MULTIDIMENSIONAL TRANSITION FRONTS FOR FISHER-KPP REACTIONS

AMIR ALWAN, ZONGLIN HAN, JESSICA LIN, ZIJIAN TAO,
AND ANDREJ ZLATOŠ

Abstract. We study entire solutions to homogeneous reaction-diffusion equations in several dimensions with Fisher-KPP reactions. Any entire solution $0 < u < 1$ is known to satisfy

$$\lim_{t\to-\infty} \sup_{|x|\le c|t|} u(t,x) = 0 \qquad \text{for each } c < 2\sqrt{f'(0)}\,,$$

and we consider here those satisfying

$$\lim_{t\to-\infty} \sup_{|x|\le c|t|} u(t,x) = 0 \qquad \text{for some } c > 2\sqrt{f'(0)}\,.$$

When f is C^2 and concave, our main result provides an almost complete characterization of transition fronts as well as transition solutions with bounded width within this class of solutions.

1. Introduction

In this paper we study entire solutions of reaction-diffusion equations

$$u_t = \Delta u + f(u) \qquad \text{on } \mathbb{R}\times\mathbb{R}^d, \tag{1.1}$$

with Fisher-KPP reaction functions $f \in C^{1+\gamma}([0,1])$ for some $\gamma > 0$. Specifically, we also assume that

$$f(0) = f(1) = 0, \qquad f'(0) = 1, \qquad 0 < f(u) \le u \text{ on } (0,1). \tag{1.2}$$

(AA) Booth School of Business, University of Chicago, 5807 S. Woodlawn Avenue, Chicago, IL 60637

(JL) Department of Mathematics and Statistics, McGill University, Burnside Hall, 805 Sherbrooke Street West, Montreal, QC H3A 0B9

(ZT) Department of Mathematics, California Institute of Technology, 1200 E. California Blvd., Pasadena, CA 91125

(ZH, AZ) Department of Mathematics, University of California San Diego, 9500 Gilman Dr. # 0112, La Jolla, CA 92093

E-mail addresses: `amir.alwan@chicagobooth.edu`, `jessica.lin@mcgill.ca`, `ztao@caltech.edu`, `zoh003@ucsd.edu`, `zlatos@ucsd.edu`.

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We note that a simple scaling argument extends our results to the general Fisher-KPP case

$$f(0) = f(1) = 0, \qquad f'(0) > 0, \qquad 0 < f(u) \le f'(0)u \text{ on } (0,1).$$

The study of (1.1) was started 80 years ago by Kolmogorov, Petrovskii, and Piskunov [13] and Fisher [6] in one dimension $d = 1$, while here we consider entire solutions $u : \mathbb{R}^{d+1} \to [0,1]$ for any $d \ge 1$. These model propagation of reactive processes such as forest fires, nuclear reactions in stars, or population dynamics. The value $u = 0$ represents the unburned (or minimal-temperature or zero-population-density) state, while $u = 1$ represents the burned (or maximal-temperature or maximal-population-density) state. Fisher-KPP reactions possess the "hair-trigger effect", meaning that for any solution $0 \le u \le 1$ except $u \equiv 0$, the asymptotically stable state $u = 1$ will invade the whole spatial domain $\mathbb{R}^d$ as $t \to \infty$ (while the state $u = 0$ is unstable). In fact, we have [1]

$$\lim_{t\to\infty} \inf_{|x|\le ct} u(t,x) = 1 \qquad \text{for each } c < 2. \tag{1.3}$$

This immediately implies that except when $u \equiv 1$, we also have

$$\lim_{t\to-\infty} \sup_{|x|\le c|t|} u(t,x) = 0 \qquad \text{for each } c < 2. \tag{1.4}$$

Note that the strong maximum principle and $0 \le u \le 1$ imply that $0 < u < 1$ whenever $u \not\equiv 0, 1$, and we will assume this from now on.

In their pioneering work [10], Hamel and Nadirashvili provided a partial characterization of such solutions of (1.1). Under the additional hypotheses of $f \in C^2([0,1])$, f being concave, and $f'(1) < 0$, they identified all solutions $u : \mathbb{R}^{d+1} \to (0,1)$ which also satisfy (cf. (1.4))

$$\lim_{t\to-\infty} \sup_{|x|\le c|t|} u(t,x) = 0 \qquad \text{for some } c > 2 \tag{1.5}$$

(we will call these *Hamel-Nadirashvili solutions*). They showed that these solutions are naturally parametrized by all finite positive Borel measures supported inside the open unit ball in $\mathbb{R}^d$ (see Remark 1 after Theorem 1.2 below). One of us later showed [27] that this infinite-dimensional manifold of solutions, parametrized by Borel measures, also exists without the additional hypotheses from [10] (see Theorem 1.2), although it is not yet known whether other solutions satisfying (1.5) can exist in this case.

It follows from (1.3) and (1.4) that all entire solutions $0 < u < 1$ for Fisher-KPP reactions satisfy

$$\lim_{t\to-\infty} u(t,x) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} u(t,x) = 1 \tag{1.6}$$

locally uniformly. Our goal here is to study the nature of this transition from 0 to 1. Aerial footage of forest fires usually shows relatively narrow lines of fire separating burned and unburned areas, and we investigate the question which entire solutions also have this property. More specifically, which are *transition fronts*, defined by Berestycki and Hamel in [2, 3] (and earlier in some special situations by Matano [15] and Shen [21]); and more generally, which are *transition solutions with bounded width*, defined by one of us in [30]. Let us now state these definitions.

For any u as above, $t \in \mathbb{R}$, and $\epsilon \in [0,1]$ let

$$\Omega_{u,\epsilon}(t) := \{x \in \mathbb{R}^d : u(t,x) \geq \epsilon\},$$
$$\Omega'_{u,\epsilon}(t) := \{x \in \mathbb{R}^d : u(t,x) \leq \epsilon\},$$

and for any $E \subseteq \mathbb{R}^d$ and $L > 0$ let

$$B_L(E) := \bigcup_{x \in E} B_L(x),$$

with the convention $B_L(\varnothing) = \varnothing$.

Definition 1.1. Let $0 < u < 1$ be an entire solution to (1.1).

