

PROPAGATION OF REACTIONS IN INHOMOGENEOUS MEDIA

ANDREJ ZLATOŠ

ABSTRACT. Consider reaction-diffusion equation $u_t = \Delta u + f(x, u)$ with $x \in \mathbb{R}^d$ and general inhomogeneous ignition reaction $f \geq 0$ vanishing at $u = 0, 1$. Typical solutions $0 \leq u \leq 1$ transition from 0 to 1 as time progresses, and we study them in the region where this transition occurs. Under fairly general qualitative hypotheses on f we show that in dimensions $d \leq 3$, the Hausdorff distance of the super-level sets $\{u \geq \varepsilon\}$ and $\{u \geq 1 - \varepsilon\}$ remains uniformly bounded in time for each $\varepsilon \in (0, 1)$. Thus, u remains uniformly in time close to the characteristic function of $\{u \geq \frac{1}{2}\}$ in the sense of Hausdorff distance of super-level sets. We also show that each $\{u \geq \varepsilon\}$ expands with average speed (over any long enough time interval) between the two spreading speeds corresponding to any x -independent lower and upper bounds on f . On the other hand, these results turn out to be false in dimensions $d \geq 4$, at least without further quantitative hypotheses on f . The proof for $d \leq 3$ is based on showing that as the solution propagates, small values of u cannot escape far ahead of values close to 1. The proof for $d \geq 4$ is via construction of a counter-example for which this fails.

Such results were before known for $d = 1$ but are new for general non-periodic media in dimensions $d \geq 2$ (some are also new for homogeneous and periodic media). They extend in a somewhat weaker sense to monostable, bistable, and mixed reaction types, as well as to transitions between general equilibria $u^- < u^+$ of the PDE, and to solutions not necessarily satisfying $u^- \leq u \leq u^+$.

1. INTRODUCTION AND MOTIVATION

Reaction-diffusion equations are used to model a variety of processes and phenomena such as combustion, chemical reactions, or population dynamics. The baseline model, which already captures a lot of the properties of the dynamics involved, is the parabolic PDE

$$u_t = \Delta u + f(x, u) \tag{1.1}$$

for $u : (t_0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$, where $t_0 \in [-\infty, \infty)$ and $d \geq 1$. If $t_0 > -\infty$, then we also let

$$u(t_0, x) = u_0(x) \quad \text{for } x \in \mathbb{R}^d. \tag{1.2}$$

The Lipschitz *reaction function* f is such that there exist two ordered *equilibria* (time-independent solutions) $u^- < u^+$ for (1.1), and one is usually interested in studying the transition of general solutions of (1.1) from one to the other as $t \rightarrow \infty$.

A prototypical situation is when $u^- \equiv 0$ and $u^+ \equiv 1$, with $f \geq 0$ vanishing at $u = 0, 1$. Here $u \in [0, 1]$ is the (normalized) temperature of fuel, concentration of a reactant, or population density. Depending on the application, f may be either an *ignition reaction* (vanishing near $u = 0$) in combustion models; or a *monostable reaction* (positive for $u \in (0, 1)$) such as *Zeldovich* and *Arrhenius reactions* with $f_u(x, 0) \equiv 0$ in models of chemical reactions and *KPP reaction* with $f(x, u) \leq f_u(x, 0)u$ for all $u \geq 0$ in population dynamics models.

For the sake of clarity of presentation, we will first study this scenario, where our main results are Theorems 2.4 and 2.5. Later we will extend these to more general situations: with general $u^- < u^+$, different types of reactions, including mixtures of ignition, monostable, and *bistable reactions* (the latter have $[f(x, u) - f(x, u^\pm(x))][u - u^\pm(x)] < 0$ for u near $u^\pm(x)$), and for solutions not necessarily satisfying $u^- \leq u \leq u^+$ (see Theorems 2.7 and 2.9). However, in order to minimize technicalities, our first result will be stated in the even simpler setting of ignition reactions with a constant ignition temperature (see Theorem 1.1).

The study of transitions between equilibria of reaction-diffusion equations has seen a lot of activity since the seminal papers of Kolmogorov, Petrovskii, Piskunov [13] and Fisher [9]. Of central interest has been long time propagation of solutions with “typical” initial data, and the related questions about traveling fronts. The first type of such initial data are *spark-like data* — compactly supported, such as in (2.14) below. The second are *front-like data* — vanishing on a half-space $\{x \cdot e \geq R\}$ for some unit vector $e \in \mathbb{R}^d$ and with $\liminf_{x \cdot e \rightarrow -\infty} u_0(x)$ close enough to 1, such as in (2.15). For ignition reaction one can also allow rapid decay to 0 as $|x| \rightarrow \infty$ or $x \cdot e \rightarrow \infty$, such as in (2.12) and (2.13). We will for now discuss these data (and also call the corresponding solutions front-like and spark-like), but later we will turn to more general ones (see, e.g., Theorem 2.5).

In both cases it was proved, first for homogeneous (x -independent) reactions in several dimensions by Aronson, Weinberger [2] and then for x -periodic ones by Freidlin, Gärtner [10, 11] and Weinberger [26], that for typical solutions, the state $u = 1$ invades $u = 0$ with a speed that is asymptotically constant (in each direction for spark-like data) as $t \rightarrow \infty$. Specifically, that for each unit $e \in \mathbb{R}^d$ there is a (*front speed*) $c_e > 0$ such that for any $\delta > 0$,

$$\lim_{t \rightarrow \infty} \inf_{x \cdot e \leq (c_e - \delta)t} u(x, t) = 1 \quad \text{and} \quad \lim_{t \rightarrow \infty} \sup_{x \cdot e \geq (c_e + \delta)t} u(x, t) = 0 \quad (1.3)$$

for front-like initial data; and there is a (*spreading speed*) $s_e \in (0, c_e]$ such that for any $\delta > 0$,

$$\lim_{t \rightarrow \infty} \inf_{x \in (1 - \delta)tS} u(x, t) = 1 \quad \text{and} \quad \lim_{t \rightarrow \infty} \sup_{x \notin (1 + \delta)tS} u(x, t) = 0 \quad (1.4)$$

for spark-like initial data, where $S := \{se \mid \|e\| = 1 \text{ and } 0 \leq s \leq s_e\}$ is the *Wulff shape* for f . Of course, for homogeneous reactions there is $c > 0$ such that $s_e = c_e = c$ for all unit $e \in \mathbb{R}^d$.

Closely related to this is the study of *traveling fronts* for x -independent f and *pulsating fronts* for x -periodic f . Traveling fronts are front-like *entire* (with $t_0 = -\infty$) solutions of (1.1) moving with a constant speed c in a unit direction $e \in \mathbb{R}^d$, of the form $u(t, x) = U(x \cdot e - ct)$ with $\lim_{s \rightarrow -\infty} U(s) = 1$ and $\lim_{s \rightarrow \infty} U(s) = 0$. Pulsating fronts, first introduced by Shigesada, Kawasaki, Teramoto [23] and proved to exist for general periodic f as above by Xin [27] and Berestycki, Hamel [4], are similar but $u(t, x) = U(x \cdot e - ct, x)$ and U is periodic in the second argument. The minimal of the speeds for which such a front exists for a given unit $e \in \mathbb{R}^d$ is then precisely c_e , and we also have $s_e = \inf_{e' \cdot e > 0} [c_{e'} / (e' \cdot e)]$.

The above results hold for fairly general $f \geq 0$, and there is a vast literature on these and many other aspects of reaction-diffusion equations in homogeneous and periodic media. Instead of a more comprehensive discussion, we refer to [4, 26] and the excellent reviews by Berestycki [3] and Xin [28].

Unsurprisingly, the picture becomes less satisfactory for non-periodic reactions, particularly in the several spatial dimensions case $d \geq 2$. The above results and the comparison principle show that if c_0 and c_1 are the (e -independent) speeds associated with homogeneous reactions f_0 and f_1 such that $f_0 \leq f \leq f_1$, then (1.3), (1.4) hold with c_e and S replaced by c_0 and $B_{c_0}(0)$ in the first statements and by c_1 and $B_{c_1}(0)$ in the second ones. That is, transition between $u \sim 0$ and $u \sim 1$ occurs inside a spatial strip or annulus whose width grows linearly in time with speed $c_1 - c_0$ (while for homogeneous and x -periodic media it grows sub-linearly, by taking $\delta \rightarrow 0$ in (1.3), (1.4)). In the general inhomogeneous case, these estimates cannot be improved, unless one includes further restrictive hypotheses on f or is willing to tolerate complicated formulas involving f .

For stationary ergodic reactions, the results should hold as originally stated, but also with $|x - x \cdot e| \leq Ct$ for any $C < \infty$ in (1.3). For $d \geq 2$ this *homogenization* result was proved only in the KPP case, by Lions, Souganidis [14]. (This case has an important advantage of a close relationship of the dynamics for (1.1) and for its linearization at $u = 0$. Other authors also exploited this link in the study of spreading for KPP reactions, e.g., Berestycki, Hamel, Nadin [7]. However, results aiming to more precisely locate the transition region for non-stationary-ergodic reactions are somewhat restricted by the necessity of more complicated hypotheses involving the reaction.) Results from the present paper can be used to approach this problem for ignition and non-KPP monostable reactions. This will be done elsewhere.

The above results hold also in the case $d = 1$, with the stationary ergodic KPP reaction result proved earlier in [11]. However, some recent developments have gone further, particularly for ignition reactions. Mellet, Nolen, Roquejoffre, Ryzhik, Sire [16, 17, 20] proved for reactions of the form $f(x, u) = a(x)f_0(u)$ (with f_0 vanishing on $[0, \theta_0] \cup \{1\}$ and positive on $(\theta_0, 1)$, and $a \geq 1$ bounded above), and Zlatoš [31] for more general ignition reactions the following. There is a unique right-moving (and a unique left-moving) transition front solution and as $t \rightarrow \infty$, each front-like solution with $e = 1$ ($e = -1$) converges in L_x^∞ to its time-translate. A similar result holds for spark-like solutions, when restricted to \mathbb{R}^+ (\mathbb{R}^-). Moreover, if f is stationary ergodic, then (1.3), (1.4) hold with some $c_e = s_e$ for $e = \pm 1$.

The *transition fronts* appearing here are a generalization of the concepts of traveling and pulsating fronts to disordered (non-periodic) media. In the one-dimensional setting they are entire solutions of (1.1) satisfying

$$\lim_{x \rightarrow \mp \infty} u(t, x) = 1 \quad \text{and} \quad \lim_{x \rightarrow \pm \infty} u(t, x) = 0 \quad (1.5)$$

for each $t \in \mathbb{R}$ (with upper sign for right-moving fronts and lower sign for left-moving fronts), as well as $\sup_{t \in \mathbb{R}} L_{u, \varepsilon}(t) < \infty$ for each $\varepsilon \in (0, \frac{1}{2})$, where $L_{u, \varepsilon}(t)$ is the length of the shortest interval containing all $x \in \mathbb{R}$ with $u(t, x) \in [\varepsilon, 1 - \varepsilon]$. This last property is called *bounded width* in [31]. The definition of transition fronts was first given in some specialized cases by Shen [22] and Matano [15], and then in a very general setting (including several dimensions) by Berestycki, Hamel in their fundamental papers [5, 6]. Existence of transition fronts in one-dimensional disordered media (but no long term asymptotics of general solutions) was also proved for bistable reactions which are small perturbations of homogeneous ones by Vakulenko, Volpert [25], for KPP reactions which are (spatially) decaying perturbations of

homogeneous ones by Nolen, Roquejoffre, Ryzhik, Zlatoš [19], for general KPP reactions by Zlatoš [30], and for general monostable reactions which are close to KPP reactions by Tao, Zhu, Zlatoš [24]. We also mention results proving existence of a critical front, once some transition front exists, by Shen [22] and Nadin [18].

While it is again not possible to improve the estimates on the length of the interval on which the transition between $u \sim 0$ and $u \sim 1$ is *guaranteed to happen* (which again grows as $(c_1 - c_0)t$ in time if $f_0 \leq f \leq f_1$), bounded width of transition fronts and the convergence-to-fronts results in [16, 31] show that for ignition reactions and typical solutions, the transition does occur within intervals whose lengths are uniformly bounded in time. Moreover, this bound depends on some bounds on the reaction but neither on the reaction itself, nor on the initial condition. In particular, this shows that after a uniform-in- (f, u, t) scaling in space, each such solution becomes, in some sense, close to the *characteristic function of a time-dependent spatial interval*. Moreover, the convergence-to-fronts results can be used to show that this interval grows in (equally scaled) time with speed within $[c_0, c_1]$.

Experience from observation of natural processes modeled by (1.1) suggests that this picture should be also valid for media in several spatial dimensions. For instance, aerial footage of forest fires spreading through (spatially inhomogeneous) regions demonstrates variously curved but usually relatively narrow lines of fire separating burned ($u \sim 1$) and unburned ($u \sim 0$) areas. However, results demonstrating such phenomena for typical solutions of (1.1) with *general inhomogeneous* reactions have not been previously obtained in dimensions $d \geq 2$.

It turns out that the multi-dimensional case is much more involved in this respect. The first issue is that it is not completely obvious how to extend the definition of bounded width of solutions of (1.1), (1.2) to the multi-dimensional setting, and some first instincts may lead to unsatisfactory results for general non-periodic media (see the discussion below). The extension we introduce here is motivated by the Berestycki-Hamel definition of transition fronts (which are entire solutions of (1.1)) in several dimensions [5, 6]. However, there are a couple of differences, and we discuss the relationship of the two concepts after stating our main results, at the end of the next section.

For solutions $u \in [0, 1]$ of the Cauchy problem for (1.1), our extension is as follows. We let $\Omega_{u,\varepsilon}(t) := \{x \in \mathbb{R}^d \mid u(t, x) \geq \varepsilon\}$ be the ε -super-level set of u at time t and define the *width of the transition zone* of u from ε to $1 - \varepsilon$ (or to be more precise, from $[\varepsilon, 1 - \varepsilon]$ to $1 - \varepsilon$) to be

$$L_{u,\varepsilon}(t) := \inf \{L > 0 \mid \Omega_{u,\varepsilon}(t) \subseteq B_L(\Omega_{u,1-\varepsilon}(t))\} \quad (1.6)$$

for $\varepsilon \in (0, \frac{1}{2})$, with $B_r(A) := \bigcup_{x \in A} B_r(x)$ and $\inf \emptyset = \infty$. Notice that this is precisely the *Hausdorff distance* of the sets $\Omega_{u,\varepsilon}(t)$ and $\Omega_{u,1-\varepsilon}(t)$. We now say that u has *bounded width* if

$$\limsup_{t \rightarrow \infty} L_{u,\varepsilon}(t) < \infty \quad (1.7)$$

for each $\varepsilon \in (0, \frac{1}{2})$. It is necessary to consider large t in (1.7) because $\sup_x u(t, x) < 1$ may hold for each t (e.g., for spark-like solutions); the \limsup will be replaced by $\sup_{t \in \mathbb{R}}$ for entire solutions (see Definition 2.1). So by (1.7), $u \in [0, 1]$ has bounded width if and only if for any $0 < \varepsilon < \varepsilon' < 1$, super-level sets $\Omega_{u,\varepsilon'}(t) \subseteq \Omega_{u,\varepsilon}(t)$ have uniformly (in large time) bounded Hausdorff distance. In particular, each of them is uniformly in (large) time close to $\Omega_{u,1/2}(t)$.

One may wonder why do we not treat the equilibria 0 and 1 in a symmetric fashion and include a similar definition involving the sub-level sets of u as well. (We do so in (2.2) and in the related definition of *doubly-bounded width*, which is for $u \in [0, 1]$ equivalent to uniformly bounded Hausdorff distance of the *boundaries* $\partial\Omega_{u,\varepsilon}(t)$ and $\partial\Omega_{u,\varepsilon'}(t)$.) While adding this requirement works well in one dimension [16, 17, 20, 31], it turns out to be too restrictive for the treatment of sufficiently general (not necessarily periodic) reactions and solutions of the Cauchy problem (1.1), (1.2) in dimensions $d \geq 2$. This is due to $u \equiv 1$ being the invading equilibrium and $u \equiv 0$ the invaded one, coupled with the possibility of arbitrary variations in the medium on arbitrarily large scales in two or more unbounded dimensions. We discuss the issues involved after stating Theorem 1.1 for reactions with a constant ignition temperature θ_0 , which is a special case of Theorem 2.4 below.

Even with a suitable definition at hand, our results proving bounded widths of typical solutions, under quite general and physically natural *qualitative* hypotheses on the reaction, only hold in dimensions $d \leq 3$. In fact, such results are false for $d \geq 4$ without the addition of further *quantitative* hypotheses (e.g., f being sufficiently close to a homogeneous reaction; see Remark 1 after Definition 2.6 below). The reason is that in $d \geq 4$, even in the constant ignition temperature case, intermediate values of u may spread faster than values close to 1 (see the discussion after Definition 2.3 and Remark 1 after Theorem 2.4 for the general ignition case). This turns out to be related to the possibility of existence of non-constant stationary solutions $p \in (0, 1)$ of (1.1) in \mathbb{R}^{d-1} (see Section 10 for details).

