

TRANSITION FRONTS FOR INHOMOGENEOUS MONOSTABLE REACTION-DIFFUSION EQUATIONS VIA LINEARIZATION AT ZERO

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ABSTRACT. We prove existence of transition fronts for a large class of reaction-diffusion equations in one dimension, with inhomogeneous monostable reactions. We construct these as perturbations of corresponding front-like solutions to the linearization of the PDE at $u = 0$. While a close relationship of the solutions to the two PDEs has been well known and exploited for KPP reactions (and our method is an extension of such ideas from [16]), to the best of our knowledge this is the first time such an approach has been used in the construction and study of fronts for non-KPP monostable reactions.

1. INTRODUCTION

We study transition fronts for the one-dimensional reaction-diffusion equation

$$u_t = u_{xx} + f(x, u), \quad (1.1)$$

with an inhomogeneous non-negative reaction $f \geq 0$ satisfying $f(x, 0) = f(x, 1) = 0$, and with $u \in [0, 1]$. Such PDEs model a host of natural processes such as combustion, chemical reactions, population dynamics and others, with u representing (normalized) temperature, concentration of a reactant, or population density.

Both $u \equiv 0$ and $u \equiv 1$ are equilibrium solutions of (1.1) and one is interested in the study of propagation of reaction in space, that is, invasion of the state $u = 0$ by the state $u = 1$. An important class of solutions modeling the propagation of reaction are transition fronts. A (*right-moving*) *transition front* is any entire solution $u : \mathbb{R}^2 \rightarrow [0, 1]$ of (1.1) which satisfies

$$\lim_{x \rightarrow -\infty} u(t, x) = 1 \quad \text{and} \quad \lim_{x \rightarrow +\infty} u(t, x) = 0 \quad (1.2)$$

for each $t \in \mathbb{R}$. In addition, we also require that for any $\varepsilon > 0$ there exists $L_\varepsilon < \infty$ such that

$$\sup_{t \in \mathbb{R}} \text{diam}\{x \in \mathbb{R} \mid \varepsilon \leq u(t, x) \leq 1 - \varepsilon\} \leq L_\varepsilon. \quad (1.3)$$

The definition of a left-moving transition front is similar, with the limits in (1.2) exchanged. We will only study right-moving fronts here because the treatment of both cases is identical, up to a reflection in x . We note that the above definition is from [3, 6, 12].

We will consider here the case of *monostable reactions*, for which $u \equiv 1$ is an asymptotically stable solution while $u \equiv 0$ is unstable. We assume that f is Lipschitz,

$$f(x, 0) = f(x, 1) = 0 \quad \text{for } x \in \mathbb{R}, \quad (1.4)$$

$$a(x) := f_u(x, 0) > 0 \quad (1.5)$$

exists for $x \in \mathbb{R}^d$, and

$$a(x)g_0(u) \leq f(x, u) \leq a(x)g_1(u) \quad \text{for } (x, u) \in \mathbb{R} \times [0, 1], \quad (1.6)$$

where $g_0, g_1 \in C^1([0, 1])$ satisfy

$$g_0(0) = g_0(1) = 0, \quad g'_0(0) = 1, \quad g_0(u) > 0 \quad \text{and} \quad g'_0(u) \leq 1 \quad \text{for } u \in (0, 1), \quad (1.7)$$

$$g_1(0) = 0, \quad g'_1(0) = 1, \quad g'_1(u) \geq 1 \quad \text{for } u \in [0, 1], \quad (1.8)$$

$$\int_0^1 \frac{g_1(u) - g_0(u)}{u^2} du < \infty. \quad (1.9)$$

Finally, we let

$$a_- := \inf_{x \in \mathbb{R}} a(x) \leq \sup_{x \in \mathbb{R}} a(x) =: a_+. \quad (1.10)$$