(i) u is a *transition solution* if it satisfies (1.6) locally uniformly.

(ii) u has *bounded width* if for each $\epsilon \in \left(0, \frac{1}{2}\right)$ there is $L_\epsilon < \infty$ such that

$$\Omega_{u,\epsilon}(t) \subseteq B_{L_\epsilon}(\Omega_{u,1-\epsilon}(t)) \qquad \text{for each } t \in \mathbb{R}. \tag{1.7}$$

(iii) u is a *transition front* if it has bounded width, for each $\epsilon \in \left(0, \frac{1}{2}\right)$ there is $L'_\epsilon < \infty$ such that

$$\Omega'_{u,1-\epsilon}(t) \subseteq B_{L'_\epsilon}(\Omega'_{u,\epsilon}(t)) \qquad \text{for each } t \in \mathbb{R}, \tag{1.8}$$

and there are n, L such that for any $t \in \mathbb{R}$, there is a union Γ_t of at most n rotated continuous graphs in $\mathbb{R}^d$ which satisfy

$$\partial\Omega_{u,1/2}(t) \subseteq B_L(\Gamma_t).$$

Remarks. 1. When f is Fisher-KPP, then all entire solutions $0 < u < 1$ are transition solutions (this is not true for more general f).

2. A *rotated continuous graph* in $\mathbb{R}^d$ is a rotation of the graph of some continuous function $h : \mathbb{R}^{d-1} \to \mathbb{R}$ (which is a subset of $\mathbb{R}^d$).

3. The original definition of transition fronts in [2, 3] was slightly different from (iii), but the two are equivalent [30].

4. In one dimension $d = 1$ the set Γ_t in (iii) is just a collection of at most n points. The special case $n = 1$ of transition fronts with a *single interface* is of particular interest and has recently been studied

extensively for various types of reactions (see, e.g., [3,5,9,11,12,14,16–29]). These are entire solutions $0 < u < 1$ satisfying

$$\lim_{x\to-\infty} u(t,x+x_t) = 1 \qquad \text{and} \qquad \lim_{x\to\infty} u(t,x+x_t) = 0 \tag{1.9}$$

uniformly in $t \in \mathbb{R}$, where $x_t := \max\{x \in \mathbb{R} : u(t,x) = \frac{1}{2}\}$ (or with 0 and 1 exchanged in (1.9)). They were introduced as a generalization of the concept of *traveling fronts*, solutions of the form $u(t,x) = U(x-ct)$ for some decreasing front profile $U : \mathbb{R} \to (0,1)$ with $\lim_{s\to-\infty} U(s) = 1$ and $\lim_{s\to\infty} U(s) = 0$, and some front speed c. (It is well-known that for (1.1) with a Fisher-KPP reaction, these exist if and only if $c \geq 2\sqrt{f'(0)}$.) Traveling fronts, which were already studied in [6, 13], only exist for homogeneous reactions, and transition fronts are their natural generalization that can exist in both homogeneous and heterogeneous (i.e., x-dependent) media. We discuss recent results concerning transition fronts for homogeneous reactions below.

5. Solutions satisfying (1.7) and (1.8) but not necessarily the closeness-to-graphs condition are said to have *doubly bounded width* [30]. Our main result (Theorem 1.3) and its proof remain unchanged when “transition fronts” are replaced by “transition solutions with doubly bounded width”.

It is easily seen that a transition solution u is a transition front if and only if the Hausdorff distance of any two *level sets* $\{x \in \mathbb{R}^d : u(t,x) = \epsilon\}$ of u stays bounded uniformly in time, and the level set $\{x \in \mathbb{R}^d : u(t,x) = \frac{1}{2}\}$ (then also any other) is at each time uniformly close to a uniformly bounded number of time-dependent rotated continuous graphs. In contrast, u is a transition solution with bounded width if and only if the Hausdorff distance of any two *super-level sets* $\Omega_{u,\epsilon}(t)$ of u stays bounded uniformly in time.

This distinction results in some notable differences. For instance, transition fronts (and transition solutions with doubly bounded width) satisfy

$$\inf_{x\in\mathbb{R}^d} u(t,x) = 0 \quad \text{and} \quad \sup_{x\in\mathbb{R}^d} u(t,x) = 1 \tag{1.10}$$

for each $t \in \mathbb{R}$, while transition solutions with bounded only verify the second limit. Also, transition solutions with bounded width in dimensions $d \geq 2$ may involve dynamics where the invading state $u \approx 1$ first encircles large regions where $u \approx 0$ (with their sizes unbounded as $t \to \infty$) and then invades them. On the other hand, such solutions cannot be transition fronts (or have doubly bounded width) because, for instance, at some time t there will be a point x with $u(t,x) = \frac{2}{3}$ near the center of such a region but points y with $u(t,y) = \frac{1}{3}$ will all lie outside of this region (and thus far away from x). Because this phenomenon does occur for various heterogeneous reactions (e.g., for stationary ergodic reactions with short-range correlations), preventing existence of transition fronts in these settings, it is important to study both these classes of solutions to (1.1). We refer to [30] for a more detailed discussion of the relevant issues.

Coming back to the homogeneous equation (1.1) with a Fisher-KPP reaction f, the first systematic study of its entire solutions was undertaken in [9,10] under some additional conditions on f. We will use here the following closely related result from [27], which concerns the main object of our study — the Hamel-Nadirashvili solutions to (1.1) — and holds for general Fisher-KPP reactions. In order to state it, first recall that if μ is a positive Borel measure on $\mathbb{R}^d$, its support $\operatorname{supp}(\mu)$ is the minimal closed set A such that $\mu(A^c) = 0$, while its essential support is any Borel set A such that $\mu(A) = \mu(\mathbb{R}^d)$ and $\mu(A') < \mu(A)$ whenever $A' \subseteq A$ and $A \setminus A'$ has positive Lebesgue measure. The collection of all essential supports of μ will be denoted $\operatorname{ess\,supp}(\mu)$. Following [27], we then define the *convex hull* of μ to be

$$\operatorname{ch}(\mu) := \bigcap_{A \in \operatorname{ess\,supp}(\mu)} \operatorname{ch}(A),$$

where $\operatorname{ch}(A)$ is the convex hull of the set A. Note that we may have $\operatorname{ch}(\mu) \notin \operatorname{ess\,supp}(\mu)$ [27]. Finally, let B_r denote the open ball $B_r(0) \subseteq \mathbb{R}^d$ with radius r and centered at 0, and let $S^{d-1} := \partial B_1$.