Theorem 1.1. *Let f be Lipschitz (with constant K) and non-increasing in u on $[1 - \theta, 1]$ for each $x \in \mathbb{R}^d$ (where $\theta > 0$). Assume that $f_0(u) \leq f(x, u) \leq f_1(u)$ for all $(x, u) \in \mathbb{R}^d \times [0, 1]$, with $f_0, f_1 : [0, 1] \rightarrow [0, \infty)$ vanishing on $[0, \theta_0] \cup \{1\}$ and positive on $(\theta_0, 1)$ (where $\theta_0 > 0$). Let c_0 and c_1 be the spreading speeds of f_0 and f_1 .*

(i) *If $d \leq 3$, then the solution of (1.1), (1.2) with any spark-like (2.12) or front-like (2.13) initial data $u_0 \in [0, 1]$ has bounded width (1.7). In fact, for any $\varepsilon \in (0, \frac{1}{2})$ there are $\ell_\varepsilon, T_\varepsilon$ such that $\sup_{t \geq T_\varepsilon} L_{u,\varepsilon}(t) \leq \ell_\varepsilon$, with ℓ_ε depending only on ε, f_0, K (T_ε also depends on θ, u_0). Finally, u propagates with global mean speed in $[c_0, c_1]$ in the sense of Definition 2.2 below, with $\tau_{\varepsilon,\delta}$ in that definition depending only on $\varepsilon, f_0, K, \delta, f_1$.*

(ii) *If $d \geq 4$, then there are f, f_0, f_1 as above such that no solution of (1.1), (1.2) with compactly supported $u_0 \in [0, 1]$ and satisfying $\limsup_{t \rightarrow \infty} \|u(t, \cdot)\|_\infty > 0$ has bounded width.*

Remarks. 1. Hence in dimensions $d \leq 3$, any typical solution u eventually becomes uniformly close (in the sense of Hausdorff distance of ε -super-level sets for each $\varepsilon \in (0, 1)$) to the characteristic function of the set $\Omega_{u,1/2}(t)$, and the latter grows with speed (averaged over long enough time intervals) essentially in $[c_0, c_1]$. So after a uniform-in- (f, u, t) space-time scaling, typical solutions look like Figure 1, with the shaded area expanding at speeds within $[c_0, c_1]$. Since this also shows that an observer at any point $x \in \mathbb{R}^d$ at which $u(t, x) = \varepsilon$ will see transition to the value $1 - \varepsilon$ within a uniformly bounded time interval (provided t is large enough), this means that the *reaction zone* (where $u \sim \frac{1}{2}$) has a uniformly bounded width in both space and time.

2. One can use [6, Theorem 1.11] to prove (i) for homogeneous ignition reactions (see [12, 21] for bistable ones), and also for x -periodic ignition reactions and front-like solutions. However,

besides disordered media, (i) is new for x -periodic media and spark-like solutions as well. In fact, some of our results are new even for homogeneous media (e.g., Theorem 2.5).

3. We will generalize Theorem 1.1 in several ways. This will include a proof that solutions eventually increase in time on each interval of values $[\varepsilon, 1 - \varepsilon]$, extensions to other types of reactions (ignition with non-constant ignition temperature, monostable, bistable, and their mixtures) and to transitions between general equilibria $u^- < u^+$, as well as a treatment of more general types of solutions (trapped between time shifts of general time-increasing solutions, and not necessarily satisfying $u^- \leq u \leq u^+$).

The reasons for the new complications for $d \geq 2$, described above, are not just technical but stem from “real world” considerations in the case of *two or more unbounded dimensions*. (Note that the result in [31] extends to the quasi-one-dimensional case of infinite cylinders in \mathbb{R}^d . The results described in Remark 2 after Theorem 1.1 also have a (quasi-)one-dimensional nature due to either radial symmetry or periodicity.)

First, one might think that the reaction zone will always coincide with a bounded neighborhood of some time-dependent hypersurface. This turns out to not be the case in general in dimensions $d \geq 2$, since without some order in the medium (such as periodicity) a fire will not always spread at roughly the same speed everywhere, so the initial spherical or hyperplanar shape of the reaction zone can become very distorted. In fact, areas of slowly burning material in the medium may cause it to propagate *around them faster than through them*, resulting in pockets of temporarily unburned material behind the leading edge of the fire. See Figure 1 for an illustration of this phenomenon, and the proof of Theorem 2.12(ii) for an extreme example of it. While these pockets will eventually burn up, variations in the medium can create arbitrarily many or even infinitely many of them (the latter for front-like solutions, although not spark-like) at a given (large) time, and they can be arbitrarily large and occur arbitrarily far behind the leading edge as $t \rightarrow \infty$. As a result of this potentially complicated geometry of the reaction zones of general solutions of (1.1), our definition of bounded width includes no requirements on the shape of the sets $\Omega_{u,\varepsilon}(t)$ or their boundaries.

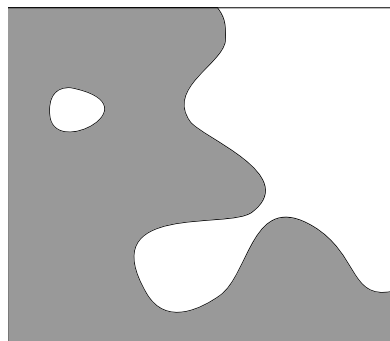


FIGURE 1. On the shaded region $u \sim 1$, and on the white region $u \sim 0$.

It is worth noting that while one might think that this issue can only arise if the medium has large spatial variations in the reaction f , this is not the case either. In fact, as long as f_0, f_1 satisfy $c_0 < c_1$, it is always possible to construct f such that $f_0 \leq f \leq f_1$ and the above

situation (arbitrarily many unburned pockets which can be arbitrarily large and arbitrarily far behind the leading edge) does indeed occur for typical solutions u . In particular, it happens almost surely for stationary ergodic media with short range correlations.

Another critical issue, related to this, arises from the consideration of what happens to such an unburned pocket far behind the leading edge of the fire. It “burns in” from its perimeter and at the time when it is just about to be burned up (say when the minimum of u on it is $\frac{3}{4}$), the nearest point where u is close to 0 (say $\leq \frac{1}{4}$) may be very far from the pocket. This shows that for general inhomogeneous media in dimensions $d \geq 2$, one may have *unbounded-in-time width of the transition zone from $u \sim 1$ to $u \sim 0$* .

On the other hand, in the situation studied here when the invaded state $u = 0$ is either stable or relatively weakly unstable (the invading state $u = 1$ clearly must be stable), pockets of burned material cannot form arbitrarily far “ahead” of the leading edge, unlike pockets of yet-unburned material “behind” the leading edge. (This is very different from the KPP case where $u = 0$ is strongly unstable; see [19] for examples of such media in one dimension, and the discussion after Definition 2.1 for what may be done in that case.) This means that typical solutions will be *pushed* (as opposed to *pulled*), their propagation being driven by intermediate (rather than small) values of u . Thus one can still hope to see a *uniformly-in-time bounded width of the transition zone from $u \sim 0$ to $u \sim 1$* . This lack of symmetry between the spatial transitions $1 \rightarrow 0$ and $0 \rightarrow 1$ is why our definition of bounded width involves the Hausdorff distance of the super-level sets of u but not of the sub-level sets (or of their boundaries).

Let us conclude this introduction with the discussion of convergence of typical solutions of the Cauchy problem to entire solutions (such as transition fronts) of (1.1) in several dimensions. In contrast to one dimension, it is unlikely that any general enough such results can be obtained for disordered media. Firstly, the disorder may result in reaction zones of solutions neither moving in a particular direction nor attaining a particular geometric shape. Secondly, in Theorem 2.12(ii) we construct media where any entire solution with uniformly-in-time bounded width of the transition zone from $u \sim 0$ to $u \sim 1$ satisfies $\lim_{t \rightarrow \infty} \inf_x u(t, x) = 1$ (while typical solutions of the Cauchy problem have $\inf_x u(t, x) = 0$ for each $t \in \mathbb{R}$).

And thirdly, if such a result existed, one should also expect the following non-linear Liouville claim to hold: If a solution u is initially between two time translates of a front-like (or spark-like) solution v (and so by the comparison principle, $v(\cdot, \cdot) \leq u(T + \cdot, \cdot) \leq v(2T + \cdot, \cdot)$ for some $T \geq 0$), then for any $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that for any $(t, x) \in [T_\varepsilon, \infty) \times \mathbb{R}^d$,

$$\|u(t, \cdot) - v(t + \tau_{t,x}, \cdot)\|_{L^\infty(B_{1/\varepsilon}(x))} < \varepsilon \quad (1.8)$$

for some $|\tau_{t,x}| \leq T$. That is, u should locally look more and more like a (possibly (t, x) -dependent) time translate of v as time progresses. Somewhat surprisingly, this claim is false in general in dimensions $d \geq 2$, even if v is required to be an entire solution. This is for non-pathological reasons and we discuss a counter-example in Section 11.

Nevertheless, despite the likely lack of sufficiently general results on convergence to transition fronts or other entire solutions in general disordered media, these solutions will still play an important role in our analysis. This is because one can use parabolic regularity to build entire solutions from those of the Cauchy problem sampled near any sequence of points (t_n, x_n) with $t_n \rightarrow \infty$, so results for the former can be used in the analysis of the latter.

Finally, let us mention that our results can be extended to some more general PDEs, with x -dependent second order terms as well as first order terms with divergence-free coefficients. This will be done elsewhere.

2. THE DEFINITION OF BOUNDED WIDTH AND THE MAIN RESULTS

Let us now turn to the relevant definitions and our main results. We will first assume that $u \in [0, 1]$ and $f \geq 0$ is Lipschitz and bounded below by some homogeneous *pure ignition* reaction f_0 (later we will consider more general situations). We will thus assume the following.

Hypothesis (H): f is Lipschitz with constant $K \geq 1$ and

$$f(x, 0) = f(x, 1) = 0 \quad \text{for } x \in \mathbb{R}^d. \quad (2.1)$$

There is also $\theta_0 \in (0, 1)$ and a Lipschitz function $f_0 : [0, 1] \rightarrow [0, \infty)$ with $f_0(u) = 0$ for $u \in [0, \theta_0] \cup \{1\}$ and $f_0(u) > 0$ for $u \in (\theta_0, 1)$ such that

$$f(x, u) \geq f_0(u) \quad \text{for } (x, u) \in \mathbb{R}^d \times [0, 1].$$

Finally, there is $\theta \in [0, \frac{1}{3}]$ such that $f(x, u) = 0$ for $(x, u) \in \mathbb{R}^d \times [0, \theta]$ and f is non-increasing in u on $[1-\theta, 1]$ for each $x \in \mathbb{R}^d$. If such $\theta > 0$ exists, then f is an ignition reaction, otherwise (in which case the last hypothesis is vacuous) f is a monostable reaction.

Remarks. 1. If there are also $\tilde{\theta}(x) \in [\theta, \theta_0]$ and $g : (0, \infty) \rightarrow (0, \infty)$ such that $f(x, u) = 0$ for $u \in [0, \tilde{\theta}(x)] \cup \{1\}$ and $f(x, u) \geq g(\text{dist}(u, \{\tilde{\theta}(x), 1\}))$ for $u \in (\tilde{\theta}(x), 1)$, then we say that f is a *pure ignition* reaction. While we will not need this extra hypothesis here, notice that reactions with a constant ignition temperature from Theorem 1.1 (the special case $f(x, u) = a(x)f_0(u)$ was considered in [16, 17, 20] in one spatial dimension) are pure ignition ones.

2. While the requirement of f being non-increasing in u on $[1-\theta, 1]$ is not always included in the definition of ignition reactions, many results for them need to assume it. This includes our main results, although the hypothesis is not needed for their slightly weaker versions (specifically, not including those statements which use Theorem 2.11(ii) below). Notice also that we can assume without loss that f_0 is non-increasing on $[1-\delta, 1]$ for some $\delta > 0$ because this can be achieved after replacing $f_0(u)$ by $\min_{v \in [1-\delta, u]} f_0(v)$. Thus f_0 is itself an ignition reaction according to the above definition.

If $u : (t_0, \infty) \times \mathbb{R}^d \rightarrow [0, 1]$ and $\varepsilon \in [0, 1]$, we let $\Omega_{u,\varepsilon}(t) := \{x \in \mathbb{R}^d \mid u(t, x) \geq \varepsilon\}$ for $t > t_0$. For $\varepsilon \in (0, \frac{1}{2})$, the *width of the transition zone* of u from ε to $1-\varepsilon$ at time $t > t_0$ is given by (1.6), while for $\varepsilon \in (\frac{1}{2}, 1)$ the corresponding width is

$$L_{u,\varepsilon}(t) := \inf \{L > 0 \mid \mathbb{R}^d \setminus \Omega_{u,\varepsilon}(t) \subseteq B_L(\mathbb{R}^d \setminus \Omega_{u,1-\varepsilon}(t))\}. \quad (2.2)$$

For $\varepsilon \in (0, \frac{1}{2})$ we also define the length of transition from $(\varepsilon, 1-\varepsilon)$ to either ε or $1-\varepsilon$ to be

$$J_{u,\varepsilon}(t) := \inf \{L > 0 \mid \mathbb{R}^d = B_L(\Omega_{u,1-\varepsilon}(t) \cup [\mathbb{R}^d \setminus \Omega_{u,\varepsilon}(t)])\}. \quad (2.3)$$

For the above to be perfectly symmetric, we could replace $\mathbb{R}^d \setminus \Omega_{u,\varepsilon}(t)$ by $\mathbb{R}^d \setminus \bigcup_{\varepsilon' > 1-\varepsilon} \Omega_{u,\varepsilon'}(t)$, but as we mentioned in the introduction, only (1.6) will play a major role here.

Definition 2.1. Let $u : (t_0, \infty) \times \mathbb{R}^d \rightarrow [0, 1]$ be a solution of (1.1) with $t_0 \in [-\infty, \infty)$. We say that u has a *bounded width (with respect to 0 and 1)* if for any $\varepsilon \in (0, \frac{1}{2})$ we have

$$L^{u,\varepsilon} := \limsup_{T \rightarrow \infty} \sup_{t > t_0 + T} L_{u,\varepsilon}(t) < \infty. \quad (2.4)$$

We say that u has a *doubly-bounded width* if (2.4) holds for any $\varepsilon \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. And we say that u has a *semi-bounded width* if for any $\varepsilon \in (0, \frac{1}{2})$ we have

$$J^{u,\varepsilon} := \limsup_{T \rightarrow \infty} \sup_{t > t_0 + T} J_{u,\varepsilon}(t) < \infty. \quad (2.5)$$

Remarks. 1. Notice that if $t_0 = -\infty$, then $L^{u,\varepsilon} = \sup_{t \in \mathbb{R}} L_{u,\varepsilon}(t)$ and $J^{u,\varepsilon} = \sup_{t \in \mathbb{R}} J_{u,\varepsilon}(t)$. For $t_0 > -\infty$, however, $L^{u,\varepsilon} = \limsup_{t \rightarrow \infty} L_{u,\varepsilon}(t)$ and $J^{u,\varepsilon} = \limsup_{t \rightarrow \infty} J_{u,\varepsilon}(t)$. One reason for this is that if $\sup_{x \in \mathbb{R}^d} u_0(x) < 1$, then $\sup_{x \in \mathbb{R}^d} u(t, x) < 1$ for any $t > t_0$. Thus for any $\varepsilon \in (0, \frac{1}{2})$, $L_{u,\varepsilon}(t)$ will equal ∞ up to some time t_ε ($\rightarrow \infty$ as $\varepsilon \rightarrow 0$).

2. We trivially have that $L^{u,\varepsilon}$ is non-increasing in $\varepsilon \in (0, \frac{1}{2})$ (as is $J^{u,\varepsilon}$) and non-decreasing in $\varepsilon \in (\frac{1}{2}, 1)$, so in fact the definition only needs to involve ε close to 0 and 1.

While the definition of bounded width involves $\varepsilon \in (0, \frac{1}{2})$, we do not make one involving only $\varepsilon \in (\frac{1}{2}, 1)$. This lack of symmetry was explained in the introduction, and is due to the possibility of existence of unburned pockets with $u \sim 0$ behind the leading edge of the reaction zone. Hence, typical solutions u in general disordered media may have $L^{u,\varepsilon} = \infty$ for $\varepsilon \in (\frac{1}{2}, 1)$. In particular, they would not have doubly-bounded widths, but may still have bounded widths, at least when $u \equiv 0$ is a stable equilibrium.

If the equilibrium $u \equiv 0$ is strongly unstable (such as for KPP f), bounded width is also too much to hope for in some situations, even when $d = 1$. Indeed, an easy extension of the construction from [19] yields media where burned pockets with $u \sim 1$ can form arbitrarily far ahead of the leading edge of the reaction zone. While we do not study this case here, we introduce the concept of semi-bounded width in Definition 2.1 because it is likely to be relevant in such situations.

We next define the propagation speed of (the reaction zone of) u (cf. [6]).