When the reaction $f(x, u) = f(u) \geq 0$ is homogeneous, a special case of transition fronts are *traveling fronts*. These are of the form $u(t, x) = U(x - ct)$, with some front speed c and front profile U such that $\lim_{s \rightarrow -\infty} U(s) = 1$ and $\lim_{s \rightarrow \infty} U(s) = 0$, and their study goes back to the seminal works of Kolmogorov, Petrovskii, and Piskunov [5], and Fisher [4]. They considered *KPP reactions*, a special case of monostable reactions with $g_1(u) = u$, and found that for each $c \geq c_0 := 2\sqrt{f'(0)}$ there is a unique traveling front $u(t, x) = U_c(x - ct)$. A simple phase-plane analysis argument (see Aronson and Weinberger [1]) shows that this turns out to be the case for general homogeneous monostable reactions, although with a different $c_0 \geq 2\sqrt{f'(0)}$. In contrast, *ignition reactions*, satisfying $f(u) = 0$ for $u \in [0, \theta] \cup \{1\}$ and $f(u) > 0$ for $u \in (\theta, 1)$ (for some *ignition temperature* $\theta \in (0, 1)$), give rise to a single speed $c_0 > 0$ and a single traveling front [1].

Despite many developments for homogeneous and space-periodic reactions in the almost eight decades since [4, 5] (see the reviews [2, 15] and references therein), transition fronts in spatially non-periodic media have only been studied relatively recently. The first existence of transition fronts result, for small perturbations of homogeneous *bistable reactions* (the latter are such that $f(u) < 0$ for $u \in (0, \theta)$ and $f(u) > 0$ for $u \in (\theta, 1)$), was obtained by Vakulenko and Volpert [14]. Existence without a hypothesis of closeness to a homogeneous reaction, for ignition reactions of the form $f(x, u) = a(x)g(u)$ with some ignition g , was proved by Mellet, Roquejoffre, and Sire [8], and by Nolen and Ryzhik [11] (see also [7] for uniqueness and stability results for these reactions). Existence, uniqueness, and stability of fronts for general inhomogeneous ignition reactions was proved by Zlatoš [17]. He also proved existence of fronts for some monostable reactions which are in some sense not too far from ignition ones (satisfying, in particular, $\sup_{x \in \mathbb{R}} f_u(x, 0) \leq \frac{1}{4}c_0^2$, with c_0 the unique speed for some ignition f_0 with $f(x, u) \geq f_0(u)$ for all $(x, u) \in \mathbb{R}^d \times [0, 1]$). All these results are based on recovering a front as a locally uniform limit, along a subsequence, of solutions u_n of the Cauchy problem with initial data $u_n(\tau_n, x) \approx \chi_{(-\infty, -n)}(x)$, where $\tau_n \rightarrow -\infty$ are such that $u_n(0, 0) = \frac{1}{2}$. Existence of a limit u on \mathbb{R}^2 is guaranteed by parabolic regularity, and the challenge is to show that u is a transition front. We note that even in the monostable case in [17], when one expects multiple transition fronts, existence of only a single transition front was obtained.

A very different approach has been used by Nolen, Roquejoffre, Ryzhik, and Zlatoš [10], and by Zlatoš [16] to prove existence of multiple transition fronts for inhomogeneous KPP reactions. It is well known that when f is KPP, then there is a close relationship between the solutions of (1.1) and those of its linearization

$$v_t = v_{xx} + a(x)v \quad (1.11)$$

at $u = 0$. The reason for this is that all KPP fronts are *pulled*, with the front speeds determined by the reaction at $u = 0$, which is due to the *reaction strength* $\frac{f(x,u)}{u}$ being largest at $u = 0$ for any fixed $x \in \mathbb{R}$. This is in stark contrast with ignition fronts, which are always *pushed* because they are “driven” by the reaction at intermediate values of u .