Theorem 1.2 ([27]). *Assume that $f \in C^{1+\gamma}([0,1])$ for some $\gamma > 0$ and satisfies (1.2), let μ be a finite positive non-zero Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq B_1$, and let*

$$v_\mu(t,x) := \int_{B_1} e^{-\xi\cdot x + (|\xi|^2+1)t}\, d\mu(\xi). \tag{1.11}$$

(i) There is an increasing function $h : [0,\infty] \to [0,1)$ with $h(0) = 0$, $h'(0) = 1$ and $\lim_{v\to\infty} h(v) = 1$, and an entire solution u_μ of (1.1) such that $(u_\mu)_t > 0$ and

$$h(v_\mu) \le u_\mu \le \min\{v_\mu, 1\}. \tag{1.12}$$

In addition, $u_\mu \not\equiv u_{\mu'}$ whenever $\mu \neq \mu'$.

(ii) We have

$$\inf_{x\in\mathbb{R}^d} u_\mu(t,x) = 0 \qquad \text{and} \qquad \sup_{x\in\mathbb{R}^d} u_\mu(t,x) = 1 \tag{1.13}$$

for each $t \in \mathbb{R}$ if and only if $0 \notin \operatorname{ch}(\mu)$.

(iii) If $0 \notin \operatorname{supp}(\mu)$, then u_μ has bounded width.

Remarks. 1. If also $f \in C^2([0,1])$ and it is concave, then [10, Theorem 1.2] shows that the solutions from (i) are precisely those entire solutions $0 < u < 1$ satisfying (1.5). We note that for such f, [10] also constructs entire solutions corresponding to some measures supported in $\bar{B}_1$ but not in B_1 (which then do not satisfy (1.5)), namely those whose restriction to S^{d-1} is a finite sum of Dirac masses.[1]

2. Note that the functions $e^{-\xi\cdot x+(|\xi|^2+1)t}$ and v_μ from (1.11) solve the linearization

$$v_t = \Delta v + v$$

of (1.1) at $u = 0$. Moreover, if we denote $c_{|\xi|} := |\xi| + \frac{1}{|\xi|}$ for $\xi \neq 0$, then

$$e^{-\xi\cdot x+(|\xi|^2+1)t} = e^{-\xi\cdot x+|\xi|c_{|\xi|}t} = e^{-\xi\cdot(x-\frac{\xi}{|\xi|}c_{|\xi|}t)}.$$

So this is an exponential that moves with speed $c_{|\xi|}$ in the direction $\frac{\xi}{|\xi|}$.

3. (ii) and (1.10) show that $0 \notin \mathrm{ch}(\mu)$ is a necessary condition for u_μ to be a transition front.

4. This result, and thus also Theorem 1.3 below, holds for f satisfying (1.2) which is only Lipschitz, as long as $f(u) \geq g(u)$ on $[0,1]$ for some $g \in C^1([0,1])$ such that $g(0) = g(1) = 0$, $g'(0) = 1$, $g(u) > 0$ and $g'(u) \leq 1$ on $(0,1)$, and $\int_0^1 \frac{u-g(u)}{u^2}du < \infty$ [27]. We note that if $f \in C^{1+\gamma}([0,1])$ satisfies (1.2), then there exists such function g with $g(u) = u - Cu^{1+\gamma}$ for some C and all small $u \geq 0$.

We now turn to our main result, an almost complete characterization of transition fronts as well as transition solutions with bounded width within the class of the solutions from Theorem 1.2. Recall that if $f \in C^2([0,1])$ is concave and $f'(1) < 0$, then this class coincides with the class of Hamel-Nadirashvili solutions. In one dimension $d = 1$ and under these extra hypotheses, a complete characterization of transition fronts among all the solutions from [10] was recently obtained by Hamel and Rossi [12]. (These solutions are then parametrized by finite positive nonzero Borel measures μ on the interval $[-1,1] = \bar{B}_1$, or on $(-2,2)^c \cup \{\infty\}$ after the transformation $\xi \mapsto (1+|\xi|^{-2})\xi$ mentioned above.) They proved that the solution u_μ is a transition front if and only if $\mathrm{supp}(\mu) \subseteq [-1,0)$ or $\mathrm{supp}(\mu) \subseteq (0,1]$. In several dimensions, this task is considerably more challenging because the geometry of B_1 is more complicated there. In fact, we are not aware of any relevant previous results for Fisher-KPP reactions. We note that transition fronts and transition solutions with bounded width for ignition and bistable reactions satisfying very mild hypotheses were proved to increase in time [3, 30], and examples

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[1] In fact, the measures in [10] are supported in $B_2^c \cup \{\infty\}$ but the map $\xi \mapsto (1+|\xi|^{-2})\xi$ establishes the relevant correspondence between $\bar{B}_1$ and $B_2^c \cup \{\infty\}$.

of transition fronts for homogeneous bistable reactions that are not traveling fronts were recently constructed in [8].

For $\zeta \in S^{d-1}$ and $\alpha \in [0,1]$, let

$$\mathcal{W}_{\alpha,\zeta} := \{x \in \mathbb{R}^d : x \cdot \zeta \geq \alpha |x|\},$$

which is a closed cone with axis ζ when $\alpha > 0$, while $\mathcal{W}_{0,\zeta}$ is the closed half-space with inner normal ζ. We will also call an *upright cone*

$$\mathcal{W}_{\alpha} := \mathcal{W}_{\alpha,e_d} = \{x \in \mathbb{R}^d : x_d \geq \alpha |x|\}. \tag{1.14}$$

Theorem 1.3. *Let f, μ, u_μ be as in Theorem 1.2.*

(i) If there are $\zeta \in S^{d-1}$ and $\alpha > 0$ such that

$$0 \notin \operatorname{supp}(\mu) \subseteq \mathcal{W}_{\alpha,\zeta}, \tag{H1}$$

then u_μ is a transition front (and hence also a transition solution with bounded width).