Definition 2.2. Let $u : (t_0, \infty) \times \mathbb{R}^d \rightarrow [0, 1]$ be a solution of (1.1) with $t_0 \in [-\infty, \infty)$, and let $0 < c \leq c' \leq \infty$. We say that u *propagates with global mean speed in $[c, c']$* if for any $\varepsilon \in (0, \frac{1}{2})$ and $\delta > 0$ there are $T_{\varepsilon,\delta}, \tau_{\varepsilon,\delta} < \infty$ such that

$$B_{(c-\delta)\tau}(\Omega_{u,\varepsilon}(t)) \subseteq \Omega_{u,1-\varepsilon}(t + \tau) \quad \text{and} \quad \Omega_{u,\varepsilon}(t + \tau) \subseteq B_{(c'+\delta)\tau}(\Omega_{u,1-\varepsilon}(t)) \quad (2.6)$$

whenever $t > t_0 + T_{\varepsilon,\delta}$ and $\tau \geq \tau_{\varepsilon,\delta}$. If any such $0 < c \leq c' \leq \infty$ exist, we also say that u *propagates with a positive global mean speed*.

Remarks. 1. If $t_0 = -\infty$, then obviously $t \in \mathbb{R}$ above is arbitrary.

2. Notice that the definition would be unchanged if we took $T_{\varepsilon,\delta} = \tau_{\varepsilon,\delta}$. However, this formulation will be more convenient for us because we will show that under certain conditions, $\tau_{\varepsilon,\delta}$ (but not necessarily $T_{\varepsilon,\delta}$) will be independent of f, u .

We now let c_0 be the front/spreading speed associated with the homogeneous reaction f_0 . That is, c_0 is the *unique* value such that (1.1) for $d = 1$ and with $f_0(u)$ in place of $f(x, u)$ has a traveling front solution $u(t, x) = U(x - c_0 t)$ with $\lim_{s \rightarrow -\infty} U(s) = 1$ and $\lim_{s \rightarrow \infty} U(s) = 0$.

We also let $f_1 : [0, 1] \rightarrow [0, \infty)$ be any Lipschitz function with constant K such that

$$f(x, u) \leq f_1(u) \quad \text{for } (x, u) \in \mathbb{R}^d \times [0, 1], \quad (2.7)$$

which is also pure ignition if $\theta > 0$ in (H) and pure monostable (i.e. $f_1(0) = f_1(1) = 0$ and $f_1(u) > 0$ for $u \in (0, 1)$) otherwise. For instance, we could pick $f_1(u) := \sup_{x \in \mathbb{R}^d} f(x, u)$, if this function is pure ignition/monostable. We also let c_1 be the front/spreading speed associated with f_1 (which is again the unique traveling front speed if f_1 is ignition, and it is the *minimal* traveling front speed if f_1 is monostable). The existence of c_0, c_1 is well known, as well as that $f_0(u) \leq f_1(u) \leq Ku$ implies $c_0 \leq c_1 \leq 2\sqrt{K}$.

Our main results below say that under appropriate (quite general and physically relevant) qualitative hypotheses on the reaction, typical solutions of (1.1) have bounded widths and (their reaction zones) propagate with global mean speeds in the interval $[c_0, c_1]$. They also eventually grow in time on any closed interval of values of u contained in $(0, 1)$. Specifically, we will prove the following conclusion for typical solutions u .

Conclusion (C): For any $\varepsilon \in (0, \frac{1}{2})$, there are $\ell_\varepsilon, m_\varepsilon, T_\varepsilon \in (0, \infty)$ such that

$$\sup_{t > t_0 + T_\varepsilon} L_{u, \varepsilon}(t) \leq \ell_\varepsilon \quad \text{and} \quad \inf_{\substack{(t, x) \in (t_0 + T_\varepsilon, \infty) \times \mathbb{R}^d \\ u(t, x) \in [\varepsilon, 1 - \varepsilon]}} u_t(t, x) \geq m_\varepsilon. \quad (2.8)$$

In particular, $L^{u, \varepsilon} \leq \ell_\varepsilon$, so u has a bounded width. Moreover, if a pure ignition f_1 satisfies (2.7), then u propagates with global mean speed in $[c_0, c_1]$.

Moreover, $\ell_\varepsilon, m_\varepsilon$ as well as $\tau_{\varepsilon, \delta}$ from Definition 2.2 will depend on some uniform bounds on the reaction, but neither on the reaction itself nor on the solution. That is, the spatial scale on which the transition from $u \sim 0$ to $u \sim 1$ happens as well as the temporal scale on which the global mean speed of (the reaction zone of) u is observed to be in $[c_0, c_1]$, will become independent of f, u after an initial time interval.

Note that such expanding sets may also be weak solutions of appropriate Hamilton-Jacobi equations. Connection of the two types of PDE is well-established in the homogenization theory for various types of media (e.g., periodic or stationary ergodic), see for instance [10, 14]. It will be explored, via our results, for general disordered media elsewhere.

Solutions of the Cauchy Problem with Bounded Widths

We will first show that for ignition reactions (i.e., with $\theta > 0$), (C) holds in dimensions $d \leq 3$, but not in dimensions $d \geq 4$ (under the same qualitative hypotheses).

A crucial additional (and necessary) hypothesis, which is automatically satisfied in the case of constant ignition temperature θ_0 , relates to the following definition (see Remarks 1 and 2 below). It says that if for any $x \in \mathbb{R}^d$ we increase u from 0 to 1, once $f(x, u)$ becomes large enough, it cannot become arbitrarily small until $u \sim 1$, as illustrated in Figure 2.

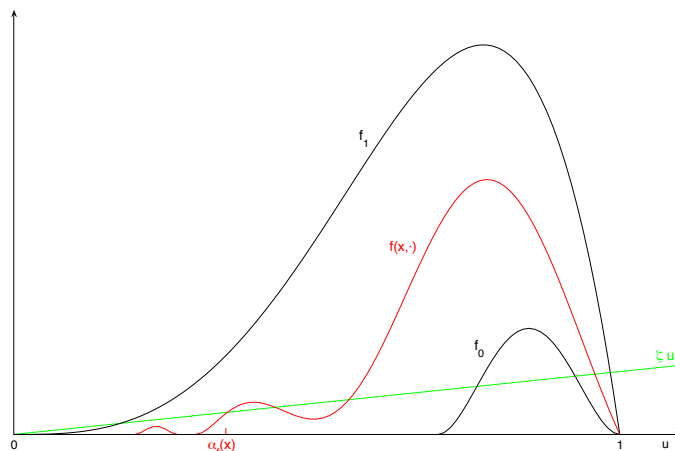


FIGURE 2. Example of a reaction from Definition 2.3 (at some fixed $x \in \mathbb{R}^d$).

Definition 2.3. Let f_0, K, θ be as in (H) and let $\zeta, \eta > 0$. If f satisfies (H), define

$$\alpha_f(x) = \alpha_f(x; \zeta) := \inf\{u \geq 0 \mid f(x, u) > \zeta u\} \quad (2.9)$$

and let $F(f_0, K, \theta, \zeta, \eta)$ be the set of all f satisfying (H) such that (recall $\inf \emptyset = \infty$)

$$\inf_{\substack{x \in \mathbb{R}^d \\ u \in [\alpha_f(x), \theta_0]}} f(x, u) \geq \eta. \quad (2.10)$$

Remarks. 1. We will require that $f \in F(f_0, K, \theta, \zeta, \eta)$ for some *not too large* $\zeta > 0$ and some $\eta > 0$. This assumption is physically relevant, as reaction should not cease at intermediate temperatures if it occurs at lower ones, and encompasses a large class of functions. A natural example is the pure ignition reaction from Remark 1 after (H), in which case $f \in F(f_0, K, \theta, \zeta, \eta)$ for any $\zeta > 0$ and small enough $\eta > 0$.

2. Note that $F(f_0, K, \theta, \zeta, \eta)$ is spatially translation invariant and closed under locally uniform convergence of functions. It is also decreasing in its odd arguments and increasing in the even ones. In particular, $F(f_0, K, \theta, \zeta, \eta) \subseteq F(f_0, K, 0, \zeta, \eta)$. These facts will be useful later, as well as the obvious $\alpha_f(x) \geq \frac{\eta}{K}$ for $f \in F(f_0, K, \theta, \zeta, \eta)$.

Without (some version of) the assumption from Remark 1, solutions of (1.1) need not have bounded widths even when $d = 1$ and f is a homogeneous ignition reaction. Indeed, assume that $f : [0, 1] \rightarrow [0, \infty)$ is such that $f(u) = 0$ for $u \in [0, \frac{1}{4}]$, $f(u) > 0$ for $u \in (\frac{1}{4}, \frac{1}{2})$, and $f(u) = 2f(u - \frac{1}{2})$ for $u \in [\frac{1}{2}, 1]$. Such f vanishes on $[\frac{1}{2}, \frac{3}{4}]$ and so belongs to $F(f_0, K, \theta, \zeta, \eta)$ *only for large* ζ (specifically $\zeta \geq \|f(u)/u\|_\infty$).

For such f , there obviously is a traveling front solution $u(t, x) = U(x - ct)$ of (1.1) connecting 0 and $\frac{1}{2}$ (i.e., such that $\lim_{s \rightarrow -\infty} U(s) = \frac{1}{2}$ and $\lim_{s \rightarrow \infty} U(s) = 0$) and another $u(t, x) = \frac{1}{2} + U(\sqrt{2}(x - \sqrt{2}ct))$ connecting $\frac{1}{2}$ and 1. Their speeds are $c > 0$ and $\sqrt{2}c (> c)$ and a simple comparison principle argument shows that all spark-like and front-like solutions

have a linearly in time growing *propagating terrace*:

$$\lim_{t \rightarrow \infty} \sup_{x \in [(c+\delta)t, (\sqrt{2}c-\delta)t]} \left| u(x, t) - \frac{1}{2} \right| = 0 \quad (2.11)$$

for any $\delta > 0$ (see [8] for further results of this nature). In particular, they do not have bounded widths. Of course, for such solutions one can separately study the transition from 0 to $\frac{1}{2}$ and that from $\frac{1}{2}$ to 1, using our results. Hence, the latter can also be applied in some situations when (2.10) is not satisfied for any $\eta > 0$ (and some not too large $\zeta > 0$).

We are now ready to state our first main result for ignition reactions, which applies to general *spark-like* and *front-like* initial data $u_0 \in [0, 1]$. Specifically, we will assume that either there are $x_0 \in \mathbb{R}^d$, $R_2 \geq R_1 > 0$, and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$(\theta_0 + \varepsilon_1)\chi_{B_{R_1}(x_0)}(x) \leq u_0(x) \leq e^{-\varepsilon_2(|x-x_0|-R_2)}, \quad (2.12)$$

or there are $e \in \mathbb{S}^{n-1}$, $R_2 \geq R_1$, and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$(\theta_0 + \varepsilon_1)\chi_{\{x \mid x \cdot e < R_1\}}(x) \leq u_0(x) \leq e^{-\varepsilon_2(x \cdot e - R_2)}. \quad (2.13)$$

In (2.12) we also assume that R_1 is large enough (depending on ε_1) to guarantee *spreading* (i.e., $\lim_{t \rightarrow \infty} u(x, t) = 1$ locally uniformly in \mathbb{R}^d), because otherwise one might have *quenching* (i.e., $\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_\infty = 0$) for ignition reactions.

Theorem 2.4. (i) Let f_0, K , and $\theta > 0$ be as in (H) and let $\eta > 0$, $\zeta \in (0, c_0^2/4)$, and $f \in F(f_0, K, \theta, \zeta, \eta)$. Let u solve (1.1), (1.2) with *spark-like* or *front-like* $u_0 \in [0, 1]$ as above. If $d \leq 3$, then (C) holds with $\ell_\varepsilon, m_\varepsilon$ depending only on $\varepsilon, f_0, K, \zeta, \eta$, and $\tau_{\varepsilon, \delta}$ in Definition 2.2 also depending on δ, f_1 .

(ii) If $d \geq 4$, then there is f as in (H) with $\theta > 0$ and $f(x, u) = 0$ for $(x, u) \in \mathbb{R}^d \times [0, \theta_0]$ (so that $f \in F(f_0, K, \theta, \zeta, \eta)$ for any $\zeta, \eta > 0$) such that all claims in (C) are false for any $u_0 \in [0, 1]$ supported in the left half-space for which $\limsup_{t \rightarrow \infty} \|u(t, \cdot)\|_\infty > 0$.

Remarks. 1. As noted before, the hypothesis $\zeta < c_0^2/4$ is crucial in (i). It guarantees that the reaction at small u (where $f(x, u) \leq \zeta u$) is not strong enough to cause spreading at speeds $\geq c_0$ (see Lemma 4.1(ii)). This is because spreading speeds for homogeneous reactions bounded above by ζu are no more than $2\sqrt{\zeta} < c_0$. Since $f \geq f_0$ has spreading speed no less than c_0 , one should then expect spreading to be driven by “intermediate” values of u (above $\alpha_f(x)$ and not too close to 1, where f is small). Thus u would be a “pushed” solution, and one can hope for it to have a bounded width, provided one can also show that values of u close to 1 do not “trail” far behind the intermediate ones. We will prove the latter for $d \leq 3$ in Lemma 4.2 but will also show in the proof of (ii) that it fails in general for $d \geq 4$, with the provided counter-example even having constant ignition temperature as in Theorem 1.1.

2. Note that the second claim in (2.8) and parabolic regularity shows that $\Omega_{u, \varepsilon}(t)$ grows with *instantaneous speed* greater than some positive constant at all times $t \geq t_0 + T_\varepsilon$ in (i). An upper bound on the instantaneous speed of growth does not exist in general, however, because for $\varepsilon \in (0, \frac{1}{2})$, $\Omega_{u, \varepsilon}(t)$ may acquire new connected components (which then soon merge with the “main” component) as time progresses.

3. As the proof of (i) shows, T_ε in (C) depends on $\varepsilon, f_0, K, \zeta, \eta, \theta, R_2 - R_1, \varepsilon_1, \varepsilon_2$, and $T_{\varepsilon, \delta}$ in Definition 2.2 also depends on δ, f_1 .

4. The result extends to monostable reactions in a weaker form. (ii) holds without change (the counter-example we construct is easily modified) but in (i) we need to assume that either there are $R_1, R_2, \varepsilon_1 > 0$ (R_1 sufficiently large, depending on ε_1) and $x_0 \in \mathbb{R}^d$ such that

$$(\theta_0 + \varepsilon_1)\chi_{B_{R_1}(x_0)}(x) \leq u_0(x) \leq \chi_{B_{R_2}(x_0)}(x), \quad (2.14)$$

or there are $R_1, R_2 \in \mathbb{R}, \varepsilon_1, \varepsilon_2 > 0$, and $e \in \mathbb{S}^{n-1}$ such that

$$(\theta_0 + \varepsilon_1)\chi_{\{x \mid x \cdot e < R_1\}}(x) \leq u_0(x) \leq (1 - \varepsilon_2)\chi_{\{x \mid x \cdot e < R_2\}}(x). \quad (2.15)$$

Then for any $\varepsilon \in (0, \frac{1}{2})$ there are $\ell_\varepsilon, T_\varepsilon \in (0, \infty)$, depending on $\varepsilon, f_0, K, \zeta, \eta, \varepsilon_1$, and either on R_2 (for (2.14)) or on $R_2 - R_1, \varepsilon_2$ (for (2.15)), such that $L_{u, \varepsilon}(t) \leq \ell_\varepsilon$ for $t > t_0 + T_\varepsilon$.

The first step in the proof of Theorem 2.4(i) will be to consider general solutions with $u_t \geq 0$. That is, such that on \mathbb{R}^d ,

$$\Delta u_0(\cdot) + f(\cdot, u_0(\cdot)) \geq 0, \quad (2.16)$$

which then guarantees $u_t \geq 0$ because $v := u_t$ solves $v_t = \Delta v + f_u(x, u(x))v$ with $v(0, x) \geq 0$. For $d \leq 3$ we will show that if the width of the reaction zone of such u is controlled at the initial time t_0 (see (2.17) below), then the conclusions of Theorem 2.4(i) continue to hold. This step is related to our proof of existence of transition fronts in [31], but will be considerably more involved, particularly for $d = 3$.

This latter result applies to any such solution u (as well as solutions trapped between time-shifts of such u), not just the spark-like or front-like ones, and is stated next. We let

$$L_{u, \varepsilon, \varepsilon'}(t) := \inf \{L > 0 \mid \Omega_{u, \varepsilon}(t) \subseteq B_L(\Omega_{u, \varepsilon'}(t))\} \quad (2.17)$$

be the width of the transition zone from ε to ε' . We will assume that $L_{u, \varepsilon, \varepsilon'}(t_0) < \infty$ for each $\varepsilon > 0$ and some fixed $\varepsilon' > \theta_0$. Here ε' can be arbitrary when $d \leq 2$, and equals $1 - \varepsilon_0$ when $d = 3$ (with $\varepsilon_0 = \varepsilon_0(f_0, K) > 0$ from Lemma 3.1 below). This choice of ε' will guarantee spreading for any solution satisfying (2.16) and $u(t_0, x) \geq \varepsilon'$ for some $x \in \mathbb{R}^d$.