One can therefore consider the simpler *front-like solutions* of (1.11), which are of the form

$$v_\lambda(t, x) = e^{\lambda t} \phi_\lambda(x). \quad (1.12)$$

Here $\phi_\lambda > 0$ is a generalized eigenfunction of the operator $\mathcal{L} := \partial_{xx} + a(x)$, satisfying

$$\phi_\lambda'' + a(x)\phi_\lambda = \lambda\phi_\lambda \quad (1.13)$$

on \mathbb{R} , which exponentially grows to ∞ as $x \rightarrow -\infty$ and exponentially decays to 0 as $x \rightarrow \infty$. If we let $\lambda_0 := \sup \sigma(\mathcal{L})$ be the supremum of the spectrum of \mathcal{L} , which satisfies $\lambda_0 \in [a_-, a_+]$ because $\sigma(\partial_{xx}) = (-\infty, 0]$, then it is a well known spectral theory result that such ϕ_λ exists precisely when $\lambda > \lambda_0$, and is unique if we also require $\phi_\lambda(0) = 1$.

For KPP reactions one can try to use these solutions to find transition fronts for (1.1) with

$$\lim_{x \rightarrow \infty} \frac{u_\lambda(t, x)}{v_\lambda(t, x)} = 1 \quad (1.14)$$

for each $t \in \mathbb{R}$, at least for some $\lambda > \lambda_0$. This has been achieved in [10] for KPP reactions which converge to a homogeneous KPP reaction as $|x| \rightarrow \infty$, and for more general KPP reactions in [16]. In both cases one needs $\lambda_0 < 2a_-$ (otherwise it is possible that no transition fronts exist [10]) and $\lambda \in (\lambda_0, 2a_-)$.

In the present paper we show that this linearization approach can be extended to general *non-KPP monostable reactions*. Our method is an extension of the (relatively simple and robust) approach from [16]. There it was discovered that while v_λ is obviously a super-solution of (1.1) when $g_1(u) = u$ (i.e., in the KPP case), one can also use v_λ to find a sub-solution of the form $\tilde{w}_\lambda(t, x) = \tilde{h}_\lambda(v_\lambda(t, x))$, for $\lambda \in (\lambda_0, 2a_-)$ and an appropriate g_0 -dependent increasing function $\tilde{h}_\lambda : [0, \infty) \rightarrow [0, 1)$ with

$$\tilde{h}_\lambda(0) = 0, \quad \tilde{h}'_\lambda(0) = 1, \quad \lim_{v \rightarrow \infty} \tilde{h}_\lambda(v) = 1, \quad \tilde{h}_\lambda(v) \leq v \quad \text{on } [0, \infty). \quad (1.15)$$

It follows that $\tilde{w}_\lambda \leq v_\lambda$, and one can then find a transition front u_λ between the two using parabolic regularity (see below).

Since this \tilde{w}_λ depends on g_0 but not on g_1 , it remains a sub-solution even for $g_1(u) \geq u$ (the latter follows from (1.8)). On the other hand, v_λ need not be anymore a super-solution. However, we prove here that one can still construct a super-solution of the form $w_\lambda(t, x) = h_\lambda(v_\lambda(t, x))$, for an appropriate increasing $h_\lambda : [0, \infty) \rightarrow [0, \infty)$ such that

$$h_\lambda(0) = 0, \quad h'_\lambda(0) = 1, \quad h''_\lambda(v) \geq 0 \quad \text{on } h_\lambda^{-1}([0, 1]). \quad (1.16)$$

Once again, we then find a transition front u_λ between \tilde{w}_λ and $\min\{w_\lambda, 1\}$.

Moreover, a result of Nadin [9] (see also [12]) shows that once some front exists, then also a (time-increasing) *critical front* exists. The latter is a transition front u_C for (1.1) such that if $u \neq u_C$ is any other transition front and $u(t, x) = u_C(t, x)$ for some $(t, x) \in \mathbb{R}^2$, then

$$[u_C(t, y) - u(t, y)](y - x) < 0$$

for all $y \neq x$. That is, a critical front is the (unique up to time translation) “steepest” transition front for (1.1), and is the inhomogeneous version of the minimal speed front for homogeneous reactions. Indeed, if f is homogeneous, then u_C is precisely the traveling front with the minimal speed c_0 .

Thus we obtain the following result.