(ii) If there are $\zeta \in S^{d-1}$ and $\alpha > 0$ such that

$$0 \in \operatorname{supp}(\mu) \subseteq \mathcal{W}_{\alpha,\zeta}, \tag{H2}$$

then u_μ is not a transition solution with bounded width (and hence also not a transition front).

(iii) If

$$\operatorname{supp}(\mu) \nsubseteq \mathcal{W}_{0,\zeta} \quad \textit{for each } \zeta \in S^{d-1}, \tag{H3}$$

then u_μ is a transition solution with bounded width but not a transition front.

Notice that the only cases of measures from Theorem 1.2 not covered by this result are those supported in some half-space $\mathcal{W}_{0,\zeta}$ but not in any cone $\mathcal{W}_{\alpha,\zeta}$ with $\alpha > 0$. We can still say something in this case: if $0 \notin \operatorname{supp}(\mu)$, then Theorem 1.2(iii) shows that u_μ is a transition solution with bounded width, and we also conjecture that u_μ is not a transition front. However, if $0 \in \operatorname{supp}(\mu)$, then determining whether u_μ is a transition front and/or a transition solution with bounded width will likely be a very delicate question.

We prove the three parts of Theorem 1.3 in the following three sections, leaving some technical lemmas for the Appendix.

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2. Proof of Theorem 1.3(i)

We may assume without loss of generality that $\zeta = e_d$, so that the cone $\mathcal{W}_{\alpha,\zeta} = \mathcal{W}_\alpha$ is upright. Then (H1) implies there is $\delta > 0$ such that

$$\operatorname{supp}(\mu) \subseteq \mathcal{W}_\alpha \cap A(\delta, 1), \tag{2.1}$$

with $A(r_1, r_2) := B_{r_2} \setminus B_{r_1}$ an annulus. In particular,

$$\inf\{x_d : x \in \operatorname{supp}(\mu)\} \geq \alpha\delta > 0.$$

Let us first show that u_μ has bounded width (recall that each u_μ is a transition solution). This follows immediately from Theorem 1.2 but our argument will also be useful in the proof that u_μ is a transition front. Let $\epsilon \in \left(0, \frac{1}{2}\right)$ and $x \in \Omega_{u_\mu,\epsilon}(t)$, and define $s := (\alpha\delta)^{-1} \ln(h^{-1}(1-\epsilon)/\epsilon) \geq 0$ and $x_s := x - se_d$. Here h is the μ-dependent function from Theorem 1.2(i) (and we note that $h^{-1}(v) \geq v$). From (2.1) we have

$$v_\mu(t, x_s) = \int_{B_1} e^{s(\xi\cdot e_d)} e^{-\xi\cdot x + (|\xi|^2+1)t}\, d\mu(\xi) \geq e^{s\alpha\delta} \int_{B_1} e^{-\xi\cdot x + (|\xi|^2+1)t}\, d\mu(\xi),$$

so the definition of s and (1.12) yield

$$v_\mu(t, x_s) \geq \frac{h^{-1}(1-\epsilon)}{\epsilon} v_\mu(t, x) \geq \frac{h^{-1}(1-\epsilon)}{\epsilon} u_\mu(t, x) \geq h^{-1}(1-\epsilon).$$

From (1.12) we now have $x_s \in \Omega_{u_\mu,1-\epsilon}(t)$, so (1.7) with $u = u_\mu$ holds for each $t \in \mathbb{R}$ and $L_\epsilon := s + 1$. Hence u_μ is a transition solution with bounded width.

The verification of (1.8) for u_μ is analogous. If $\epsilon \in \left(0, \frac{1}{2}\right)$ and $x \in \Omega'_{u_\mu,1-\epsilon}(t)$, then the above argument for $x_s := x + se_d$ yields

$$v_\mu(t, x_s) \leq \frac{\epsilon}{h^{-1}(1-\epsilon)} v_\mu(t, x) \leq \frac{\epsilon}{h^{-1}(1-\epsilon)} h^{-1}(u_\mu(t, x)) \leq \epsilon.$$

From (1.12) we now have $x_s \in \Omega'_{u_\mu,\epsilon}(t)$, so (1.8) with $u = u_\mu$ holds for each $t \in \mathbb{R}$ and $L'_\epsilon := L_\epsilon$.

Finally, the last claim in Definition 1.1(iii) is satisfied with $\Gamma_t := \{x \in \mathbb{R}^d : v_\mu(t, x) = \frac{1}{2}\}$ (which is a graph of a function of $(x_1, \ldots, x_{d-1})$ because $\operatorname{supp}(\mu) \subseteq \mathcal{W}_\alpha \cap A(\delta, 1)$ implies $(v_\mu)_{x_d} < 0$) and any $L > L_{h(1/2)}$. Indeed, if $u_\mu(t, x) = \frac{1}{2}$, then $x \in \Omega'_{u_\mu,1-h(1/2)}(t)$, so (1.12) and the above arguments show that $v_\mu(t, x) \geq \frac{1}{2}$ as well as

$$v_\mu(t, x + L_{h(1/2)}e_d) \leq h^{-1}\left(u_\mu(t, x + L_{h(1/2)}e_d)\right) \leq h^{-1}\left(h\left(\frac{1}{2}\right)\right) = \frac{1}{2}.$$

Hence there is $l \in [0, L_{h(1/2)}]$ such that $x + le_d \in \Gamma_t$, and it follows that u_μ is indeed a transition front.

3. Proof of Theorem 1.3(ii)

We again assume without loss that $\zeta = e_d$, so the cone $\mathcal{W}_{\alpha,\zeta} = \mathcal{W}_\alpha$ is upright, and let h be the μ-dependent function from Theorem 1.2(i). We will now show that the width of the transition zone of u_μ becomes unbounded as $t \to \infty$, violating Definition 1.1(ii). Thus, u_μ is neither a transition solution with bounded width nor a transition front.