Theorem 2.5. *Let $d \leq 3$, let f_0, K , and $\theta > 0$ be as in (H), and let $\eta > 0, \zeta \in (0, c_0^2/4)$, and $f \in F(f_0, K, \theta, \zeta, \eta)$. Let u solve (1.1), (1.2) with $u_0 \in [0, 1]$ satisfying (2.16).*

(i) *If ε' is as above and $L_{u, \varepsilon, \varepsilon'}(t_0) < \infty$ for each $\varepsilon > 0$, then (C) holds with $\ell_\varepsilon, m_\varepsilon$ depending only on $\varepsilon, f_0, K, \zeta, \eta$, and $\tau_{\varepsilon, \delta}$ in Definition 2.2 also depending on δ, f_1 .*

(ii) *If u is as in (i), and a solution v of (1.1) satisfies*

$$u(t_0, \cdot) \leq v(t_0 + \tau, \cdot) \leq u(t_0 + 2\tau, \cdot)$$

for some $\tau > 0$, then (C) holds for v with $\ell_\varepsilon, m_\varepsilon, \tau_{\varepsilon, \delta}$ as in (i) (so independent of τ).

Remarks. 1. In (i), T_ε in (C) depends on $\varepsilon, f_0, K, \zeta, \eta, \theta, u_0$, the dependence on u_0 being only via the number $L_{u, h, \varepsilon'}(t_0)$ with $h := \min \{\theta(c_0^2 - 4\zeta)(c_0^2 + 4\zeta)^{-1}, \frac{\eta}{4K}, 1 - \varepsilon', \frac{\varepsilon}{2}\}$ (see the proof); $T_{\varepsilon, \delta}$ in Definition 2.2 also depends on δ, f_1 . In (ii) they also depend on τ .

2. (i) extends to monostable f if we also assume $\sup_{\varepsilon \in (0, 1)} \varepsilon e^{\sqrt{\zeta} L_{u, \varepsilon, \varepsilon'}(t_0)} < \infty$, but with global mean speed in $[c_0, c'_Y]$, where c'_Y is from (4.7) below.

3. (ii) also extends to monostable f if we assume $\sup_{\varepsilon \in (0,1)} \varepsilon e^{\sqrt{\zeta} L_{u,\varepsilon,\varepsilon'}(t_0)} < \infty$, but with τ -dependent $\ell_\varepsilon, \tau_{\varepsilon,\delta}$, without the second claim in (2.8), and with global mean speed in $[c_0, c'_Y]$.

Notice that in (ii), the bounds in (C) are independent of the time shift τ . To prove this, we will first need to show such solution-independent bounds for entire solutions with bounded widths when $d \leq 3$. In particular, as long as such a solution has a bounded width, the bound on $\sup_{t \in \mathbb{R}} L_{u,\varepsilon}(t)$ (for $\varepsilon \in (0, \frac{1}{2})$) will in fact *only depend on* $\varepsilon, f_0, K, \zeta, \eta$. It will then suffice to show, using parabolic regularity, that the solutions from (ii) asymptotically look like entire solutions with bounded widths, where the bounds involved will be allowed to depend on τ .

A crucial ingredient in this will be the proof that entire solutions with bounded widths satisfy $u_t \geq 0$ (in all dimensions). Such a result was previously proved in [6] for transition fronts in a closely related setting. This and the uniform bounds for entire solutions with bounded widths are stated in Theorem 2.11 below.

Theorem 2.4(i) is proved similarly to Theorem 2.5(ii), but the solution will be sandwiched between time-shifts of a time-increasing solution, perturbed by certain exponentially in space decreasing functions. We will therefore also need to prove stability of spark-like and front-like time-increasing solutions with respect to such perturbations. This could be extended to other situations where time-increasing solutions with some specific profiles are stable with respect to appropriate (exponentially decreasing) perturbations. For instance, one could handle in this way *cone-like* solutions, with initial data exponentially decreasing inside a d -dimensional cone and converging to 1 outside it. We will not pursue this direction here.

We also note that these results cannot be extended to arbitrary spreading solutions, even for homogeneous pure ignition reactions $f(x, u) = f_0(u)$ and $d = 1$. Indeed, the author showed [29] that then there exists a unique $M > 0$ such that the solution of (1.1), (1.2) with $u_0 := \chi_{[-M, M]}$ converges locally uniformly to θ_0 as $t \rightarrow \infty$. If we now let $R \gg 1$ and $u_0 := \chi_{[-R, R]} + \sum_{n=1}^{\infty} \chi_{[a_n - M, a_n + M]}$ with sufficiently rapidly growing a_n , the solution u will have increasingly long plateaus as $t \rightarrow \infty$. Specifically, there will be $t_n, b_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \sup_{x \in [a_n - b_n, a_n + b_n]} |u(t_n, x) - \theta_0| = 0.$$

Such u therefore does not even have a *semi-bounded* width!

Finally, most of the argument for $d \leq 3$ also applies if $d \geq 4$, the one exception being Lemma 4.2 below. The reason it fails for $d \geq 4$ lies in Lemma 3.4, which only excludes existence of equilibrium solutions to (1.1) which are independent of one coordinate when $d \leq 3$. Such solutions will be the basis of the counter-example proving Theorem 2.4(ii).

Extensions to More General Reactions, Equilibria, and Solutions

Let us now discuss the more general case when typical solutions transition from some equilibrium u^- to another equilibrium u^+ (instead from 0 to 1), with $u^- < u^+$ and

$$0 < \inf_{x \in \mathbb{R}^d} [u^+(x) - u^-(x)] \leq \sup_{x \in \mathbb{R}^d} [u^+(x) - u^-(x)] < \infty \quad (2.18)$$

(the case $u^- > u^+$ is identical, as one can consider the equation for $-u$ instead). Our goal is to extend the positive results in Theorems 2.4(i) and 2.5 to such situations. In addition,

we will also consider more general reactions than ignition, specifically, mixtures of ignition, monostable, and bistable ones.

We will assume $u^- \equiv 0$ without loss, because the general case is immediately reduced to this by taking $v := u - u^-$, which solves (1.1) with f replaced by

$$g(x, v) := f(x, v + u^-(x)) - f(x, u^-(x)).$$

Obviously, we can also assume $u^+ \leq 1$, by (2.18) and after scaling in u .

Thus we will now assume the following generalization of (H).

Hypothesis (H'): f is Lipschitz with constant $K \geq 1$ and

$$f(x, 0) = 0 \quad \text{for } x \in \mathbb{R}^d.$$

There are also $0 < \theta_0 < \theta_1 \leq 1$ and Lipschitz $f_0 : [0, \theta_1] \rightarrow \mathbb{R}$ with $f_0(0) = f_0(\theta_0) = f_0(\theta_1) = 0$, $f_0(u) < 0$ for $u \in (0, \theta_0)$, and $f_0(u) > 0$ for $u \in (\theta_0, \theta_1)$, such that $\int_0^{\theta_1} f_0(u) du > 0$ and

$$f(x, u) \geq f_0(u) \quad \text{for } (x, u) \in \mathbb{R}^d \times [0, \theta_1].$$

Furthermore, we assume that there is an equilibrium solution u^+ of (1.1) with

$$\theta_0 < \inf_{x \in \mathbb{R}^d} u^+(x) \leq \sup_{x \in \mathbb{R}^d} u^+(x) \leq 1, \quad (2.19)$$

and we have

$$f(x, u) \geq 0 \text{ when } u < 0 \quad \text{and} \quad f(x, u) \leq f(x, u^+(x)) \text{ when } u > u^+(x) \quad (2.20)$$

Finally, there is $\theta \in [0, \frac{\theta_0}{3}]$ such that f is non-increasing in u on $[0, \theta]$ and on $[u^+(x) - \theta, u^+(x)]$ for each $x \in \mathbb{R}^d$ ($\theta = 0$ obviously always works but we will obtain stronger results when $\theta > 0$).

That is, f_0 is now a *pure bistable reaction* (while f_1 in (2.7) will still be pure ignition or pure monostable), so f could be any mix of different reaction types. The hypothesis $\int_0^{\theta_1} f_0(u) du > 0$ is necessary for solutions of (1.1), (1.2), with reaction f_0 and large enough $u_0 \in [0, \theta_1]$, to spread (i.e., $\lim_{t \rightarrow \infty} u(t, x) = \theta_1$ locally uniformly). In fact, it guarantees that the front/spreading speed c_0 for this f_0 (which corresponds to the traveling front for f_0 connecting 0 and θ_1 , and is unique just as for ignition reactions) is positive. Thus, typical non-negative solutions of (1.1) transition *away from* $u = 0$. Transition *to* u^+ is, however, not guaranteed by (H') only. Finally, (2.20) will be needed in Theorem 2.9 to extend our results to solutions which are not necessarily between 0 and u^+ .

We next need to generalize Definitions 2.1–2.3 to the case at hand. We will first consider solutions $0 \leq u \leq u^+$ (henceforth denoted $u \in [0, u^+]$), when (2.20) is of no consequence. Definitions 2.1 and 2.2 are unchanged for such u , but use (for $\varepsilon \in (0, \frac{1}{2})$)

$$\Omega_{u, \varepsilon}(t) := \{x \in \mathbb{R}^d \mid u(t, x) \geq \varepsilon\}, \quad (2.21)$$

$$\Omega_{u, 1-\varepsilon}(t) := \{x \in \mathbb{R}^d \mid u(t, x) \geq u^+(x) - \varepsilon\}. \quad (2.22)$$

Definition 2.3, on the other hand, needs to be changed because $f(x, u^+(x)) \not\equiv 0$ in general. The motivation for this new form comes from the proofs of Theorems 2.4(i) and 2.5, specifically from the use of Lemma 3.4 below in the proof of the $d = 3$ case of Lemma 4.2.

Definition 2.6. Let f_0, K, θ be as in (H') and $\zeta, \eta > 0$. If f satisfies (H'), define $\alpha_f(x; \zeta)$ as in (2.9). Finally, let $F'(f_0, K, \theta, \zeta, \eta)$ be the set of all (f, u^+) satisfying (H') such that $\alpha_f(x; \zeta) \geq \eta$ for all $x \in \mathbb{R}^d$ and any equilibrium solution p of (1.1) with $0 < p < u^+$ satisfies

$$\sup_{x_0 \in \mathbb{R}^d} \sum_{n \geq 1} \frac{1}{1 + d(x_0, \mathcal{C}_n)} \leq \frac{1}{\eta}. \quad (2.23)$$

Here $d(\cdot, \cdot)$ is the distance in \mathbb{R}^d and $\mathcal{C}_1, \mathcal{C}_2, \dots$ are all (distinct) unit cubes in \mathbb{R}^d , whose corners have integer coordinates, such that $p(x) > \alpha_f(x; \zeta)$ for some $x \in \mathcal{C}_n$.

Remarks. 1. The advantage of (2.10), relative to (2.23), is that the former is a local condition while the latter is not. Thus (2.23) is more difficult to check. An obvious sufficient condition is when $p(\cdot) \leq \alpha_f(\cdot; \zeta)$ for each equilibrium $0 < p < u^+$ (with $\zeta < c_0^2/4$, so that our results apply), which may be proved under some *quantitative* local hypotheses on f . A simple such example is when $d = 1 = \theta_1$ and f is sufficiently close to a homogeneous reaction f_0 as in (H') with $\int_0^\beta f_0(u) du > 0$, where $\beta \in (\theta_0, 1)$ is smallest number such that $f_0(\beta) = \beta c_0^2/4$.

2. Lemma 3.4 shows that in the setting of (H), (2.10) implies (2.23) when $d \leq 3$ (but not when $d \geq 4$), although with a different $\eta > 0$.

3. (2.23) will cause typical solutions between 0 and u^+ to transition to u^+ (instead of to some other equilibrium $p < u^+$), and also to have a bounded width. The latter need not be true without a condition like (2.23), as is demonstrated by the example in the proof of Theorem 2.4(ii), for which the sum in (2.23) diverges, albeit slowly (as $\log n$).

Note that unlike $F(f_0, K, \theta, \zeta, \eta)$, the set $F'(f_0, K, \theta, \zeta, \eta)$ may be neither spatially translation invariant (although it would be if the \mathcal{C}_n were integer translations of any fixed unit cube \mathcal{C} , and the sup in (2.23) were also taken over all such \mathcal{C}) nor closed with respect to locally uniform convergence (i.e., locally uniform convergence for $q(x, u) := (f(x, u), u^+(x))$ on $\mathbb{R}^d \times \mathbb{R}$). Since these properties will be essential in our analysis, in the following generalization of Theorems 2.4(i) and 2.5 we will work with subsets $\mathcal{F} \subseteq F'(f_0, K, \theta, \zeta, \eta)$ which possess them both (an example of particular interest is the closure of all translations of a given (f, u^+) with respect to locally uniform convergence). We will denote $\mathcal{F}_\theta := \mathcal{F} \cap F'(f_0, K, \theta, \zeta, \eta)$ for $\theta \geq 0$, which then also has the same properties.

Theorem 2.7. Let f_0, K , and $\theta > 0$ be as in (H') and let $\eta > 0$, $\zeta \in (0, c_0^2/4)$, and $\mathcal{F} \subseteq F'(f_0, K, \theta, \zeta, \eta)$ be spatially translation invariant and closed with respect to locally uniform convergence. Let $(f, u^+) \in \mathcal{F}_\theta$ and let u solve (1.1), (1.2) with $u_0 \in [0, u^+]$.

(i) If $d \geq 1$ and u_0 satisfies (2.12) or (2.13), then (C) holds with $1 - \varepsilon$ replaced by $u^+(x) - \varepsilon$ in (2.8), with $\ell_\varepsilon, m_\varepsilon$ depending only on ε, \mathcal{F} , and $\tau_{\varepsilon, \delta}$ in Definition 2.2 also depending on δ, f_1 .

(ii) If $d \geq 1$, u_0 satisfies (2.16), and $L_{u, \varepsilon, 1 - \varepsilon_0}(t_0) < \infty$ for $\varepsilon_0 > 0$ from Lemma 8.1 and each $\varepsilon > 0$, then (C) holds for u and for v as in Theorem 2.5(ii), with $1 - \varepsilon$ replaced by $u^+(x) - \varepsilon$ in (2.8), with $\ell_\varepsilon, m_\varepsilon$ depending only on ε, \mathcal{F} , and $\tau_{\varepsilon, \delta}$ in Definition 2.2 also depending on δ, f_1 .

Remarks. 1. Here T_ε in (C) and $T_{\varepsilon, \delta}$ in Definition 2.2 depend on the same parameters as in Theorems 2.4 (in (i)) and 2.5 (in (ii)), but with f_0, K, ζ, η replaced by \mathcal{F} . This is also the case in Theorem 2.9 below, but there T_ε and $T_{\varepsilon, \delta}$ depend also on $\|u_0\|_\infty$.

2. These results again extend to the case $\theta = 0$ in the slightly weaker form from Remark 4 after Theorem 2.4 and Remarks 2,3 after Theorem 2.5.

Next, we consider extensions of our results to solutions that are not necessarily between the equilibria which they connect. We first need to extend Definitions 2.1 and 2.2 in a physically relevant manner to such solutions (we will do so for general u^\pm). Namely, we will consider u to be ε -close to u^\pm at (t, x) if $|u(t, y) - u^\pm(y)| < \varepsilon$ for all y in a ball centered at x , whose size grows to ∞ as $\varepsilon \rightarrow 0$. It will therefore be useful to define for a set $A \subseteq \mathbb{R}^d$ its r -interior

$$r\text{-int } A := \{x \in A \mid B_r(x) \subseteq A\}.$$

Definition 2.8. Let u^\pm be equilibrium solutions of (1.1) with bounded Lipschitz f , satisfying (2.18). For a solution u of (1.1) on $(t_0, \infty) \times \mathbb{R}^d$, define (for $\varepsilon \in (0, \frac{1}{2})$)

$$\begin{aligned} \Omega_{u,\varepsilon}(t) &:= \{x \in \mathbb{R}^d \mid |u(t, x) - u^-(x)| \geq \varepsilon\}, \\ \Omega_{u,1-\varepsilon}(t) &:= \{x \in \mathbb{R}^d \mid |u(t, x) - u^+(x)| \leq \varepsilon\}, \\ L_{u,\varepsilon}(t) &:= \inf \{L > 0 \mid \Omega_{u,\varepsilon}(t) \subseteq B_L(\tfrac{1}{\varepsilon}\text{-int } \Omega_{u,1-\varepsilon}(t))\}, \\ L_{u,1-\varepsilon}(t) &:= \inf \{L > 0 \mid \mathbb{R}^d \setminus \Omega_{u,1-\varepsilon}(t) \subseteq B_L(\tfrac{1}{\varepsilon}\text{-int } [\mathbb{R}^d \setminus \Omega_{u,\varepsilon}(t)])\}, \\ J_{u,\varepsilon}(t) &:= \inf \{L > 0 \mid \mathbb{R}^d = B_L(\tfrac{1}{\varepsilon}\text{-int } \Omega_{u,1-\varepsilon}(t) \cup \tfrac{1}{\varepsilon}\text{-int } [\mathbb{R}^d \setminus \Omega_{u,\varepsilon}(t)])\}. \end{aligned} \quad (2.24)$$

We say that u has a *bounded width* (with respect to u^\pm) if (2.4) holds for any $\varepsilon \in (0, \frac{1}{2})$, a *doubly-bounded width* if (2.4) holds for any $\varepsilon \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, and a *semi-bounded width* if (2.5) holds for any $\varepsilon \in (0, \frac{1}{2})$. Definition 2.2 remains the same but with these new $\Omega_{u,\varepsilon}(t)$.