Theorem 1.1. *Assume (1.4)–(1.9), let $\nu := \sup_{u \in (0, 1]} \frac{g_1(u)}{u} \geq 1$, and let the supremum of the spectrum of $\mathcal{L} := \partial_{xx} + a(x)$ be $\lambda_0 := \sup \sigma(\mathcal{L}) \in [a_-, a_+]$. If $\lambda \in (\lambda_0, 2a_-)$ satisfies*

$$\lambda \leq 2a_- - \frac{2\sqrt{\nu - 1}}{\sqrt{\nu} + \sqrt{\nu - 1}}a_+, \quad (1.17)$$

then (1.1) has a transition front u_λ with $(u_\lambda)_t > 0$, satisfying (1.14). In particular, if λ_0 is smaller than the right-hand side of (1.17), then a critical front u_C also exists and $(u_C)_t > 0$.

Remarks. 1. If $a(x) = f_u(x, 0)$ is constant on \mathbb{R} , then $\lambda_0 = a_- = a_+$, so the right-hand side of (1.17) is always greater than λ_0 . Thus a transition front exists for any g_0, g_1 in this case.

2. The front u_λ does not have a constant speed in general, but when f is stationary ergodic in x , then it almost surely has an *asymptotic speed* $c_\lambda > 0$ in the sense that if $X(t)$ is the rightmost point such that $u(t, X(t)) = \frac{1}{2}$, then

$$\lim_{|t| \rightarrow \infty} \frac{X(t)}{t} = c_\lambda.$$

This is because the same claim holds for v_λ [16] and $\tilde{h}_\lambda(v_\lambda) \leq u_\lambda \leq h_\lambda(v_\lambda)$.

3. The result also holds with v_λ replaced by more general solutions of (1.11) of the form $v_\mu(t, x) \equiv \int_{\mathbb{R}} v_\lambda(t, x) d\mu(\lambda)$, with μ a finite non-negative non-zero Borel measure supported on a compact subset of $(\lambda_0, 2a_- - 2\sqrt{\nu - 1}(\sqrt{\nu} + \sqrt{\nu - 1})^{-1}a_+]$ (or of $(\lambda_0, 2a_-)$ if $\nu = 1$).

4. The result also applies to the more general equation

$$u_t = (A(x)u_x)_x + q(x)u_x + f(x, u)$$

with

$$0 < A_- \leq A(x) \leq A_+ < \infty \quad \text{and} \quad |q(x)| \leq q_+ < \infty$$

for all $x \in \mathbb{R}$, provided that $q_+ \leq 2\sqrt{(aA)_-}$ with $(aA)_- := \inf_{x \in \mathbb{R}} [a(x)A(x)]$, where

$$\lambda_0 := \sup_{\psi \in H^1(\mathbb{R})} \frac{\int_{\mathbb{R}} [-A(x)\psi'(x)^2 + q(x)\psi'(x)\psi(x) + a(x)\psi(x)^2] dx}{\int_{\mathbb{R}} \psi(x)^2 dx} \quad (\geq a_-)$$

and $2a_-$ is replaced in (1.17) by

$$\lambda_1 := \inf_{x \in \mathbb{R}} \left\{ a(x) + \sqrt{(aA)_-} \left[\sqrt{(aA)_-} - |q(x)| \right] A(x)^{-1} \right\} \quad (\leq 2a_-).$$

We indicate the proofs of Remarks 2–4 after the proof of the theorem.

Our construction of the super-solution w_λ is of independent interest and extends to more general equations in several dimensions, possibly with time-dependent coefficients. Hence we state it here as a separate result.