First consider the case $\mu(\{0\}) > 0$ and let $t_0 := \ln(2\mu(\{0\}))$. Then from the Lebesgue dominated convergence theorem, we have

$$\lim_{x_d \to \infty} v_\mu(-t_0, x) = \mu(\{0\}) e^{-t_0} = \frac{1}{2} \qquad (\le v_\mu(-t_0, x) \text{ for all } x \in \mathbb{R}^d)$$

locally uniformly in $(x_1, \dots, x_{d-1})$. This and Theorem 1.2(i) show that there is $M < \infty$ such that $u_\mu(-t_0, x) \in [h(\frac{1}{2}), \frac{2}{3}]$ whenever $x_d > M$. Thus $L_{\min\{h(1/2),1/4\}}$ from (1.7) with $u = u_\mu$ cannot be finite and we are done.

Let us now assume $\mu(\{0\}) = 0$, and fix any $\varepsilon \in (0, \frac{1}{4})$ such that $h^{-1}(\varepsilon) \le \frac{1}{4}$. For each $t \in \mathbb{R}$, let $X(t) = (0, \dots, 0, s_t)$ be such that $v_\mu(t, X(t)) = h^{-1}(\varepsilon)$. This point is unique because $\operatorname{supp}(\mu) \subseteq \mathcal{W}_\alpha$ and $\mu(\{0\}) = 0$ imply $(v_\mu)_{x_d} < 0$.

Fix any $\delta \in (0,1)$ and let $\delta' \in (0, \delta)$ be such that $c_{\delta'} \ge \frac{3}{\alpha} c_\delta$. For instance, $\delta' = \frac{\alpha\delta}{6}$ works. Next let

$$\begin{aligned} v_1(t,x) &:= \int_{A(\delta,1)} e^{-\xi\cdot x + |\xi| c_{|\xi|} t} \, d\mu(\xi), \\ v_2(t,x) &:= \int_{B_\delta} e^{-\xi\cdot x + |\xi| c_{|\xi|} t} \, d\mu(\xi), \\ v_3(t,x) &:= \int_{B_{\delta'}} e^{-\xi\cdot x + |\xi| c_{|\xi|} t} \, d\mu(\xi), \end{aligned}$$

so that $v_\mu = v_1 + v_2$. Note also that $(v_j)_{x_d} < 0$ for $j = 1,2,3$.

Let now $r_t := \frac{2}{\alpha} c_\delta t$ and $Y(t) = (0, \dots, 0, r_t)$. Then from $c_{|\xi|}$ being decreasing in $|\xi| \in (0,1]$, we obtain for any $\xi \in \mathcal{W}_\alpha \cap A(\delta, 1)$ and $t \ge 0$,

$$-\xi \cdot Y(t) + |\xi| c_{|\xi|} t \le -|\xi| (\alpha r_t - c_\delta t) \le -\delta c_\delta t \le -t.$$

On the other hand, for $\xi \in \mathcal{W}_\alpha \cap B_{\delta'}$ and $t > 0$ we obtain

$$-\xi \cdot Y(t) + |\xi| c_{|\xi|} t \ge |\xi| (c_{\delta'} t - r_t) \ge \frac{|\xi| c_\delta}{\alpha} t.$$

From these, $\mu([\mathcal{W}_\alpha \cap B_{\delta'}] \setminus \{0\}) > 0$, and the Lebesgue dominated convergence theorem it follows that

$$\lim_{t\to\infty} v_1(t, Y(t)) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} v_3(t, Y(t)) = \infty.$$

Therefore $s_t > r_t$ and $\lim_{t\to\infty} v_1(t, X(t)) = 0$. But then from $v_\mu = v_1 + v_2$, $|\nabla v_1| \leq \sqrt{d} v_1$, $|\nabla v_2| \leq \sqrt{d}\delta v_2$, and $v_\mu(t, X(t)) \leq \frac{1}{4}$ it follows that

$$\lim_{t\to\infty} \sup_{y\in B_{d^{-1/2}\delta^{-1}\ln 2}} v_\mu(t, X(t)+y) \leq \frac{1}{2}.$$

Then since $v_\mu(t, X(t)) = h^{-1}(\varepsilon)$, applying Theorem 1.2(i) shows that $u_\mu(t, X(t)) \geq \varepsilon$ and

$$\lim_{t\to\infty} \sup_{y\in B_{d^{-1/2}\delta^{-1}\ln 2}} u_\mu(t, X(t)+y) \leq \frac{1}{2}.$$

This shows that L_ε from (1.7) with $u = u_\mu$ must satisfy $L_\varepsilon \geq d^{-1/2}\delta^{-1}\ln 2$. Since $\delta > 0$ was arbitrary, such $L_\varepsilon < \infty$ cannot exist and we are done.

4. PROOF OF THEOREM 1.3(III)

Throughout this section, $\mathrm{int}(E)$ and ∂E denote the interior and boundary of a set $E \subseteq \mathbb{R}^d$. We split the proof in two parts.

4.1. Proof that u_μ is not a transition front.

This follows immediately from Theorem 1.2(ii) and the following result.

Proposition 4.1. *If μ satisfies (H3), then $0 \in \mathrm{ch}(\mu)$.*

The proof of Proposition 4.1 uses several results from convex analysis:

Lemma 4.2 (Section 9, Chapter 6, Theorem 3 in [4])**.** *Let $S \subseteq \mathbb{R}^d$ be a nonempty compact set. Then $0 \notin \mathrm{ch}(S)$ if and only if there exists an $\zeta \in S^{d-1}$ such that $S \subseteq \mathrm{int}(\mathcal{W}_{0,\zeta})$.*

Lemma 4.3 (Theorem Δ_n in [7])**.** *If $S \subseteq \mathbb{R}^d$ and $x \in \mathrm{int}(\mathrm{ch}(S))$, then there is $S^* \subseteq S$ such that $\mathrm{card}(S^*) \leq 2d$ and $x \in \mathrm{int}(\mathrm{ch}(S^*))$.*

Finally, we need a technical result concerning the stability of the convex hull of a finite set of points, which we prove in the Appendix.