Parabolic regularity and strong maximum principle show that if $u^- \leq u \leq u^+$, then this new definition of bounded/doubly-bounded/semi-bounded width is equivalent to the one using (1.6), (2.2), (2.3) and these new $\Omega_{u,\varepsilon}(t)$ (which are those from (2.21), (2.22) if also $u^- = 0$). In fact, while the new $L_{u,\varepsilon}(t)$ is larger than the original one for such u , it is finite for all $\varepsilon \in (0, \frac{1}{2})$ resp. all $\varepsilon \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ as long as the same is true for the original $L_{u,\varepsilon}(t)$.

Finally, let us extend the definition of *spark-like* and *front-like* initial data as follows. We will assume that either there are $x_0 \in \mathbb{R}^d$, $R_2 \geq R_1 > 0$, and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$(\theta_0 + \varepsilon_1)\chi_{B_{R_1}(x_0)}(x) - e^{-\varepsilon_2(|x-x_0|-R_2)}\chi_{\mathbb{R}^d \setminus B_{R_1}(0)}(x) \leq u_0(x) \leq e^{-\varepsilon_2(|x-x_0|-R_2)} \quad (2.25)$$

(with R_1 sufficiently large, depending on $\varepsilon_1, \varepsilon_2, R_2 - R_1$, to guarantee spreading), or there are $e \in \mathbb{S}^{n-1}$, $R_2 \geq R_1$, and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$(\theta_0 + \varepsilon_1)\chi_{\{x \mid x \cdot e < R_1\}}(x) - e^{-\varepsilon_2(x \cdot e - R_2)}\chi_{\{x \mid x \cdot e \geq R_1\}}(x) \leq u_0(x) \leq e^{-\varepsilon_2(x \cdot e - R_2)}. \quad (2.26)$$

Theorem 2.9. Consider the setting of Theorem 2.7 but with u_0 only bounded.

(i) Theorem 2.7(i) holds with (2.12)/(2.13) replaced by (2.25)/(2.26), provided that in the case of (2.26), “ \leq ” is replaced by “ $<$ ” in (2.20) for all $(f, u^+) \in \mathcal{F}$.

(ii) Theorem 2.7(ii) holds, provided that “ \leq ” and “ \geq ” are replaced by “ $<$ ” and “ $>$ ” in (2.20) for all $(f, u^+) \in \mathcal{F}$.

Remarks. 1. The extra condition in (i) guarantees $\limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^d} [u(t, x) - u^+(x)] \leq 0$ for any bounded u_0 , uniformly in \mathcal{F} . This as well as (i) also hold for u_0 from (2.26) if instead we assume $\limsup_{x \rightarrow -\infty} [u_0(x) - u^+(x)] \leq 0$, but then $T_\varepsilon, T_{\varepsilon, \delta}$ in (i) depend on u_0 also via the rate of this decay (cf. Remark 1 after Theorem 2.7).

2. The extra condition in (ii) guarantees $\limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^d} [u(t, x) - u^+(x)] \leq 0$ and $\liminf_{t \rightarrow \infty} \inf_{x \in \mathbb{R}^d} u(t, x) \geq 0$ for any bounded u_0 , uniformly in \mathcal{F} .

3. Theorems 2.4(i) and 2.5 extend similarly to solutions u not necessarily in $[0, 1]$.

Entire Solutions with Bounded Widths

Finally, let us turn to the discussion of the above-mentioned entire solutions of (1.1).

Definition 2.10. Let u^\pm be equilibrium solutions of (1.1) satisfying (2.18). A *transition solution* (connecting u^- to u^+) for (1.1) is a bounded entire solution u of (1.1) which satisfies

$$\lim_{t \rightarrow \pm\infty} u(t, x) = u^\pm(x) \quad (2.27)$$

locally uniformly on \mathbb{R}^d .

As above, in the following we will assume $u^- \equiv 0$ without loss.

Theorem 2.11. Let $u^- \equiv 0$ and u^+ satisfy (2.18) and be equilibrium solutions of (1.1) with some Lipschitz f , satisfying (2.20) (but not necessarily (H')). Let $u \not\equiv 0, u^+$ be a bounded entire solution of (1.1) which has a bounded width with respect to $0, u^+$.

(i) We have $0 < u < u^+$.

(ii) If u propagates with a positive global mean speed, then u is a transition solution. If, in addition, there is $\theta > 0$ such that f is non-increasing in u on $[0, \theta]$ and on $[u^+(x) - \theta, u^+(x)]$ for each $x \in \mathbb{R}^d$, then $u_t > 0$.

(iii) Assume f_0, K , and $\theta > 0$ are as in (H') and $\eta > 0, \zeta \in (0, c_0^2/4), \mathcal{F} \subseteq F'(f_0, K, 0, \zeta, \eta)$ is spatially translation invariant and closed with respect to locally uniform convergence. If $(f, u^+) \in \mathcal{F}_\theta$, then (C) holds for u , with $t_0 + T_\varepsilon$ replaced by $-\infty$ and $1 - \varepsilon$ by $u^+(x) - \varepsilon$ in (2.8), with $\ell_\varepsilon, m_\varepsilon$ depending only on ε, \mathcal{F} , and $\tau_{\varepsilon, \delta}$ in Definition 2.2 also depending on δ, f_1 .

Remarks. 1. (i,ii) were proved in [6, Theorem 1.11], in a more general setting and for a smaller class of entire solutions called *invasions*. The latter have doubly-bounded widths and their reaction zones satisfy an additional geometric requirement (2.31) (see the discussion below). Our proof proceeds along similar lines, using a version of the sliding method.

2. (ii) will play a crucial role in the proofs of Theorems 2.4(i), 2.5, 2.7, and 2.9.

3. Notice that as long as u has a bounded width in (iii), we actually have the u -independent bound $\sup_{t \in \mathbb{R}} L_{u, \varepsilon}(t) \leq \ell_\varepsilon$.

The hypothesis (2.20) is necessary in Theorem 2.11, even for homogeneous f and $d = 1$. It is well known that, for instance, if $0 \leq f(u) \leq f'(0)u$ for $u \in [0, 1]$ (i.e., f is a *KPP reaction* with $f'(0) > 0$) and $f(u) = f'(0)u$ for $u < 0$, then for any $c \in (0, 2\sqrt{f'(0)})$ there is a traveling front solution $u(t, x) = U_c(x - ct)$ of (1.1) on $\mathbb{R} \times \mathbb{R}$ with $\lim_{s \rightarrow -\infty} U_c(s) = 1$, $\lim_{s \rightarrow \infty} U_c(s) = 0$, and $\inf_{s \in \mathbb{R}} U_c(s) < 0$. This solution satisfies neither (i) nor (ii). Counterexamples with ignition f also exist.

Theorem 2.11 suggests a couple of interesting questions.

Open problems. 1. Does $u_t > 0$ hold in Theorem 2.11(ii) when $\theta = 0$?

2. Does Theorem 2.11(ii) and/or Theorem 2.11(iii) hold if we drop the hypotheses of bounded width and positive global mean speed and instead only assume that $u \in [0, u^+]$ is a transition solution? Of course, bounded width and positive global mean speed would then follow from the claim of Theorem 2.11(iii).

A natural question is whether solutions considered in Theorem 2.11 must always exist. The following result answers this in the affirmative under the hypotheses of Theorem 2.7, even when $\theta = 0$ in (H'). It also shows that transition solutions with *doubly-bounded width* need not exist for $d \geq 2$ even for ignition reactions, as was discussed in the introduction.

Theorem 2.12. (i) If $(f, u^+) \in \mathcal{F}$, with \mathcal{F} as in Theorem 2.7 (so $\theta = 0$), then there exists a transition solution $u \in (0, u^+)$ for (1.1) with $u_t > 0$ and a bounded width.

(ii) If $d \geq 2$, then there exists f as in Theorem 2.4 such that any bounded entire solution $u \not\equiv 0, 1$ for (1.1) with bounded width is a transition solution $u \in (0, 1)$ satisfying $u_t > 0$ and $\lim_{t \rightarrow \infty} \inf_{x \in \mathbb{R}^2} u(t, x) = 1$. In particular, there exists no transition solution with a doubly-bounded width for (1.1) (and hence also no transition front — see the discussion below).

Remarks. 1. The hypothesis $\zeta < c_0^2/4$ is at least qualitatively necessary in (i), as counterexamples with $\zeta > c_0^2/2$ exist even for $d = 1$ [19].

2. Note that for $d = 1$, transition fronts always exist under the hypotheses in (ii) [31]. The first example of non-existence of fronts was given in [19] for *KPP reactions* (and $d = 1$). It is based on the construction of f for which the equilibrium $u \equiv 0$ is *strongly unstable* in some region of space, so that arbitrarily small amounts of heat diffusing far ahead of the reaction zone quickly ignite on their own inside this region. (ii) is the first non-existence result for ignition reactions (so it does not rely on this strong instability property of KPP reactions).

Before proving the above results, let us note that while the concepts of bounded and semi-bounded width of solutions to (1.1) are new, the concept of doubly-bounded width is closely related to the Berestycki-Hamel definition of transition fronts from [5, 6], which motivated this work. The latter definition is more geometric in nature and its scope is slightly different from ours. It involves entire solutions rather than solutions of the Cauchy problem, and is also stated for wider classes of PDEs and spatial domains, and vector-valued solutions with possibly time-dependent coefficients and u^\pm . This is beyond the scope of the present paper (although the corresponding generalizations are rather straightforward), so we will only discuss the case at hand: (1.1) on $\mathbb{R} \times \mathbb{R}^d$ with bounded Lipschitz f and time-independent u^\pm satisfying (2.18).

In this setting, the definition in [6] says that a *transition front* connecting u^- and u^+ is an entire solution u such that for each $t \in \mathbb{R}$ there are open non-empty sets $\Omega_t^\pm \subseteq \mathbb{R}^d$ satisfying

$$\Omega_t^- \cap \Omega_t^+ = \emptyset, \quad \partial\Omega_t^- = \partial\Omega_t^+ =: \Gamma_t, \quad \Omega_t^- \cup \Gamma_t \cup \Omega_t^+ = \mathbb{R}^d, \quad (2.28)$$

$$\sup \{d(y, \Gamma_t) \mid y \in \Omega_t^\pm \cap \partial B_r(x)\} \rightarrow \infty \text{ as } r \rightarrow \infty, \text{ uniformly in } t \in \mathbb{R} \text{ and } x \in \Gamma_t, \quad (2.29)$$

$$u(t, x) - u^\pm(x) \rightarrow 0 \text{ as } d(x, \Gamma_t) \rightarrow \infty \text{ and } x \in \Omega_t^\pm, \text{ uniformly in } t \in \mathbb{R}, \quad (2.30)$$

and there is $n \geq 1$ such that for each $t \in \mathbb{R}$,

$$\Gamma_t \text{ is a subset of } n \text{ (rotated in } \mathbb{R}^d) \text{ graphs of functions from } \mathbb{R}^{d-1} \text{ to } \mathbb{R}. \quad (2.31)$$

While we have to forgo geometric conditions, such as (2.31), in our definitions (as was explained earlier), it is not difficult to see that (2.28)–(2.30) for an entire solution $u \not\equiv u^\pm$ are in fact *equivalent* to u having a doubly-bounded width! Indeed, if $u \not\equiv u^\pm$ has a doubly-bounded width (in the sense of Definition 2.8 if $u \notin [u^-, u^+]$), one only needs to take

$$\Omega_t^+ := \text{int} \left\{ x \in \mathbb{R}^d \left| u(t, x) \geq \frac{u^+(x) + u^-(x)}{2} \right. \right\}, \quad (2.32)$$

$\Gamma_t := \partial\Omega_t^+$, and $\Omega_t^- := \mathbb{R}^d \setminus \bar{\Omega}_t^+$. (Of course, the sets Ω_t^\pm, Γ_t from (2.28) are not unique!) On the other hand, when (2.28)–(2.30) holds, it is easy to see that Γ_t and the boundary of the set from (2.32) are within a (uniformly in t) bounded distance of each other. So transition fronts are precisely those entire solutions with doubly-bounded widths which also satisfy (2.31) in the sense that for some $n \geq 1$, the boundary of the set in (2.32) is uniformly (in time) close to the union of n t -dependent graphs. In particular, Theorems 2.11 and 2.12(ii) apply to them, the latter also showing that transition fronts need not always exist in dimensions $d \geq 2$.

We note that the condition (2.27) for our transition solutions also has a counterpart in [6]. There an *invasion* of u^- by u^+ is defined to be a transition front connecting u^\pm for which

$$\Omega_s^+ \subseteq \Omega_t^+ \text{ when } s \leq t \quad \text{and} \quad \lim_{r \rightarrow \infty} \inf_{|t-s|=r} d(\Gamma_t, \Gamma_s) = \infty. \quad (2.33)$$

This condition, together with (2.28)–(2.30), implies (2.27) but is stronger than our definition of transition solutions with doubly-bounded widths. Nevertheless, if we relax (2.33) to the existence of T such that

$$\Omega_s^+ \subseteq \Omega_t^+ \text{ when } s + T \leq t \quad \text{and} \quad \lim_{r \rightarrow \infty} \inf_{|t-s|=r} d(\Gamma_t, \Gamma_s) = \infty, \quad (2.34)$$

then (2.28)–(2.30), (2.34) are in fact *equivalent* to our definition of transition solutions with doubly-bounded widths which also propagate with a positive global mean speed. Indeed, notice that (2.34) implies that $\inf_{|t-s|=r} d(\Gamma_t, \Gamma_s)$ grows at least linearly as $r \rightarrow \infty$, so we can again use (2.32) to define Ω_t^+ .

Organization of the Paper and Acknowledgements

In Section 3 we prove some preliminary results. Section 4 is the heart of the argument proving bounded widths of solutions for $d \leq 3$ (the proof of Lemma 4.2 is considerably more complicated for $d = 3$, so it is postponed until Section 7). Theorem 2.5(i) will then be obtained in the short Section 5, and a more involved argument (along with Theorem 2.11(ii)) will be needed to prove Theorem 2.5(ii) and Theorem 2.4(i) in Section 6. All these arguments are extended in Section 8 to obtain proofs of Theorems 2.7 and 2.9, and in Section 9 we prove Theorem 2.11 (the proof of its parts (i,ii) only uses Lemma 3.3 below and, in particular, not the results proved in Section 6). In Section 10 we prove Theorem 2.4(ii) by means of a counter-example (the proof is also independent of the rest of the paper) and Theorem 2.12 is proved in Section 11.

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3. PRELIMINARIES (CASE $u^+ \equiv 1$)

In Sections 3–7 we consider the setting of Theorems 2.4 and 2.5, with f_0, K, θ as in (H), $u^+ \equiv 1$, and $u \in [0, 1]$. We will extend the results below to the setting of (H') in Section 8.

Let us start with some useful preliminary lemmas.

Lemma 3.1. *There is $\varepsilon_0 = \varepsilon_0(f_0, K) > 0$ such that for each $c < c_0$ and $\varepsilon > 0$ there is $\tau = \tau(f_0, K, c, \varepsilon) \geq 0$ such that the following holds. If $u \in [0, 1]$ solves (1.1), (1.2) with f from (H), and $u(t_1, x) \geq 1 - \varepsilon_0$ for some $(t_1, x) \in [t_0 + 1, \infty) \times \mathbb{R}^d$, then for each $t \geq t_1 + \tau$,*

$$\inf_{|y-x| \leq c(t-t_1)} u(t, y) \geq 1 - \varepsilon. \quad (3.1)$$

The same result holds if the hypothesis $u(t_1, x) \geq 1 - \varepsilon_0$ is replaced by

$$u(t_1, \cdot) \geq \frac{1 + \theta_0}{2} \chi_{B_R(x)}(\cdot) \quad (3.2)$$

for some $(t_1, x) \in [t_0, \infty) \times \mathbb{R}^d$ and a large enough $R = R(f_0) > 0$.

Proof. The second claim is proved in [2] when $f(y, \cdot) = f_0(\cdot)$ for all $y \in \mathbb{R}^d$ and follows for general f by the comparison principle.

The first claim holds because (3.2) follows from $u(t_1, x) \geq 1 - \varepsilon_0$, provided $\varepsilon_0 > 0$ is sufficiently small (depending on f_0, K). Indeed, assume that for each $n \in \mathbb{N}$ there were f_n satisfying (H) and u_n solving (1.1) on $(-1, \infty) \times \mathbb{R}^d$ with $f = f_n$, such that $u_n(0, 0) \geq 1 - \frac{1}{n}$ and $\inf_{y \in B_R(0)} u_n(0, y) < \frac{1}{2}(1 + \theta_0)$ (note that we can shift (t_1, x) to $(0, 0)$ without loss, and then $t_0 \leq -1$). By parabolic regularity, there is a subsequence $\{n_j\}_{j \geq 1}$ with u_{n_j} and f_{n_j} locally uniformly converging to $u \in [0, 1]$ and f such that f satisfies (H) and u solves (1.1) on $(-1, \infty) \times \mathbb{R}^d$, with $u(0, 0) = 1$ and $\inf_{y \in B_R(0)} u(0, y) < 1$. But this contradicts the strong maximum principle, and we are done. \square

The first claim of this result immediately shows that solutions with bounded widths propagate with global mean speed in $[c_0, \infty]$. It turns out that bounded width also makes the global mean speed not exceed c_1 , at least in the ignition case. This can be proved by a separate argument and we state both these results in the following lemma.