Lemma 1.2. *Let the function $f(t, x, u) \geq 0$, positive definite matrix $A(t, x)$, and vector field $q(t, x)$ be all Lipschitz, with $(t, x, u) \in (t_0, t_1) \times \mathbb{R}^d \times [0, 1]$ and some $-\infty < t_0 < t_1 \leq \infty$. Assume that $a(t, x) \equiv f_u(t, x, 0) > 0$ exists, (1.4)–(1.9) hold with $(t, x) \in (t_0, t_1) \times \mathbb{R}^d$ in place of $x \in \mathbb{R}^d$, and define $\nu := \sup_{u \in (0, 1]} \frac{g_1(u)}{u} \geq 1$. Let $v > 0$ be a solution of*

$$v_t = \nabla \cdot (A(t, x) \nabla v) + q(t, x) \cdot \nabla v + a(t, x)v$$

on $(t_0, t_1) \times \mathbb{R}^d$. If $\nu > 1$ and for some $\alpha \leq (\sqrt{\nu} - \sqrt{\nu - 1})^2$ (or for some $\alpha < 1$ if $\nu = 1$),

$$\nabla v(t, x) \cdot A(t, x) \nabla v(t, x) \leq \alpha a(t, x)v(t, x)^2 \quad (1.18)$$

holds for all $(t, x) \in (t_0, t_1) \times \mathbb{R}^d$, then there exist increasing functions \tilde{h} satisfying (1.15) and h satisfying (1.16) such that $\tilde{w} := \tilde{h}(v)$ is a sub-solution of

$$u_t = \nabla \cdot (A(t, x) \nabla u) + q(t, x) \cdot \nabla u + f(t, x, u) \quad (1.19)$$

on $(t_0, t_1) \times \mathbb{R}^d$ and $w := h(v)$ is a super-solution on $[(t_0, t_1) \times \mathbb{R}^d] \cap \{(t, x) \mid w(t, x) \leq 1\}$. Therefore, if u solves (1.19) with

$$\tilde{w}(t_0, x) \leq u(t_0, x) \leq \min\{w(t_0, x), 1\} \quad (1.20)$$

for all $x \in \mathbb{R}^d$, then for all $(t, x) \in (t_0, t_1) \times \mathbb{R}^d$ we have

$$\tilde{w}(t, x) \leq u(t, x) \leq \min\{w(t, x), 1\}. \quad (1.21)$$

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2. PROOF OF THEOREM 1.1 (USING LEMMA 1.2)

Let $\lambda \in (\lambda_0, 2a_-)$ and $v = v_\lambda$ be from (1.12), with $\phi = \phi_\lambda > 0$ from (1.13) with $\lim_{x \rightarrow \infty} \phi(x) = 0$ and $\phi(0) = 1$. It is proved in [16, Proof of Theorem 1.1] that there exists a unique such ϕ and it satisfies

$$\phi'(x)^2 \leq \alpha a_- \phi(x)^2 \quad (2.1)$$

for $\alpha := 1 - (2a_- - \lambda)a_+^{-1} < 1$ and all $x \in \mathbb{R}$, as well as

$$\phi(x) \geq 2\phi(y) \quad (2.2)$$

for some $L < \infty$ and any $y - x \geq L$.

Since $\alpha \leq (\sqrt{\nu} - \sqrt{\nu - 1})^2$ is, by the definition of α , equivalent to

$$\lambda \leq 2a_- - \left[1 - (\sqrt{\nu} - \sqrt{\nu - 1})^2\right] a_+ = 2a_- - \frac{2\sqrt{\nu - 1}}{\sqrt{\nu} + \sqrt{\nu - 1}} a_+$$

(which is (1.17)), Lemma 1.2 applies to v and (1.1). Thus we have (1.21) and a standard limiting argument now recovers an entire solution to (1.1) between \tilde{w} and $\min\{w, 1\}$. We let u_n be the solution of (1.1) on $(-n, \infty) \times \mathbb{R}$ with $u_n(-n, x) := \tilde{w}(-n, x)$. Since $\tilde{w}(t, x) \leq \min\{w(t, x), 1\}$ because $h(v) \geq v$ for $v \in h^{-1}([0, 1])$, (1.20) is satisfied with $t_0 := -n$ and we have (1.21) on $(-n, \infty) \times \mathbb{R}$. By parabolic regularity, there is a subsequence of $\{u_n\}$ which converges, locally uniformly on \mathbb{R}^2 , to an entire solution u of (1.1). We obviously have