Proposition 4.4. *If $S^* = \{x_1, \dots, x_k\} \subseteq \mathbb{R}^d$ and $0 \in \mathrm{int}(\mathrm{ch}(S^*))$, then there is $\epsilon > 0$ such that for all $y_i \in B_\epsilon(x_i)$, we have $0 \in \mathrm{ch}(\{y_1, \dots, y_k\})$.*

Proof of Proposition 4.1. By (H3), $\mathrm{supp}(\mu) \notin \mathrm{int}(\mathcal{W}_{0,\zeta})$ for any $\zeta \in S^{d-1}$. Since $\mathrm{supp}(\mu)$ is compact, Lemma 4.2 implies that $0 \in \mathrm{ch}(\mathrm{supp}(\mu))$. We cannot have $0 \in \partial(\mathrm{ch}(\mathrm{supp}(\mu)))$ because then the convexity of $\mathrm{ch}(\mathrm{supp}(\mu))$ would imply existence of a supporting hyperplane H of $\mathrm{ch}(\mathrm{supp}(\mu))$ such that $0 \in H$ (and then $H = \partial\mathcal{W}_{0,\zeta}$ for some $\zeta \in S^{d-1}$). This implies that $\mathrm{supp}(\mu) \subseteq \mathrm{ch}(\mathrm{supp}(\mu)) \subseteq \mathcal{W}_{0,\zeta}$, yielding a contradiction. Therefore $0 \in \mathrm{int}(\mathrm{supp}(\mu))$, and Lemma 4.3 shows that there exist $k \leq 2d$ points $\{x_1, \dots, x_k\} \subseteq \mathrm{supp}(\mu)$ such that

$$0 \in \mathrm{int}(\mathrm{ch}(\{x_1, \dots, x_k\})).$$

By Proposition 4.4, there is $\epsilon > 0$ such that $0 \in \operatorname{ch}(\{y_1, y_2, \dots, y_k\})$ whenever $y_i \in B_\epsilon(x_i)$ for each $i = 1, \dots, k$. Since any $A \in \operatorname{ess\,supp}(\mu)$ satisfies $A \cap B_\epsilon(x_i) \neq \varnothing$ for each $i = 1, \dots, k$ (because $x_i \in \operatorname{supp}(\mu)$ and so $\mu(B_\epsilon(x_i)) > 0$), it follows that $0 \in \operatorname{ch}(A)$. Therefore, $0 \in \operatorname{ch}(\mu)$. ∎

4.2. Proof that u_μ is a transition solution with bounded width.

Let us start with some preliminary lemmas. Note that we obviously have $\mu(\mathcal{W}^c_{0,\zeta}) > 0$ for any $\zeta \in S^{d-1}$.

Lemma 4.5. *If μ satisfies (H3), then*

$$a^* := \inf_{\zeta \in S^{d-1}} \mu(\mathcal{W}^c_{0,\zeta}) > 0.$$

Proof. If $a^* = 0$, then there is a sequence $\{\zeta_n\} \subseteq S^{d-1}$ with $\mu(\mathcal{W}^c_{0,\zeta_n}) < 2^{-n}$ for each n. By compactness of S^{d-1}, after passing to a subsequence we can assume that $\zeta_n \to \zeta \in S^{d-1}$. But

$$\mathcal{W}^c_{0,\zeta} \subseteq \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \mathcal{W}^c_{0,\zeta_n}$$

then yields $\mu(\mathcal{W}^c_{0,\zeta}) = 0$, a contradiction with (H3). ∎

For $N \geq 1$, let

$$Z_N := \{\zeta \in S^{d-1} : \mu(C_{N,\zeta}) > 0\},$$

where for $\zeta \in S^{d-1}$ we let

$$C_{N,\zeta} := \operatorname{int}(\mathcal{W}_{N^{-1},-\zeta} \cap A(N^{-1}, 1)).$$

Lemma 4.6. *If μ satisfies (H3), then $Z_N = S^{d-1}$ for some $N \geq 1$.*

Proof. Note that Z_N is open in S^{d-1} for each $N \geq 1$ because we have $C_{N,\zeta} \subseteq \bigcup_{n=1}^{\infty} C_{N,\zeta_n}$ whenever $\zeta_n \to \zeta$. Since obviously $Z_N \subseteq Z_{N+1}$ for each N, it follows that $\{Z_N^c\}_{N=1}^{\infty}$ is a decreasing sequence of compact sets. If none of these is empty, then there exists $\zeta \in S^{d-1} \setminus \bigcup_{N=1}^{\infty} Z_N$, which contradicts $\mathcal{W}^c_{0,\zeta} = \bigcup_{N=1}^{\infty} C_{N,\zeta}$ and $\mu(\mathcal{W}^c_{0,\zeta}) > 0$. ∎

From this, similarly to Lemma 4.5, we obtain the following.

Lemma 4.7. *If μ satisfies (H3) and N is from Lemma 4.6, then*

$$b^* := \inf_{\zeta \in S^{d-1}} \mu(C_{N,\zeta}) > 0.$$

From now on, we fix N from Lemma 4.6 and b^* from Lemma 4.7 (both depending on μ). We will now prove (1.7) for u_μ, first considering all large negative t.

Lemma 4.8. *If μ satisfies (H3) and $\epsilon \in (0, \frac{1}{2})$, then there are $K, T > 0$ such that $u_\mu(t,x) \geq 1 - \epsilon$ whenever $t \leq -T$ and $|x| \geq K|t|$.*

Proof. Let $K := 3N^2$ and $T := \ln \frac{h^{-1}(1-\epsilon)}{b^*}$ (it suffices to consider $\epsilon > 0$ such that $1 - \epsilon > h(b^*)$). Since for any $x \in \mathbb{R}^d \smallsetminus \{0\}$ we obviously have

$$\inf_{\xi \in C_{N,x|x|^{-1}}} \left(-\xi \cdot \frac{x}{|x|} \right) \geq \frac{1}{N^2},$$

for any $t \leq -T$ and $x \in \mathbb{R}^d$ with $|x| \geq K|t|$ we obtain

$$\begin{aligned} v_\mu(t,x) &= \int_{B_1} e^{-\xi \cdot x + (|\xi|^2+1)t} \, d\mu(\xi) \\ &\geq \int_{C_{N,x|x|^{-1}}} e^{|x|N^{-2}+2t} \, d\mu(\xi) \\ &\geq e^{(KN^{-2}-2)|t|} \mu(C_{N,x|x|^{-1}}) \\ &\geq e^T b^* \\ &= h^{-1}(1-\epsilon). \end{aligned}$$