Lemma 3.2. *Let f_0, K be as in (H) and f_1 be pure ignition. For each $\varepsilon \in (0, \frac{1}{2})$ and $\delta > 0$ there is $\varepsilon' > 0$ and $\tau < \infty$ such that the following holds. If $u \in [0, 1]$ solves (1.1) on $(t_0, \infty) \times \mathbb{R}^d$ with ignition f from (H) satisfying (2.7), and $\sup_{t \in [t_0+1, t_3]} L_{u, \varepsilon'}(t) \leq L$, then*

$$B_{(c_0-\delta)(t_2-t_1)-L}(\Omega_{u, \varepsilon}(t_1)) \subseteq \Omega_{u, 1-\varepsilon}(t_2) \quad \text{and} \quad \Omega_{u, \varepsilon}(t_2) \subseteq B_{(c_1+\delta)(t_2-t_1)+L}(\Omega_{u, 1-\varepsilon}(t_1))$$

whenever $t_1 \geq t_0 + 1$ and $t_2 \in [t_1 + \tau, t_3]$.

Proof. The first inclusion is immediate for any $\varepsilon' \in (0, \min\{\varepsilon, \varepsilon_0\}]$, with τ from Lemma 3.1 with ε and $c := c_0 - \delta$. Indeed, if $x \in \Omega_{u,\varepsilon}(t_1)$, then $\bar{B}_L(x) \cap \Omega_{u,1-\varepsilon_0}(t_1) \neq \emptyset$, so Lemma 3.1 yields the result (even for monostable f).

Let us now consider the second inclusion. Extend f_1 by 0 to $\mathbb{R} \setminus [0, 1]$. It is well known that for any $\delta > 0$ there is $\varepsilon' \in (0, \frac{\varepsilon}{2})$ and a traveling front for some $f_2 \geq f_1 (\geq 0)$ with $f_2 \equiv 0$ on $[0, 2\varepsilon'] \cup \{1 + \varepsilon'\}$, which has speed $c_2 \in [c_1, c_1 + \frac{\delta}{3}]$ and connects ε' and $1 + \varepsilon'$. That is, there is a solution of $U'' + c_2 U' + f_2(U) = 0$ on \mathbb{R} with $U' < 0$, $U(-\infty) = 1 + \varepsilon'$ and $U(\infty) = \varepsilon'$ (and we can also assume $U(0) = 2\varepsilon'$ after translation). Indeed, one only needs to take ε' small enough and f_2 close enough to f_1 .

Let $z_1 := \frac{6d}{\delta}$, $z_2 := \frac{6d+7}{\delta}$ and let $h : [0, \infty) \rightarrow [0, \infty)$ be any C^2 function with $h \equiv 0$ on $[0, z_1]$, $h' \equiv 1$ on $[z_2, \infty)$, and $h' \leq 1$ and $h'' \in [0, \frac{\delta}{6}]$ on $[z_1, z_2]$. We now claim that

$$v(t, x) := U \left(z_2 - h(|x|) - \left(c_2 + \frac{\delta}{3} \right) t \right) \quad (3.3)$$

satisfies

$$v_t \geq \Delta v + f_2(v) \quad \text{on } (-\infty, 0) \times \mathbb{R}^d. \quad (3.4)$$

Indeed, for $|x| \leq z_1$ the argument of U is positive (so $f_2(U) = 0$) and we have

$$v_t - \Delta v - f_2(v) = - \left(c_2 + \frac{\delta}{3} \right) U' \geq 0.$$

For $|x| \geq z_1$ we get

$$-v_t + \Delta v + f_2(v) = \left[\left(c_2 + \frac{\delta}{3} \right) - h''(|x|) - \frac{d-1}{|x|} h'(|x|) \right] U' + (h'(|x|))^2 U'' + f_2(U) =: (*).$$

If $|x| \geq z_2$, then

$$(*) = \left[\left(c_2 + \frac{\delta}{3} \right) - \frac{d-1}{|x|} \right] U' + U'' + f_2(U) = \left[\frac{\delta}{3} - \frac{d-1}{|x|} \right] U' \leq 0.$$

If $|x| \in [z_1, z_2]$, then again the argument of U is positive (so $f_2(U) = 0$) and we have

$$\begin{aligned} (*) &= \left[\left(c_2 + \frac{\delta}{3} \right) - h''(|x|) - \frac{d-1}{|x|} h'(|x|) - c_2 (h'(|x|))^2 \right] U' \\ &= \left[c_2 \left(1 - (h'(|x|))^2 \right) + \left(\frac{\delta}{6} - h''(|x|) \right) + \left(\frac{\delta}{6} - \frac{d-1}{|x|} h'(|x|) \right) \right] U'. \end{aligned}$$

Each of the three terms in the last square bracket is non-negative, so again $(*) \leq 0$ and (3.4) holds.

We now let $\tau := \frac{3}{\delta}(2z_2 - U^{-1}(1))$ and consider arbitrary $y \notin B_{(c_1+\delta)(t_2-t_1)+L}(\Omega_{u,1-\varepsilon}(t_1))$. By the hypothesis and $\varepsilon > \varepsilon'$, we have $u(t, x) < \varepsilon'$ for all $x \in B_{(c_1+\delta)(t_2-t_1)}(y)$. The function

$$w(t, x) := v(t - t_2, x - y) \quad (\geq \varepsilon')$$

is obviously a super-solution of (1.1) on $(t_1, t_2) \times \mathbb{R}^d$, and for $x \notin B_{(c_1+\delta)(t_2-t_1)}(y)$ we have

$$w(t_1, x) \geq U \left(2z_2 - |x - y| - \left(c_1 + \frac{2\delta}{3} \right) (t_1 - t_2) \right) \geq U \left(2z_2 - \frac{\delta}{3}(t_2 - t_1) \right) \geq U \left(2z_2 - \frac{\delta}{3}\tau \right) = 1$$

by $U' < 0$, $h(z) \geq z - z_2$, $c_2 \leq c_1 + \frac{\delta}{3}$, and $t_2 - t_1 \geq \tau$. Hence $w(t_1, \cdot) \geq u(t_1, \cdot)$ and so

$$u(t_2, y) \leq w(t_2, y) = v(0, 0) = U(z_2) < 2\varepsilon' < \varepsilon.$$

Thus $y \notin \Omega_{u,\varepsilon}(t_2)$ and we are done. \square

During the proofs of our main results, we will sometimes need to pass to limits along subsequences of $\{(f_n, u_n)\}$, where f_n satisfy (H) and $u_n \in [0, 1]$ have uniform-in- n bounds on their widths. The following will be useful.

For $\varepsilon \in (0, \frac{1}{2})$, $\ell > 0$, and $t_0 \in [-\infty, \infty)$, let $S_{t_0, \varepsilon, \ell} = S_{t_0, \varepsilon, \ell}(f_0, K, \theta)$ be the set of all pairs (f, u) such that f satisfies (H) with the given f_0, K, θ and $u \in [0, 1]$ solves (1.1) on $(t_0, \infty) \times \mathbb{R}^d$ and satisfies $L_{u, \varepsilon'}(t) \leq \ell$ for all $\varepsilon' \in (\varepsilon, \frac{1}{2})$ and all $t > t_0$. For non-increasing and left-continuous $L : (0, \frac{1}{2}) \rightarrow (0, \infty)$, let

$$S_{t_0, L} = S_{t_0, L}(f_0, K, \theta) := \bigcap_{\varepsilon \in (0, 1/2)} S_{t_0, \varepsilon, L(\varepsilon)}(f_0, K, \theta).$$

(so $(f, u) \in S_{t_0, L}$ implies $L_{u, \varepsilon}(t) \leq L(\varepsilon)$ for $\varepsilon \in (0, \frac{1}{2})$ and $t > t_0$, by left-continuity of L) and

$$S_L = S_L(f_0, K, \theta) := \{(f, u) \mid (f, u) \in S_{-\infty, L}(f_0, K, \theta) \text{ and } u \not\equiv 0, 1\}.$$

Thus any entire solution $u \in [0, 1]$ of (1.1) with bounded width, except $u \equiv 0, 1$, appears in some S_L . Of course, strong maximum principle gives $u \in (0, 1)$ if $(f, u) \in S_L$.

Lemma 3.3. *Fix f_0, K, θ and L as above and let $t_0 \in [-\infty, \infty)$.*

(i) *If for $\varepsilon \in (0, \frac{1}{2})$ and $\ell > 0$ we have $(f_n, u_n) \in S_{t_n, \varepsilon, \ell}(f_0, K, \theta)$ and $\limsup_{n \rightarrow \infty} t_n \leq t_0$, then there is $n_j \rightarrow \infty$ (as $j \rightarrow \infty$) and $(f, u) \in S_{t_0, \varepsilon, \ell}(f_0, K, \theta)$ such that $f_{n_j} \rightarrow f$ locally uniformly on $\mathbb{R}^d \times [0, 1]$ and $u_{n_j} \rightarrow u$ locally uniformly on $(t_0, \infty) \times \mathbb{R}^d$.*

(ii) *If for each $\varepsilon \in (0, \frac{1}{2})$ we have $(f_n, u_n) \in S_{t_n(\varepsilon), \varepsilon, L(\varepsilon)}(f_0, K, \theta)$ and $\limsup_{n \rightarrow \infty} t_n(\varepsilon) \leq t_0$, then there is $n_j \rightarrow \infty$ (as $j \rightarrow \infty$) and $(f, u) \in S_{t_0, L}(f_0, K, \theta)$ such that $f_{n_j} \rightarrow f$ locally uniformly on $\mathbb{R}^d \times [0, 1]$ and $u_{n_j} \rightarrow u$ locally uniformly on $(t_0, \infty) \times \mathbb{R}^d$.*

(iii) *If $\varepsilon \in (0, 2\varepsilon_0]$ and $\ell > 0$, then*

$$\inf \left\{ u_t(t, x) \mid (f, u) \in S_{0, \varepsilon/2, \ell}(f_0, K, 0), u_t \geq 0 \text{ on } (0, \infty) \times \mathbb{R}^d, t \geq 1, u(t, x) \in [\varepsilon, 1 - \varepsilon] \right\} > 0 \quad (3.5)$$

Proof. (i) The properties of f_n , uniform boundedness of u_n , and standard parabolic regularity for u_n prove existence of locally uniform limits f, u along a subsequence $\{n_j\}_{j \geq 1}$, as well as that f satisfies (H) (with the same f_0, K, θ) and u solves (1.1). Locally uniform convergence $u_{n_j} \rightarrow u$ then yields $L_{u, \varepsilon'}(t) \leq \ell$ for all $\varepsilon' \in (\varepsilon, \frac{1}{2})$ and all $t > t_0$ (just pick any $\varepsilon'' \in (\varepsilon, \varepsilon')$ and then a large enough j). Thus $(f, u) \in S_{t_0, \varepsilon, \ell}$.

(ii) The proof is identical to (i).

(iii) Assume that the inf in (3.5) is 0. Then there are $(f_n, u_n) \in S_{0, \varepsilon/2, \ell}$ with $(u_n)_t \geq 0$ and $(t_n, x_n) \in [1, \infty) \times \mathbb{R}^d$ such that $u_n(t_n, x_n) \in [\varepsilon, 1 - \varepsilon]$ and $(u_n)_t(t_n, x_n) \in [0, \frac{1}{n}]$. After shifting (t_n, x_n) to $(1, 0)$ and applying (i), we obtain $(f, u) \in S_{0, \varepsilon/2, \ell}$ with $u(1, 0) \in [\varepsilon, 1 - \varepsilon]$ and $u_t \geq 0 = u_t(1, 0)$. The strong maximum principle for the linear PDE $v_t = \Delta v + f_u(x, u(t, x))v$, satisfied by u_t , then implies $u_t \equiv 0$. This however contradicts Lemma 3.1, which yields $\lim_{t \rightarrow \infty} u(t, 0) = 1 (> u(1, 0))$ because $\sup_{x \in B_\ell(0)} u(1, x) \geq 1 - \frac{\varepsilon}{2} (\geq 1 - \varepsilon_0)$. \square

An important role in the proof of Theorems 2.4 and 2.5 will be played by equilibrium solutions of (1.1).

Lemma 3.4. *Let $f \geq 0$ be Lipschitz and $v \in [0, 1]$ satisfy*

$$\Delta v + f(x, v) = 0 \quad (3.6)$$

on \mathbb{R}^d . If $d \leq 2$, then v is constant and $f(x, v(x)) \equiv 0$. If $d \geq 3$, then

$$\int_{\mathbb{R}^d} |x|^{2-d} f(x, v(x)) dx \leq (d-2) |\partial B_1(0)|. \quad (3.7)$$

Proof. Integrating (3.6) over $B_r := B_r(0)$ and using the divergence theorem yields

$$\int_{B_r} f(x, v(x)) dx = - \int_{\partial B_r} \nabla v(x) \cdot n(x) d\sigma_r(x) = -r^{d-1} \int_{\partial B_1} \tilde{v}_\rho(r, y) d\sigma_1(y)$$

where n is the unit outer normal and σ_r the surface measure for ∂B_r , and $\tilde{v}(\rho, y) = v(\rho y)$ for $(\rho, y) \in (0, \infty) \times \partial B_1$. Multiplying by r^{1-d} and integrating in $r \in [0, r_0]$ gives

$$\int_{B_{r_0}} [l(r_0) - l(|x|)] f(x, v(x)) dx = \int_0^{r_0} r^{1-d} \int_{B_r} f(x, v(x)) dx dr = \int_{\partial B_1} [\tilde{v}(0, y) - \tilde{v}(r_0, y)] d\sigma_1(y),$$

where $l(r) = \ln r$ if $d = 2$ and $l(r) = r^{2-d}/(2-d)$ otherwise. Taking $r_0 \rightarrow \infty$ finally yields

$$\int_{\mathbb{R}^d} [l(\infty) - l(|x|)] f(x, v(x)) dx = |\partial B_1(0)| v(0) - \lim_{r \rightarrow \infty} r^{1-d} \int_{\partial B_r} v(x) d\sigma_r(x).$$

Since $v \in [0, 1]$, either $f(x, v(x)) \equiv 0$ (and then v is constant) or $d \geq 3$ and (3.7) holds. \square

Lemma 3.5. *For $\zeta > 0$, let $\Psi(x) = \psi(|x|)$ be the radially symmetric solution of*

$$\Delta \Psi = \zeta \Psi \quad (3.8)$$

on \mathbb{R}^d with $\Psi(0) = 1$. Then $\psi, \psi' > 0$ on $(0, \infty)$ and

$$\lim_{r \rightarrow \infty} \left(\sqrt{\zeta} r \right)^{(d-1)/2} e^{-\sqrt{\zeta} r} \psi^{(k)}(r) = \zeta^{k/2} l_d \quad (3.9)$$

for some $l_d \in (0, \infty)$ and $k = 0, 1$. In particular,

$$\lim_{r \rightarrow \infty} \psi'(r) \psi(r)^{-1} = \sqrt{\zeta} \quad (3.10)$$

Remark. We only need $k = 0, 1$ here but (3.9) holds for any $k \geq 0$.

Proof. Here ψ is the unique solution of $\psi'' + \frac{d-1}{r} \psi' = \zeta \psi$ on $(0, \infty)$, with $\psi(0) = 1$ and $\psi'(0) = 0$, which is obviously positive along with ψ' . If $d = 1$, one easily checks that

$$\psi(r) = \frac{e^{\sqrt{\zeta} r} + e^{-\sqrt{\zeta} r}}{2}, \quad (3.11)$$

so (3.9) holds with $l_1 = \frac{1}{2}$. If $d \geq 2$, then $\phi(r) := r^{(d-2)/2} \psi(\zeta^{-1/2} r)$ satisfies

$$\phi'' + \frac{1}{r} \phi' - \left[1 + \frac{(d-2)^2}{4r^2} \right] \phi = 0$$

on $(0, \infty)$, with $\lim_{r \rightarrow 0} r^{(2-d)/2} \phi(r) = 1$ and $\lim_{r \rightarrow 0} \frac{d}{dr} [r^{(2-d)/2} \phi(r)] = 0$. Thus by [1, p.375], $\phi = c_d I_{(d-2)/2}$ for I_ν ($\nu \in \mathbb{C}$) the modified Bessel function of the first kind and some $c_d > 0$ (in fact, $c_d = 2^{(d-2)/2} \Gamma(\frac{d}{2})$). But now (3.9) follows from $\lim_{r \rightarrow \infty} \sqrt{r} e^{-r} I_\nu^{(k)}(r) = (2\pi)^{-1/2}$ for $k = 0, 1$ [1, pp. 377 and 378], with $l_d := (2\pi)^{-1/2} c_d$. \square

4. BOUNDED WIDTHS FOR SOLUTIONS $u \in [0, 1]$ WITH $u_t \geq 0$ (CASE $u^+ \equiv 1$)

Again we consider f_0, K, θ as in (H), $u^+ \equiv 1$, $u \in [0, 1]$, and also $\eta > 0$ and $\zeta \in (0, c_0^2/4)$. All constants in this section will depend on f_0, K, ζ, η (but not on θ , unless explicitly noted!).