$$\tilde{w} \leq u \leq \min\{w, 1\}, \quad (2.3)$$

and (1.14) for $u_\lambda := u$ follows from $\tilde{h}'(0) = h'(0) = 1$. We also have $u_t \geq 0$, because $(u_n)_t \geq 0$ due to $\tilde{w}_t = \tilde{h}'(v)v_t \geq 0$ and the maximum principle for $(u_n)_t$ (which satisfies a linear equation and is non-negative at $t = -n$). The strong maximum principle then gives $u_t > 0$ because obviously $u_t \not\equiv 0$. Finally, u is a transition front because the second limit in (1.2) follows from $\lim_{x \rightarrow \infty} \phi(x) = 0$ and (1.16), and (1.3) holds with

$$L_\varepsilon := L \left[\log_2 \left(\tilde{h}^{-1}(1 - \varepsilon) - h^{-1}(\varepsilon) \right) \right]$$

due to (2.3) and (2.2) (with \tilde{h}, h from the lemma). The first limit in (1.2) is then obvious from $u \leq 1$, and the proof is finished by using the result from [9] for critical fronts.

The claim in Remark 2 is proved as an analogous statement in [16, Theorem 1.2].

The claim in Remark 3 holds because L can be chosen uniformly for all λ in the support of μ [16] and so (2.2) holds with $\phi(\cdot)$ replaced by $v_\mu(t, \cdot)$. Also, v_μ satisfies (2.1) with α corresponding to $\lambda := \sup \text{supp } \mu$.

The claim in Remark 4 holds because (2.1) and (2.2) continue to hold in that case, albeit with $2a_-$ replaced by λ_1 in the definition of α [16].

3. PROOF OF LEMMA 1.2

Lemma 2.1 in [16] shows that there is an increasing $\tilde{h} = \tilde{h}_\lambda$ as in (1.15) such that $\tilde{w}(t, x) := \tilde{h}(v)$ is a sub-solution of (1.1) with the reaction $\min\{f(x, u), a(x)u\}$ (which is a KPP reaction). Then \tilde{w} is also a sub-solution of (1.1), which yields the first inequality in (1.21).

We will next find an increasing $h = h_\lambda$ as in (1.16) such that $w(t, x) := h(v(t, x))$ will be a super-solution on the space-time domain where $w(t, x) \leq 1$, which will yield the second inequality in (1.21) because $u \leq 1$ by the hypotheses. Our proof will be a super-solution counterpart to the sub-solution argument in [16, Lemma 2.1]; it was a little surprising to us that such a counterpart argument can be found for non-KPP reactions.

If h is as in (1.16), then (1.18) shows that

$$\begin{aligned} w_t - \nabla \cdot (A \nabla w) - q \cdot \nabla w &= h'(v)[v_t - \nabla \cdot (A \nabla v) - q \cdot \nabla v] - h''(v) \nabla v \cdot A \nabla v \\ &= h'(v)av - h''(v) \nabla v \cdot A \nabla v \\ &\geq a[vh'(v) - \alpha v^2 h''(v)] \end{aligned}$$

when $w(t, x) \leq 1$. We can then conclude that w is a super-solution of (1.1) where $w(t, x) \leq 1$ once we show that on $h^{-1}([0, 1])$ we also have

$$vh'(v) - \alpha v^2 h''(v) \geq g_1(h(v)). \quad (3.1)$$

It therefore remains to find h satisfying (1.16) and (3.1). We let $c := \alpha^{1/2} + \alpha^{-1/2}$ and notice that since $\gamma + \gamma^{-1} \geq 2\sqrt{\nu}$ for all positive $\gamma \leq \sqrt{\nu} - \sqrt{\nu - 1}$, the hypothesis $\alpha \leq (\sqrt{\nu} - \sqrt{\nu - 1})^2$ yields $c \geq 2\sqrt{\nu}$. Next let U be the unique solution to the ODE

$$U'' + cU' + g_1(U) = 0 \quad (3.2)$$

on $[s_0, \infty)$, with

$$U(s_0) = 1 \quad \text{and} \quad U'(s_0) = -\sqrt{\alpha}g_1(1), \quad (3.3)$$

where $s_0 \in \mathbb{R}$ will be chosen later. (This is the ODE that would be satisfied by the traveling front profile with speed c for the homogeneous reaction $g_1(u)$ if we had $g_1(1) = 0$; this profile would then also satisfy $\lim_{s \rightarrow -\infty} U(s) = 1$ and $\lim_{s \rightarrow -\infty} U'(s) = 0$ instead of (3.3).)

