of f, and for the integral

$$L_1^0 := \int_{\mathbb{R}^2} \frac{h(y)(g(x)-g(y))^2}{(x-y)^2+(\pi-f(x)+\pi-f(y))^2}\,dydx$$

(which is analogous to L_1^+) we also obtain (recall that $h=h_{x_0}$ and $g=f'''$)

$$L_1^+ + L_1^0 - L_1^- \le \int_{\mathbb{R}^2} \frac{h(y)(g(x)-g(y))^2}{(x-y)^2+\pi^2}\,dydx \le C\|g\|^2_{\tilde{L}^2_{x_0}}$$

instead of the first inequality in (3.10), which holds because $0\le f\le\pi$ implies

$$\begin{aligned}
\min\{f(x)+f(y), 2\pi-(f(x)+f(y))\} &\ge |f(x)-f(y)|,\\
\max\{f(x)+f(y), 2\pi-(f(x)+f(y))\} &\ge \pi.
\end{aligned}$$

On the other hand, all the integrals arising from differentiation of

$$PV\int_{\mathbb{R}} (f_x(x)\pm f_x(x-y))\,\Theta^\pm(y, f(x)\pm f(x-y))\,dy$$

in x three times can be estimated much more easily. First, those involving fourth derivatives of f (analogous to $J_1^\pm$ and $J_2^\pm$) are rewritten in terms of only third (and lower order) derivatives

of f via integration by parts (cf. (3.2) and (3.9)):

$$\begin{aligned}\int_{\mathbb{R}} h(x)g(x)\, PV \int_{\mathbb{R}} g'(x)\, \Theta^{\pm}(y, f(x) \pm f(x-y))\, dydx \\ = -\frac{1}{2}\int_{\mathbb{R}} h(x)g(x)^2\, PV \int_{\mathbb{R}} \frac{d}{dx}\Theta^{\pm}(y, f(x) \pm f(x-y))\, dydx \\ -\frac{1}{2}\int_{\mathbb{R}} h'(x)g(x)^2\, PV \int_{\mathbb{R}} \Theta^{\pm}(y, f(x) \pm f(x-y))\, dydx \end{aligned} \tag{12.4}$$

and

$$\begin{aligned}\int_{\mathbb{R}} h(x)g(x)\, PV \int_{\mathbb{R}} g'(y)\, \Theta^{\pm}(x-y, f(x) \pm f(y))\, dydx \\ = -\int_{\mathbb{R}} h(x)g(x)\, PV \int_{\mathbb{R}} g(y)\, \frac{d}{dy}\Theta^{\pm}(x-y, f(x) \pm f(y))\, dydx.\end{aligned}$$

The other terms are left as they are, and then in each term we separately estimate the integrals over $|y| \leq 1$ and over $|y| \geq 1$ (in the last integral we first change variables $y \leftrightarrow x-y$). For the former, we use the sum form of $\Theta^{\pm}$ from (12.3) because each of these sums (for each appearing derivative of $\Theta^{\pm}$) is clearly uniformly bounded in $|y| \leq 1$. On the other hand, the integrals over $|y| \geq 1$ are estimated using the non-sum form of $\Theta^{\pm}$ from (12.3), similarly to the analogous estimates involving derivatives of $\frac{y}{y^2+(f(x)\pm f(x-y))^2}$ at $|y| \geq 1$. This time we even have bounded f, which makes the estimation easier, with the only difference being that $\Theta_\pi(y,r) \to \pm\frac{1}{2}$ (instead of 0) as $y \to \pm\infty$. But since derivatives of $\Theta_\pi(y,r)$ decay rapidly to 0 as $|y| \to \infty$, this is only relevant to the only term where $\Theta^{\pm}$ is not differentiated, namely (12.4). The part of it that comes from Θ_π is

$$-\frac{1}{2}\int_{\mathbb{R}} h'(x)g(x)^2\, PV \int_{|y|\geq 1} \Theta_\pi(y, f(x) \pm f(x-y))\, dydx$$

and this is bounded by $C\|g\|^2_{\tilde{L}^2_{x_0}}$ due to

$$\sup_{r\in\mathbb{R}} \left|\Theta_\pi(y,r) - \frac{1}{2}\right| \leq \frac{1}{e^{|y|}-2}$$

whenever $|y| \geq 1$.

It follows that we again obtain (2.12), and Sections 7–11 then extend to (1.14) in a straightforward manner, except for the proof of Theorem 1.2(i). The only adjustment needed is in (10.2), where we instead choose

$$f_\varepsilon(\cdot, 0) = (1 - 4\varepsilon\pi^{-1})\, \phi_\varepsilon * \psi + 2\varepsilon,$$

which means that we also have $\sup f_\varepsilon(\cdot, 0) \leq \pi - 2\varepsilon$. And then the proof of the last claim in Theorem 1.3(ii) (with $l = \pi$) is also virtually identical to that of Theorem 1.2(ii).

It remains to prove the claim of Theorem 1.2(i) for (1.14). When $\psi - \frac{l}{2} \in H^3(\mathbb{R})$ and $\|\psi - \frac{l}{2}\|_{L^\infty} < \frac{l}{2}$, then this was proved in [12, Theorem 5]. To obtain the result for general $\psi \in \tilde{H}^3(\mathbb{R})$ with $0 \leq \psi \leq l$, it suffices to apply Theorem 1.2(iii) for (1.14) with $\tilde{\psi}$ being $\tilde{\psi}_{n,\varepsilon} := (1-\varepsilon)(\psi - \frac{l}{2})\chi_n + \frac{l}{2}$ and $\mu := n$, where χ_n is a smooth characteristic function of $[-n, n]$. Then Theorem 1.2(i) for (1.14) follows from this and [12, Theorem 5] applied to the

solutions with initial data $\tilde{\psi}_{n,\varepsilon}$ (after taking first $n \to \infty$ and then $\varepsilon \to 0$) because we have (1.7) and (1.8) holds uniformly in (n, ε).

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