Theorem 1.2(i) now finishes the proof. ■

Lemma 4.9. *If μ satisfies (H3) and $\epsilon \in (0, \frac{1}{2})$, then the following holds for any $a > 0$. There are $T_a, \delta_a > 0$ such that if $(t,x) \in (-\infty, -T_a] \times \mathbb{R}^d$ and $u_\mu(t,x) < 1 - \epsilon$, then*

$$\int_{B_{\delta_a}} e^{-\xi \cdot x + (|\xi|^2+1)t} \, d\mu(\xi) \leq a.$$

Proof. We can assume without loss that $a \leq \mu(B_1)$. Let K, T be from Lemma 4.8 and define

$$\begin{aligned} T_a &:= \max\left\{ T, 1 + \left| \ln \frac{a}{\mu(B_1)} \right| \right\}, \\ \delta_a &:= \frac{1}{K} \left(1 - \frac{\left| \ln \frac{a}{\mu(B_1)} \right|}{T_a} \right) > 0. \end{aligned}$$

Since $T_a \geq T$, Lemma 4.8 shows that for any (t,x) as above, we must have $|x| < K|t|$. We also have $\delta_a K - 1 < 0$, hence for any $t \leq -T_a$ we find

$$(\delta_a K - 1)|t| \leq (\delta_a K - 1) T_a \leq \ln \frac{a}{\mu(B_1)}.$$

It follows that for (t,x) as above we obtain

$$\int_{B_{\delta_a}} e^{-\xi \cdot x + (|\xi|^2+1)t} \, d\mu(\xi) \leq \int_{B_{\delta_a}} e^{\delta_a K|t| + t} \, d\mu(\xi) \leq e^{(\delta_a K - 1)|t|} \mu(B_1) \leq a,$$

and the proof is finished. ■

We can now prove (1.7) for $u = u_\mu$.

Proposition 4.10. *If μ satisfies (H3) and $\epsilon \in (0, \frac{1}{2})$, then there is $L_\epsilon < \infty$ such that for each $t \in \mathbb{R}$,*

$$\Omega_{u_\mu,\epsilon}(t) \subseteq B_{L_\epsilon}(\Omega_{u_\mu,1-\epsilon}(t)).$$

Proof. For any $\zeta \in S^{d-1}$, let

$$Y_\zeta := \left\{ \xi \in B_1 : \zeta \cdot \xi \geq \frac{|\xi|}{2} \right\} = \mathcal{W}_{1/2,\zeta} \cap B_1.$$

Let also $a := \frac{\epsilon}{2}|Y_\zeta||B_1|^{-1}$ (note that $|Y_\zeta|$ is independent of ζ) and let δ_a, T_a be from Lemma 4.9.

We will first consider times $t \leq -T_a$. Fix any such t and let x be such that $u_\mu(t,x) \geq \epsilon$. Since $v_\mu(t,x) \geq u_\mu(t,x) \geq \epsilon$, there must be $\zeta \in S^{d-1}$ such that

$$\int_{Y_\zeta} e^{-\xi\cdot x + (|\xi|^2+1)t}\, d\mu(\xi) \geq 2a$$

(otherwise integrate the opposite inequality in $\zeta \in \mathbb{S}^{d-1}$ and get a contradiction). If $u_\mu(t,x) < 1-\epsilon$, then Lemma 4.9 shows that

$$\begin{aligned}\int_{Y_\zeta \cap A(\delta_a,1)} e^{-\xi\cdot x + (|\xi|^2+1)t}\, d\mu(\xi) &\geq \int_{Y_\zeta} e^{-\xi\cdot x + (|\xi|^2+1)t}\, d\mu(\xi) - \int_{B_{\delta_a}} e^{-\xi\cdot x + (|\xi|^2+1)t}\, d\mu(\xi)\\ &\geq 2a - a = a,\end{aligned}$$

hence for $L_\epsilon^- := \frac{2}{\delta_a} \ln \frac{h^{-1}(1-\epsilon)}{a} > 0$ (recall that $h(a) \leq a \leq \frac{\epsilon}{2} < 1-\epsilon$) we have

$$\begin{aligned} v_\mu(t, x - L_\epsilon^-\zeta) &= \int_{B_1} e^{-\xi\cdot(x - L_\epsilon^-\zeta) + (|\xi|^2+1)t}\, d\mu(\xi)\\ &\geq \int_{Y_\zeta \cap A(\delta_a,1)} e^{L_\epsilon^-(\xi\cdot\zeta)} e^{-\xi\cdot x + (|\xi|^2+1)t}\, d\mu(\xi)\\ &\geq e^{L_\epsilon^-\delta_a/2} a = h^{-1}(1-\epsilon).\end{aligned}$$

So either $u_\mu(t,x) \geq 1-\epsilon$ or $v_\mu(t, x - L_\epsilon^-\zeta) \geq 1-\epsilon$. If we now choose $L_\epsilon \geq L_\epsilon^-$, from Theorem 1.2(i) we obtain the claim for all $t \leq -T_a$.

Let us now consider $t > -T_a$. For each $\zeta \in S^{d-1}$ we obviously have

$$\inf_{\xi \in C_{N,\zeta}} (-\xi \cdot \zeta) \geq \frac{1}{N^2}.$$

Then for each $s \geq L_\epsilon^+ := N^2 \left(\left| \ln \frac{h^{-1}(1-\epsilon)}{b^*} \right| + 2T_a \right)$ and $t > -T_a$ we have

$$v_\mu(t, s\zeta) \geq \int_{C_{N,\zeta}} e^{-\xi\cdot s\zeta - 2T_a}\, d\mu(\xi) \geq e^{sN^{-2} - 2T_a} \mu(C_{N,\zeta}) \geq e^{sN^{-2} - 2T_a} b^* \geq h^{-1}(1-\epsilon).$$

Theorem 1.2(i) then yields $u_\mu(t, s\zeta) \geq 1 - \epsilon$ for all $\zeta \in S^{d-1}$, $s \geq L_\epsilon^+$, and $t > -T_a$. Hence

$$B^c_{L_\epsilon^+} \subseteq \Omega_{u_\mu, 1-\epsilon}(t)$$

for all $t > -T_a$, and the result follows with $L_\epsilon := \max\{L_\epsilon^-, L_\epsilon^+\}$. ∎

Since each u_μ is a transition solution it follows that u_μ is indeed a transition solution with bounded width.