Notice that $U'(s_0) \geq -\frac{c}{2}$ because $g_1(1) \leq \nu$ and

$$\sqrt{\alpha} \leq (\sqrt{\nu} + \sqrt{\nu - 1})^{-1} \leq \nu^{-1/2}.$$

Let $V(s) := U'(s)$, and consider the curve $\theta := \{(U(s), V(s))\}_{s \geq s_0}$. It is easy to see that θ cannot leave the closed triangle T in the (U, V) plane with sides $V = 0$, $U = 1$, and $V = -\frac{c}{2}U$. This is because $(U(s_0), V(s_0)) \in T$ and on ∂T , the vector field $(V, -cV - g_1(U))$ either points inside T or is parallel to ∂T . Here we use $c \geq 2\sqrt{\nu}$ to obtain on the third side

$$\left(\frac{c}{2}, 1\right) \cdot \left(-\frac{c}{2}U, -c\left(-\frac{c}{2}U\right) - g_1(U)\right) = \frac{c^2}{4}U - g_1(U) \geq \nu U - g_1(U) \geq 0. \quad (3.4)$$

It follows that $U'(s) < 0$ on $[s_0, \infty)$, and since $g_1(U(s)) > 0$, $U(s)$ cannot have local minima on $[s_0, \infty)$. Hence $\lim_{s \rightarrow \infty} U(s)$ exists and $\lim_{s \rightarrow \infty} U'(s) = 0$. Finally, $g_1 > 0$ on $(0, 1]$ yields $\lim_{s \rightarrow \infty} U(s) = 0$.

We now define $h(0) := 0$ and

$$h(v) := U(-\alpha^{-1/2} \ln v) \quad (3.5)$$

for $v \in (0, e^{-\sqrt{\alpha}s_0}]$, so h is increasing and continuous at 0, with $h(e^{-\sqrt{\alpha}s_0}) = 1$ (we then extend h onto $[0, \infty)$ arbitrarily, only requiring that it be increasing). Since $c > 2\sqrt{g_1'(0)} = 2$ and

$$\int_0^1 \frac{g_1(u) - u}{u^2} du < \infty$$

by (1.7) and (1.9), a result of Uchiyama [13, Lemma 2.1] shows $\lim_{s \rightarrow \infty} U(s)e^{\sqrt{\alpha}s} \in (0, \infty)$. (This result assumes $g_1(1) = 0$ but we can extend g_1, U to $[0, 2]$ so that $g_1(2) = 0$ and U satisfies (3.2), and then apply [13] to $\tilde{g}(u) := \frac{1}{2}g_1(2u)$ and the function $\tilde{U}(s) := \frac{1}{2}U(s)$.)

If we now pick the unique s_0 in (3.3) such that $\lim_{s \rightarrow \infty} U(s)e^{\sqrt{\alpha}s} = 1$ (notice that (3.2) is an autonomous ODE), we obtain $h'(0) = 1$. We also have (3.1) on $[0, e^{-\sqrt{\alpha}s_0}]$ because on that interval, (3.2) immediately yields

$$\alpha v^2 h''(v) - v h'(v) + g_1(h(v)) = 0. \quad (3.6)$$

It therefore remains to show that $h''(v) \geq 0$ on $[0, e^{-\sqrt{\alpha}s_0}]$. Due to (3.6) and (3.5), this is equivalent to