We define $\zeta' := \frac{c_0^2}{8} + \frac{\zeta}{2} \in (\zeta, c_0^2/4)$ and choose any

$$h \in \left[0, \min \left\{ \theta \frac{c_0^2 - 4\zeta}{c_0^2 + 4\zeta}, \frac{\eta}{4K} \right\} \right] \quad (4.1)$$

(obviously $h = 0$ when $\theta = 0$). This yields $\zeta'(\theta - h) \geq \zeta\theta$, which guarantees $\zeta'(u - h) \geq \zeta u$ for all $u \geq \theta$. Hence, any $f \in F(f_0, K, \theta, \zeta, \eta)$ satisfies

$$f(x, u) \leq \zeta'(u - h) \quad \text{for } x \in \mathbb{R}^d \text{ and } u \in [h, \alpha_f(x)]. \quad (4.2)$$

Here, and always, $\alpha_f(x) = \alpha_f(x; \zeta)$ (not $\alpha_f(x; \zeta')$). Let us also take ε_0 from Lemma 3.1 and ψ from Lemma 3.5 corresponding to ζ' . Below, $\psi^{-1}(\cdot)$ is the inverse function to $\psi(\cdot)$ on $[0, \infty)$ while $\psi(\cdot)^{-1} = 1/\psi(\cdot)$.

In the following we will assume that $f \in F(f_0, K, \theta, \zeta, \eta) (\subseteq F(f_0, K, 0, \zeta, \eta))$ and $u \in [0, 1]$ solves (1.1), (1.2). For any $(t, y) \in [t_0, \infty) \times \mathbb{R}^d$ we define

$$Z_y(t) := \inf_{u(t, x) \geq 1 - \varepsilon_0} |x - y| \quad (\in [0, \infty]), \quad (4.3)$$

$$Y_y^h(t) := \sup \{ \rho \mid u(t, \cdot) \leq h + \psi(\rho)^{-1} \psi(|\cdot - y|) \} \quad (\in [0, \infty]), \quad (4.4)$$

and $\gamma_y^h(t) := \psi(Y_y^h(t))^{-1}$. That is, $Z_y(t)$ is the distance from y to the nearest point with value of u sufficiently close to 1, while $Y_y^h(t)$ is the distance from y to the points where the best upper bound of the form $h + \gamma\psi(|\cdot - y|)$ on u takes the value $1 + h$ (both at time t), and $\gamma_y^h(t)$ is the γ from that bound. The latter is clearly non-increasing in h , hence $Y_y^h(t)$ is non-decreasing in h . Note that (3.10) immediately shows

$$Y_y^h(t) \leq Z_y(t) + M \quad (4.5)$$

for some $(\theta, h$ -independent) $M \geq 0$.

Let us also fix any c_Y, c_Z such that

$$2\sqrt{\zeta'} < c_Y < c_Z < c_0, \quad (4.6)$$

for instance, $c_Y := \frac{1}{4}c_0 + \frac{3}{2}\sqrt{\zeta'}$ and $c_Z := \frac{3}{4}c_0 + \frac{1}{2}\sqrt{\zeta'}$. Let $\tau_Z \geq 0$ correspond to $c = c_Z$ and $\varepsilon = \varepsilon_0$ in Lemma 3.1 and let $r_Y \geq 0$ be such that

$$\frac{\psi'(r)}{\psi(r)} \geq \frac{4\zeta'}{c_Y + 2\sqrt{\zeta'}} \quad \left(> \frac{2\zeta'}{c_Y} \right)$$

for $r \geq r_Y$ (which exists by (3.10), with ζ' in place of ζ , and $c_Y > 2\sqrt{\zeta'}$). Finally, let

$$c'_Y := \frac{(K + \zeta')c_Y}{2\zeta'} \quad \left(> \frac{(K + \zeta')}{\sqrt{\zeta'}} \geq 2\sqrt{K} \geq c_1 \right). \quad (4.7)$$

The choice of Y_y^h is motivated by the following result.

Lemma 4.1. *Let $(t_1, y) \in [t_0, \infty) \times \mathbb{R}^d$.*

(i) *If $t \geq t_1$ is such that $Y_y^h(t_1) - c'_Y(t - t_1) \geq r_Y$, then*

$$Y_y^h(t) \geq Y_y^h(t_1) - c'_Y(t - t_1). \quad (4.8)$$

(ii) *If $t_2 \geq t_1$ is such that $Y_y^h(t_1) - c_Y(t_2 - t_1) \geq r_Y$ and $u(t, x) \leq \alpha_f(x)$ on the set $A := \{(t, x) \mid t \in [t_1, t_2] \text{ and } |x - y| \leq Y_y^h(t_1) - c_Y(t - t_1)\}$, then*

$$Y_y^h(t) \geq Y_y^h(t_1) - c_Y(t - t_1) \quad (4.9)$$

for any $t \in [t_1, t_2]$.

(iii) *If $t_1 \geq t_0 + 1$ and $t \geq t_1 + \tau_Z$, then*

$$Z_y(t) \leq [Z_y(t_1) - c_Z(t - t_1)]_+. \quad (4.10)$$

Remark. The point here is that (ii) and (iii), together with $c_Y < c_Z$, will keep $Z_y(t) - Y_y^h(t)$ uniformly bounded above. This is done in Lemma 4.2 below. It turns out, however, that the hypothesis of (ii) is too strong to make this idea directly applicable for $d \geq 3$. Lemma 4.2 nevertheless still holds for $d = 3$, albeit with a considerably more involved proof (see Section 7 below). For $d \geq 4$ the lemma is false in general.

Proof. (i) Since $w(t, x) := h + e^{(K+\zeta')(t-t_1)}\gamma_y^h(t_1)\psi(|x-y|)$ is a super-solution of (1.1) due to $f(x, u) \leq K(u - \theta) \leq K(u - h)$, the comparison principle gives

$$\gamma_y^h(t) \leq e^{(K+\zeta')(t-t_1)}\gamma_y^h(t_1) \quad (4.11)$$

for any $t \geq t_1$. From this and (4.4) we obtain

$$\ln \psi(Y_y^h(t)) \geq \ln \psi(Y_y^h(t_1)) - (K + \zeta')(t - t_1).$$

Since $\frac{d}{dr}[\ln \psi(r)] \geq 2\zeta'/c_Y$ for $r \geq r_Y$, it follows that

$$Y_y^h(t) \geq Y_y^h(t_1) - \frac{(K + \zeta')c_Y}{2\zeta'}(t - t_1) = Y_y^h(t_1) - c'_Y(t - t_1)$$

for all $t \in [t_1, t_2]$, where $t_2 \geq t_1$ is the first time such that $Y_y^h(t_2) = r_Y$. Thus $r_Y \geq Y_y^h(t_1) - c'_Y(t_2 - t_1)$, so $t \leq t_2$ due to $Y_y^h(t_1) - c'_Y(t - t_1) \geq r_Y$, and we are done.

(ii) Let $\beta(t)$ be such that $w(t, x) := h + e^{\beta(t)}\gamma_y^h(t_1)\psi(|x-y|)$ equals $1 + h$ when $t \in [t_1, t_2]$ and $|x - y| = Y_y^h(t_1) - c_Y(t - t_1)$. Then $\beta(t_1) = 0$ and from $\frac{d}{dr}[\ln \psi(r)] \geq 2\zeta'/c_Y$ for $r \geq r_Y$ we obtain $\beta'(t) \geq 2\zeta'$ on $[t_1, t_2]$. Thus we have

$$w_t \geq \Delta w + \zeta'(w - h).$$

From $w \geq h$, (4.2), and the hypothesis it follows that w is a super-solution of (1.1) on A . Since $u(t, x) \leq 1 \leq w(t, x)$ when $t \in [t_1, t_2]$ and $|x - y| \geq Y_y^h(t_1) - c_Y(t - t_1)$, we obtain $w \geq u$

for $t \in [t_1, t_2]$ because $u(t_1, \cdot) \leq w(t_1, \cdot)$. Therefore $\gamma_y^h(t) \leq e^{\beta(t)} \gamma_y^h(t_1)$ for $t \in [t_1, t_2]$ and the result follows.

(iii) This is immediate from Lemma 3.1. \square

The following crucial lemma, which requires $u_t \geq 0$, will enable us to prove the claim in the remark after Lemma 4.1. It essentially shows that Y_y^h cannot decrease faster than at speed c_Y ($< c_Z$) whenever Z_y is much larger than Y_y^h .

Lemma 4.2. *Let $d \leq 3$. There are $(\theta, h, f, u_0$ -independent) $T_Y > 0$ and $\tau_Y \geq T_Y + 1$ such that we have the following whenever (2.16) holds on \mathbb{R}^d . If*

$$Z_y(t_1 + \tau_Y) \geq Y_y^h(t_1) \quad (4.12)$$

for some $(t_1, y) \in [t_0, \infty) \times \mathbb{R}^d$ and $Y_y^h(t_1) - c_Y T_Y \geq r_Y$, then

$$Y_y^h(t_1 + T_Y) \geq Y_y^h(t_1) - c_Y T_Y. \quad (4.13)$$

Remarks. 1. For $d \leq 2$ we can take any $T_Y > 0$. For $d = 3$ any large enough T_Y works.

2. When $d \geq 4$, this result fails in general! The same is true for $d \geq 1$ if f satisfies (H) but we do not require $f \in F(f_0, K, \theta, \zeta, \eta)$.

Proof. We split the proof in two cases, $d \leq 2$ and $d = 3$, due to Lemma 3.4.

Case $d \leq 2$: We first claim that there is $\tau \geq 1$ such that if a solution $u \in [0, 1]$ of (1.1) on $(0, \infty) \times \mathbb{R}^d$ satisfies $u_t \geq 0$ and $u(0, 0) > \alpha_f(0)$, then $u(\tau, 0) > 1 - \varepsilon_0$. Assume that for each $\tau = 1, 2, \dots$ there is some couple $f_\tau \in F(f_0, K, 0, \zeta, \eta)$ and u_τ contradicting this statement with $(f, u) = (f_\tau, u_\tau)$. Then parabolic regularity shows that there is a sequence $\tau_j \rightarrow \infty$ such that f_{τ_j} and u_{τ_j} converge locally uniformly on $\mathbb{R}^d \times [0, 1]$ and on $(0, \infty) \times \mathbb{R}^d$ to some $f \in F(f_0, K, 0, \zeta, \eta)$ and some solution $u \in [0, 1]$ of (1.1) such that $u_t \geq 0$ and $\lim_{t \rightarrow \infty} u(t, 0) \leq 1 - \varepsilon_0$. Moreover, $u_{\tau_j}(0, 0) \geq \alpha_{f_{\tau_j}}(0)$ and $f_{\tau_j} \in F(f_0, K, 0, \zeta, \eta)$ guarantee that $f(0, \cdot) \geq f_0(\cdot) + \eta \chi_{[u(0,0), \theta_0]}(\cdot)$. But then $v(x) := \lim_{t \rightarrow \infty} u(t, x)$ satisfies (3.6) on \mathbb{R}^d (so it is constant by Lemma 3.4) with $f(0, v(0)) > 0$, a contradiction.

We now pick any $T_Y > 0$ and apply this claim with the point $(0, 0)$ shifted to $(t_1 + T_Y, x)$, for any $x \in B_{Y_y^h(t_1)}(y)$. If we let $\tau_Y := T_Y + \tau$, it follows from (4.12) that $u(t_1 + T_Y, x) \leq \alpha_f(x)$, and thus $u(t, x) \leq \alpha_f(x)$ for all $(t, x) \in [t_1, t_1 + T_Y] \times B_{Y_y^h(t_1)}(y)$. Lemma 4.1(ii) now yields (4.13).

Case $d = 3$: This case is considerably more involved, due to the limitation in Lemma 3.4. We postpone its proof until Section 7 in order to not interrupt the flow of the presentation. \square

Note that in the case $d \leq 2$, this result holds even if (2.10) is replaced by

$$\inf_{\substack{x \in \mathbb{R}^d \\ u \in [\alpha_f(x), \theta_0]}} \sup_{y \in B_R(x)} f(y, u) \geq \eta$$

for some $R < \infty$, because we still obtain $f(x, v(x)) > 0$ for the constant function v and some $|x| \leq R$. Theorems 2.4(i) and 2.5 also extend accordingly.

The following result is at the heart of the proofs of our main results.

Theorem 4.3. *Let $d \leq 3$, let f_0, K be as in (H), and let $\eta > 0$ and $\zeta \in (0, c_0^2/4)$.*

(i) There is $M > 0$ such that if $\theta \geq 0$, h satisfies (4.1), $f \in F(f_0, K, \theta, \zeta, \eta)$, $u_0 \in [0, 1]$ satisfies (2.16), and u solves (1.1), (1.2) on $(t_0, \infty) \times \mathbb{R}^d$, then for any $(t, y) \in (t_0, \infty) \times \mathbb{R}^d$ we have

$$Z_y(t) - Y_y^h(t) \leq M + \left[Z_y(t_0) - Y_y^h(t_0) - \left(\frac{c_0}{2} - \sqrt{\zeta'} \right) (t - t_0) \right]_+. \quad (4.14)$$

Moreover, for any $\varepsilon \in (h, \frac{1}{2})$ there is $(\theta, h, f, u_0$ -independent) $\tau_\varepsilon > 0$, continuous and non-increasing in $\varepsilon > 0$, such that

$$L_{u,\varepsilon}(t) \leq M_{\varepsilon-h} + \left[\sup_{y \in \mathbb{R}^d} (Z_y(t_0) - Y_y^h(t_0)) - \left(\frac{c_0}{2} - \sqrt{\zeta'} \right) (t - t_0) \right]_+ \quad (4.15)$$

for any $t \geq t_0 + \tau_\varepsilon$, with $M_\delta := M + c'_Y \tau_\delta + \psi^{-1}(\delta^{-1})$.

(ii) If $\theta, h, M, M_\delta, \tau_\varepsilon, f, u$ are from (i) and $v \in [0, 1]$ satisfies

$$u(t - T, \cdot) - \frac{\varepsilon}{2} \leq v(t, \cdot) \leq u(t + T, \cdot) + \frac{\varepsilon}{2} \quad (4.16)$$

for some $\varepsilon \in (2h, \frac{1}{2})$, $T \geq 0$, and $t \geq t_0 + T + \tau_{\varepsilon/2}$, then for such t ,

$$L_{v,\varepsilon}(t) \leq M_{\varepsilon/2-h} + 3c'_Y T + \left[\sup_{y \in \mathbb{R}^d} (Z_y(t_0) - Y_y^h(t_0)) - \left(\frac{c_0}{2} - \sqrt{\zeta'} \right) (t - t_0) \right]_+ \quad (4.17)$$

Remarks. 1. Recall also the bound from below in (4.5).

2. $\frac{1}{2}c_0 - \sqrt{\zeta'}$ can be replaced by any $c < c_0 - 2\sqrt{\zeta'}$, and then M, M_δ also depend on c .

3. Obviously M_δ is continuous and decreasing in $\delta > 0$.

Proof. (i) Let us start with (4.14). Assume, without loss, that $y = 0$ and $t_0 = 0$, and denote $Y_0^h = Y$ and $Z_0 = Z$. Recall that $c_Z = \frac{3}{4}c_0 + \frac{1}{2}\sqrt{\zeta'}$ and $c_Y = \frac{1}{4}c_0 + \frac{3}{2}\sqrt{\zeta'}$, so that $c_Z - c_Y = \frac{1}{2}c_0 - \sqrt{\zeta'}$, and then pick $c'_Y, \tau_Z, r_Y, T_Y, \tau_Y$ as above (all these constants are independent of θ, h).

We can assume $Z(t) > 0$ because otherwise the claim is obvious. It is also sufficient to prove the claim for t such that $Y(t) \geq c'_Y(\tau_Y + \tau_Z) + r_Y$ because then the result follows for all $t > 0$ after increasing M by $c'_Y(\tau_Y + \tau_Z) + r_Y$. This is because Z (and also Y) is non-increasing due to (2.16). We also note that Y is then continuous by Lemma 4.1(i), while Z is right-continuous and lower-semi-continuous by continuity of u on $(0, \infty) \times \mathbb{R}^d$. Finally, we can assume that $t > \tau_Y + \tau_Z$, because for $t \in (0, \tau_Y + \tau_Z]$ the estimate follows for any $M \geq (c'_Y + \frac{1}{2}c_0 - \sqrt{\zeta'})(\tau_Y + \tau_Z)$ due to Lemma 4.1(i), Z and Y being non-increasing, and the assumption $Y(t) \geq c'_Y(\tau_Y + \tau_Z) + r_Y$.