5. Appendix

In this appendix, we prove Proposition 4.4. The proof uses two auxiliary lemmas:

Lemma 5.1. *If* $0 \in \operatorname{int}(\operatorname{ch}(\{x_1, \dots, x_k\}))$, *then there are* $c_i > 0$ *such that*

$$\sum_{i=1}^k c_i x_i = 0.$$

Proof. There obviously are $a_i \geq 0$ with $\sum_{i=1}^k a_i = 1$ such that $\sum_{i=1}^k a_i x_i = 0$. Since $0 \in \operatorname{int}(\operatorname{ch}(\{x_1, \dots, x_k\}))$, there is $\delta > 0$ such that we have $-\delta x_i \in \operatorname{ch}(\{x_1, \dots, x_k\})$ for each i. Thus, each $-\delta x_i$ may be written as a convex combination $-\delta x_i = \sum_{j=1}^k b_{ij} x_j$, with $b_{ij} \geq 0$ and $\sum_{j=1}^k b_{ij} = 1$. Then

$$\begin{aligned} 0 &= \sum_{i=1}^k a_i x_i + \sum_{i=1}^k \delta x_i + \sum_{i=1}^k -\delta x_i \\ &= \sum_{i=1}^k (a_i + \delta) x_i + \sum_{i=1}^k \sum_{j=1}^k b_{ij} x_j \\ &= \sum_{i=1}^k (a_i + \delta + \sum_{j=1}^k b_{ji}) x_i. \end{aligned}$$

Hence we can take $c_i := a_i + \delta + \sum_{j=1}^k b_{ji} > 0$. ∎

Lemma 5.2. *If* $0 \in \operatorname{int}(\operatorname{ch}(\{x_1, \dots, x_k\}))$, *then for any* $r > 0$ *there is* $\epsilon > 0$ *such that any* $p \in B_\epsilon$ *can be written as*

$$p = \sum_{i=1}^k a_i x_i, \qquad \textit{where } |a_i| \leq r.$$

Proof. There is $\delta > 0$ such that $B_\delta \subseteq \operatorname{ch}(\{x_1, \dots, x_k\})$. Then each $z \in B_\delta$ can be written as $z = \sum_{i=1}^k b_i x_i$, where $b_i \geq 0$ and $\sum_{i=1}^k b_i = 1$. Given $r > 0$, let $\epsilon := r\delta$. Then for any $p \in B_\epsilon$ we have $p = rz$ for some $z \in B_\delta$, so $p = \sum_{i=1}^k (rb_i) x_i$ with some $b_i \in [0, 1]$. The proof is finished. ∎

Proof of Proposition 4.4. Let $c_i > 0$ ($i = 1, \dots, k$) be as in Lemma 5.1. Consider the system of linear equations

$$A\Theta := \begin{bmatrix} 1+a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & 1+a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & 1+a_{kk} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_k \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

with some given $a_{ij} \in \mathbb{R}$. The determinant

$$\det A = \begin{vmatrix} 1+a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & 1+a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & 1+a_{kk} \end{vmatrix}$$

is a continuous function of the a_{ij} and equals 1 when they all vanish. Thus, there is $r_0 > 0$ such that $\max_{i,j} |a_{ij}| \le r_0$ implies $\det A > 0$. Similarly,

$$\det M_l = \begin{vmatrix} 1+a_{11} & \dots & a_{1(l-1)} & c_1 & a_{1(l+1)} & \dots & a_{1k} \\ a_{21} & \dots & a_{2(l-1)} & c_2 & a_{2(l+1)} & \dots & a_{2k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{k1} & \dots & a_{k(l-1)} & c_k & a_{k(l+1)} & \dots & 1+a_{kk} \end{vmatrix}$$

depends continuously on the a_{ij} and equals $c_l > 0$ when they all vanish. Thus, there is $r_1 > 0$ such that $\max_{i,j} |a_{ij}| \le r_1$ implies $\max_l \det M_l > 0$.

Let $r := \min\{r_0, r_1\}$, and let $\epsilon > 0$ be as in Lemma 5.2. Let $y_j \in B_\epsilon(x_j)$ be arbitrary and denote $p_j := y_j - x_j$. Then Lemma 5.2 shows that each p_j can be written as

$$p_j = \sum_{i=1}^{k} a_{ij} x_i, \qquad \text{with } |a_{ij}| \le r.$$

Finally, for each $j = 1, \dots, k$ let

$$\theta_j := \frac{\det M_j}{\det A} > 0,$$

so that $\Theta = (\theta_1, \ldots, \theta_k)$ is the (unique) solution of the above system (by Cramer's rule). Then

$$\begin{aligned}
\sum_{j=1}^{k} \theta_j y_j &= \sum_{j=1}^{k} \theta_j (x_j + p_j) \\
&= \sum_{j=1}^{k} \theta_j x_j + \sum_{j=1}^{k} \theta_j \sum_{i=1}^{k} a_{ij} x_i \\
&= \sum_{i=1}^{k} \theta_i x_i + \sum_{i=1}^{k} \Big[\sum_{j=1}^{k} a_{ij} \theta_j x_i \Big] \\
&= \sum_{i=1}^{k} \Big[\theta_i + \sum_{j=1}^{k} a_{ij} \theta_j \Big] x_i \\
&= \sum_{i=1}^{k} c_i x_i = 0.
\end{aligned}$$

Normalizing now yields the desired convex combination

$$0 = \sum_{j=1}^{k} \frac{\theta_j}{\sum_{i=1}^{k} \theta_i} y_j,$$

and the proof is finished. ∎

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