$$-U'(s) \geq \sqrt{\alpha} g_1(U(s)) \quad (3.7)$$

for $s \geq s_0$. Thus we need to show that θ stays at or below $\psi := \{(U(s), -\sqrt{\alpha} g_1(U(s)))\}_{s \geq s_0}$. This is true at $s = s_0$ by the definition of U , so it is sufficient to show that on ψ , the vector field $(V, -cV - g_1(U))$ points either below or is parallel to ψ . This holds because the normal vector to ψ pointing down is $(\sqrt{\alpha} g_1'(U)V, V)$, so on ψ we have

$$\begin{aligned} (V, -cV - g_1(U)) \cdot (\sqrt{\alpha} g_1'(U)V, V) &= \alpha^{1/2} g_1'(U) V^2 - (\alpha^{1/2} + \alpha^{-1/2}) V^2 - \alpha^{1/2} g_1(U) \alpha^{-1/2} V \\ &= \alpha^{1/2} (g_1'(U) - 1) V^2, \end{aligned}$$

which is non-negative due to (1.8) and $U(s) \leq 1$ for $s \geq s_0$. It follows that $h''(v) \geq 0$ on $[0, e^{-\sqrt{\alpha} s_0}]$ and so h satisfies (1.16) and (3.1). The proof is finished.

REFERENCES

- [1] D.G. Aronson and H.F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math. **30** (1978), 33–76.
- [2] H. Berestycki, *The influence of advection on the propagation of fronts in reaction-diffusion equations*, Nonlinear PDEs in Condensed Matter and Reactive Flows, NATO Science Series C, 569, H. Berestycki and Y. Pomeau eds, Kluwer, Dordrecht, 2003.
- [3] H. Berestycki and F. Hamel, *Generalized transition waves and their properties*, Comm. Pure Appl. Math **65** (2012), 592–648.
- [4] R. Fisher, *The wave of advance of advantageous genes*, Ann. Eugenics **7** (1937), 355–369.
- [5] A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov, *Étude de l'équation de la chaleur de matière et son application à un problème biologique*, Bull. Moskov. Gos. Univ. Mat. Mekh. **1** (1937), 1–25.
- [6] H. Matano, talks at several conferences.
- [7] A. Mellet, J. Nolen, J.-M. Roquejoffre and L. Ryzhik, *Stability of generalized transition fronts*, Commun. PDE **34** (2009), 521–552.
- [8] A. Mellet, J.-M. Roquejoffre and Y. Sire, *Generalized fronts for one-dimensional reaction-diffusion equations*, Discrete Contin. Dyn. Syst. **26** (2010), 303–312.
- [9] G. Nadin, *Critical travelling waves for general heterogeneous one dimensional reaction-diffusion equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.
- [10] J. Nolen, J.-M. Roquejoffre, L. Ryzhik, and A. Zlatoš, *Existence and non-existence of Fisher-KPP transition fronts*, Arch. Ration. Mech. Anal. **203** (2012), 217–246.
- [11] J. Nolen and L. Ryzhik, *Traveling waves in a one-dimensional heterogeneous medium*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 1021–1047.
- [12] W. Shen, *Traveling waves in diffusive random media*, J. Dynam. Differ. Equat. **16** (2004), 1011–1060.
- [13] K. Uchiyama, *The behavior of solutions of some non-linear diffusion equations for large time*, J. Math. Kyoto Univ. **18** (1978), 453–508.
- [14] S. Vakulenko and V. Volpert, *Generalized travelling waves for perturbed monotone reaction-diffusion systems*, Nonlinear Anal. **46** (2001), 757–776.
- [15] J. Xin, *Front propagation in heterogeneous media*, SIAM Rev. **42** (2000), 161–230.
- [16] A. Zlatoš, *Transition fronts in inhomogeneous Fisher-KPP reaction-diffusion equations*, J. Math. Pures Appl. **98** (2012), 89–102.
- [17] A. Zlatoš, *Generalized traveling waves in disordered media: Existence, uniqueness, and stability*, Arch. Ration. Mech. Anal. **208** (2013), 447–480.

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