We will now prove (4.14) assuming $t > \tau_Y + \tau_Z$, $Z(t) > 0$ and $Y(t) \geq c'_Y(\tau_Y + \tau_Z) + r_Y$, with

$$M := c_Z \tau_Y + c'_Y(\tau_Y + \tau_Z).$$

Let t_2 be the smallest number in $[0, t - \tau_Y]$ such that $Z(t_1 + \tau_Y) \geq Y(t_1)$ for all $t_1 \in (t_2, t - \tau_Y)$. Lower-semi-continuity of $Z(\cdot + \tau_Y) - Y(\cdot)$ now shows the following. If $t_2 = 0$, then $Z(\tau_Y) \geq Y(0)$; if $t_2 \in (0, t - \tau_Y)$, then $Z(t_2 + \tau_Y) = Y(t_2)$; and if $t_2 = t - \tau_Y$, then $Z(t) \leq Y(t - \tau_Y)$.

If $t_2 = 0$, let $N := \lfloor t/T_Y \rfloor$. Applying Lemma 4.1(i) once and then Lemma 4.2 N times, we obtain using $Y(t) - c'_Y T_Y \geq r_Y$ (recall that $\tau_Y > T_Y$),

$$Y(t) \geq Y(NT_Y) - c'_Y T_Y \geq Y(0) - N c_Y T_Y - c'_Y T_Y \geq Y(0) - c_Y t - c'_Y T_Y.$$

On the other hand, Lemma 4.1(iii) and $t \geq \tau_Y + \tau_Z$ yield

$$Z(t) \leq Z(\tau_Y) - c_Z(t - \tau_Y) \leq Z(0) - c_Z t + c_Z \tau_Y$$

(notice that $Z(\tau_Y) - c_Z(t - \tau_Y) > 0$ because otherwise $Z(t) = 0$ by Lemma 4.1(iii)). Thus

$$Z(t) - Y(t) \leq c_Z \tau_Y + c'_Y T_Y + Z(0) - Y(0) - (c_Z - c_Y)t \leq M + Z(0) - Y(0) - (c_Z - c_Y)t.$$

If $t_2 \in (0, t - \tau_Y - \tau_Z)$, then let $N := \lfloor (t - t_2)/T_Y \rfloor$. An identical argument now yields

$$Y(t) \geq Y(t_2) - c_Y(t - t_2) - c'_Y T_Y$$

and

$$Z(t) \leq Z(t_2 + \tau_Y) - c_Z(t - t_2 - \tau_Y).$$

Thus $Z(t_2 + \tau_Y) = Y(t_2)$ yields

$$Z(t) - Y(t) \leq c_Z \tau_Y + c'_Y T_Y + Z(t_2 + \tau_Y) - Y(t_2) - (c_Z - c_Y)(t - t_2) \leq c_Z \tau_Y + c'_Y T_Y \leq M.$$

If $t_2 \in [t - \tau_Y - \tau_Z, t - \tau_Y]$, then $Z(t_2 + \tau_Y) \leq Y(t_2)$, so that

$$Z(t) - Y(t) \leq Z(t_2 + \tau_Y) - Y(t) \leq Y(t_2) - Y(t) \leq c'_Y(\tau_Y + \tau_Z) \leq M.$$

by Lemma 4.1(i). The proof of (4.14) is finished.

Let us now turn to (4.15) and again assume $t_0 = 0$. Let $c := \frac{1}{2}c_0$, and for any $\varepsilon \in (h, \frac{1}{2})$ let $\tau_\varepsilon := \tau + 1$, with τ from Lemma 3.1 (this can obviously be chosen continuous and non-increasing in $\varepsilon > 0$). For $t \geq \tau_\varepsilon$, let $x \in \Omega_{u,\varepsilon}(t)$. Then $Y_x^h(t) \leq \psi^{-1}((\varepsilon - h)^{-1})$, so

$$Y_x^h(t - \tau) \leq \psi^{-1}((\varepsilon - h)^{-1}) + c'_Y \tau \leq \psi^{-1}((\varepsilon - h)^{-1}) + c'_Y \tau_\varepsilon$$

by Lemma 4.1(i), and (4.14) gives

$$Z_x(t - \tau) \leq \psi^{-1}((\varepsilon - h)^{-1}) + c'_Y \tau_\varepsilon + M + \left[\sup_{y \in \mathbb{R}^d} (Z_y(0) - Y_y^h(0)) - \left(\frac{c_0}{2} - \sqrt{\zeta'} \right) (t - \tau) \right]_+.$$

Lemma 3.1 with $t_1 := t - \tau$ now shows that there is y with $u(t, y) \geq 1 - \varepsilon$ and

$$|y - x| \leq \psi^{-1}((\varepsilon - h)^{-1}) + c'_Y \tau_\varepsilon + M + \left[\sup_{y \in \mathbb{R}^d} (Z_y(0) - Y_y^h(0)) - \left(\frac{c_0}{2} - \sqrt{\zeta'} \right) (t - \tau) \right]_+ - \frac{c_0}{2} \tau.$$

This yields (4.15) for $t \geq \tau_\varepsilon$ (recall that $t_0 = 0$ and $\tau \geq 0$).

(ii) Again assume $t_0 = 0$. An identical argument to the last one shows that if $t \geq T + \tau_{\varepsilon/2}$ and $x \in \Omega_{u,\varepsilon/2}(t + T)$ (the latter holds when $v(t, x) \geq \varepsilon$), then

$$Y_x^h(t - T - \tau) \leq \psi^{-1} \left(\left(\frac{\varepsilon}{2} - h \right)^{-1} \right) + c'_Y(2T + \tau_{\varepsilon/2}),$$

and ultimately that there is y with $u(t - T, y) \geq 1 - \frac{\varepsilon}{2}$ (which yields $v(t, y) \geq 1 - \varepsilon$) and

$$|y - x| \leq \psi^{-1} \left(\left(\frac{\varepsilon}{2} - h \right)^{-1} \right) + c'_Y(2T + \tau_{\varepsilon/2}) + M + \left[\sup_{y \in \mathbb{R}^d} (Z_y(0) - Y_y^h(0)) - \left(\frac{c_0}{2} - \sqrt{\zeta'} \right) (t - T - \tau) \right]_+ - \frac{c_0}{2} \tau.$$

This proves (4.17) because $\frac{1}{2}c_0 - \sqrt{\zeta'} \leq c'_Y$. \square

Before moving onto the proofs of our main results, we state an important corollary of Theorem 4.3. For a solution u of (1.1) on $(t_0, \infty) \times \mathbb{R}^d$ and for $t \geq t_0$, we let

$$\Lambda_u^h(t) := \sup_{y \in \mathbb{R}^d} (Z_y(t) - Y_y^h(t)) \quad (\leq \infty) \quad (4.18)$$

for h from (4.1). Then Λ_u^h is non-increasing and right-continuous in h , by definition of Y_y^h .

Notice that if Λ_u^h is finite, then it controls $L_{u,\varepsilon}$ for any $\varepsilon \in (h, \frac{1}{2})$. Indeed, the argument proving (4.15) from (4.14) applies to any u (even if $u_t \not\geq 0$) and, with the notation from Theorem 4.3, yields for $t \geq t_0 + \tau_\varepsilon$,

$$L_{u,\varepsilon}(t) \leq M_{\varepsilon-h} - M + \Lambda_u^h(t - \tau_\varepsilon + 1). \quad (4.19)$$

For ignition f we also have the opposite direction. For any $(t, y) \in (t_0, \infty) \times \mathbb{R}^d$ and $h \in (0, \min \{ \theta(c_0^2 - 4\zeta)(c_0^2 + 4\zeta)^{-1}, \frac{\eta}{4K}, \varepsilon_0 \}]$, we have

$$\sup_{|x-y| < Y_y^h(t)} u(t, x) > h$$

because otherwise we would have $u(t, \cdot) \leq h + \gamma\psi(|\cdot|)$ for some $\gamma < \psi(Y_y^h(t))^{-1}$, contradicting the definition of $Y_y^h(t)$. But then $Z_y(t) \leq Y_y^h(t) + L_{u,h}(t)$ by $h \leq \varepsilon_0$, so

$$\Lambda_u^h(t) \leq L_{u,h}(t). \quad (4.20)$$

We now have the following result for entire solutions with $u_t \geq 0$.

Corollary 4.4. *Let $d \leq 3$, let f_0, K , and $\theta \geq 0$ be as in (H), and let $\eta > 0$, $\zeta \in (0, c_0^2/4)$, and $f \in F(f_0, K, \theta, \zeta, \eta)$. Assume that $u \in [0, 1]$ solves (1.1) and satisfies $u_t \geq 0$ on $\mathbb{R} \times \mathbb{R}^d$.*

(i) If h is as in (4.1) and $\limsup_{t \rightarrow -\infty} \Lambda_u^h(t) < \infty$, then in fact $\sup_{t \in \mathbb{R}} \Lambda_u^h(t) \leq M$ and $\sup_{t \in \mathbb{R}} L_{u,\varepsilon}(t) \leq M_{\varepsilon-h}$ for any $\varepsilon \in (h, \frac{1}{2})$ (here M, M_δ are from Theorem 4.3). We also have

$$\inf_{u(t,x) \in [\varepsilon, 1-\varepsilon]} u_t(t, x) \geq \mu_{\varepsilon, M_{\varepsilon/2-h}} \quad (4.21)$$

for any $\varepsilon \in (2h, 2\varepsilon_0]$, where $\mu_{\varepsilon,\ell} > 0$ is the inf in (3.5).

(ii) If $\theta > 0$ and u has bounded width, then $\sup_{t \in \mathbb{R}} \Lambda_u^0(t) \leq M$ (and so (i) holds with $h = 0$ and $\varepsilon \in (0, \frac{1}{2})$). Moreover, if a pure ignition f_1 satisfies (2.7), then u propagates with global mean speed in $[c_0, c_1]$, with $\tau_{\varepsilon,\delta}$ in Definition 2.2 depending only on $\delta, f_1, \varepsilon, f_0, K, \zeta, \eta$.

Remark. Recall that $M, M_\varepsilon, \mu_\varepsilon$ depend on f_0, K, ζ, η (and ε) but not on θ, h, f, u . This and Theorem 2.11(ii) will be the key to the independence of the bounds in Theorem 2.4(i) on u_0 .

Proof. (i) The first claim is immediate from (4.14) after letting $u_0(x) := u(t_0, x)$ and then sending $t_0 \rightarrow -\infty$. The second then follows from (4.19), and the third from Lemma 3.3(iii) with $\theta = 0$, applied to u shifted in time by $1 - t$.

(ii) For any $h \in (0, \min \{ \theta(c_0^2 - 4\zeta)(c_0^2 + 4\zeta)^{-1}, \frac{\eta}{4K}, \varepsilon_0 \}]$, (4.20) and (i) show $\sup_{t \in \mathbb{R}} \Lambda_u^h(t) \leq M$, and right-continuity of Λ_u^h in h then yields $\sup_{t \in \mathbb{R}} \Lambda_u^0(t) \leq M$. The second claim now follows from Lemma 3.2 with $\delta/2$ instead of δ , using the bound $L_{u, \varepsilon'}(t) \leq M_{\varepsilon'}$ for ε' from that lemma (which holds by (4.19) with $h = 0$). Indeed, we only need to take $\tau_{\varepsilon, \delta} \geq \max\{2\delta^{-1}M_{\varepsilon'}, \tau\}$ in Definition 2.2, with τ from Lemma 3.2. \square

Open problem. It is an interesting question whether there is a transition solution u satisfying all the hypotheses of Corollary 4.4(ii), except of the hypothesis of bounded width, such that Λ_u^0 is unbounded (cf. Open problem 2 after Theorem 2.11). It is obvious from (4.14) that in that case one would have $\liminf_{t \rightarrow -\infty} |t|^{-1} \Lambda_u^0(t) > 0$.

5. PROOF OF THEOREM 2.5(I)

We can assume $u_0 \not\equiv 0, 1$ because then the result holds trivially. As in Section 4, all constants will depend on f_0, K, ζ, η (but not on θ from (H), unless explicitly noted).

The second claim in (2.8) follows immediately from the first. Indeed: it is sufficient to prove it for $\varepsilon \in (0, 2\varepsilon_0]$; if $\mu_{\varepsilon, \ell} > 0$ is the inf in (3.5) for such ε and $\ell_\varepsilon, T_\varepsilon$ are from the first claim in (2.8), then the second claim follows with $m_\varepsilon := \mu_{\varepsilon, \ell_{\varepsilon/2}}$ and T_ε replaced by $T_{\varepsilon/2} + 1$, after applying Lemma 3.3(iii) to u shifted in time by $-(t_0 + T_{\varepsilon/2})$.

Similarly, the claim about global mean speed also follows from the first claim in (2.8). Indeed, if ε', τ are from Lemma 3.2 with $\delta/2$ instead of δ , then that lemma shows that we only need $T_{\varepsilon, \delta} \geq T_{\varepsilon'} + 1$ and $\tau_{\varepsilon, \delta} \geq \max\{2\delta^{-1}\ell_{\varepsilon'}, \tau\}$ in Definition 2.2.

We are left with proving the first claim in (2.8) (which also proves that u has a bounded width). We will do so with $\ell_\varepsilon := M_{\varepsilon/2}$ from Theorem 4.3. We define Z_y, Y_y^h as in Section 4 and split the proof into two cases.

Case $d = 3$: Let $\varepsilon' := 1 - \varepsilon_0$ (which depends on f_0, K) and given any $\varepsilon \in (0, \frac{1}{2})$, let $h := \min \{ \theta(c_0^2 - 4\zeta)(c_0^2 + 4\zeta)^{-1}, \frac{\eta}{4K}, \varepsilon_0, \frac{\varepsilon}{2} \}$. The argument which proves (4.20) now shows $Z_y(t_0) \leq Y_y^h(t_0) + L_{u, h, \varepsilon'}(t_0)$ for each $y \in \mathbb{R}^3$. Hence the right-hand side of (4.15) equals $M_{\varepsilon-h} (\leq M_{\varepsilon/2})$ for all large enough t and we are done.

Case $d \leq 2$: First, there is $\tau \geq 1$ such that if a solution $u \in [0, 1]$ of (1.1) on $(t_0, \infty) \times \mathbb{R}^d$ with f as in the theorem satisfies $u_t \geq 0$ and $u(t_0, x) \geq \varepsilon'$, then $u(t_0 + \tau, x) > 1 - \varepsilon_0$. This is proved just as a similar claim in the proof of Lemma 4.2.

Define now Z'_y as Z_y but with ε' in place of $1 - \varepsilon_0$, and given any $\varepsilon \in (0, \frac{1}{2})$, let $h := \min \{ \theta(c_0^2 - 4\zeta)(c_0^2 + 4\zeta)^{-1}, \frac{\eta}{4K}, 1 - \varepsilon', \frac{\varepsilon}{2} \}$. The argument which proves (4.20) now shows $Z'_y(t_0) \leq Y_y^h(t_0) + L_{u, h, \varepsilon'}(t_0)$, and then the previous paragraph and Lemma 4.1(i) yield

$$Z_y(t_0 + \tau) \leq Y_y^h(t_0) + L_{u, h, \varepsilon'}(t_0) \leq Y_y^h(t_0 + \tau) + c'_Y \tau + L_{u, h, \varepsilon'}(t_0).$$

This holds for all $y \in \mathbb{R}^d$, so we conclude as in the first case.

The proof of Theorem 2.5(i) is finished.

Proof of Remark 2 after Theorem 2.5. The second claim in (2.8) follows from the first as above. To prove the first claim in (2.8), again consider two cases.

Case $d = 3$: Let $\varepsilon' := 1 - \varepsilon_0$. The hypothesis and (3.9) for $k = 0$ and ζ' in place of ζ imply

$$C := \sup_{\varepsilon \in (0,1), r \geq 0} \varepsilon \psi(r)^{-1} \psi(r + L_{u,\varepsilon,\varepsilon'}(t_0)) < \infty.$$

Assume that $u_0 \neq 0$ because otherwise the result holds trivially. By the definition of Y_y^h , for any $y \in \mathbb{R}^3$, there is $x \in \mathbb{R}^3$ such that

$$u(t_0, x) = \psi(Y_y^0(t_0))^{-1} \psi(|x - y|) \quad (=:\varepsilon > 0).$$

Then there is $x' \in B_{L_{u,\varepsilon,\varepsilon'}(t_0)}(x)$ with $u(t_0, x') \geq \varepsilon' (= 1 - \varepsilon_0)$, and we have

$$\psi(|x' - y|) \leq \psi(|x - y| + L_{u,\varepsilon,\varepsilon'}(t_0)) \leq C\varepsilon^{-1} \psi(|x - y|) = C\psi(Y_y^0(t_0)).$$

Since C is independent of y , this means $\sup_{y \in \mathbb{R}^3} (Z_y(t_0) - Y_y^0(t_0)) < \infty$. We conclude as in the ignition case, using (4.15).

Case $d \leq 2$: The argument from the first case shows $\sup_{y \in \mathbb{R}^d} (Z'_y(t_0) - Y_y^0(t_0)) < \infty$, with Z'_y from the ignition case $d \leq 2$. As in the ignition case $d \leq 2$, and with the same τ , we obtain $\sup_{y \in \mathbb{R}^d} (Z_y(t_0 + \tau) - Y_y^0(t_0 + \tau)) < \infty$ and the result follows as before.

Finally, the first inclusion of the claim about global mean speed follows from the first claim in (2.8) as in the ignition case because the first inclusion in Lemma 3.2 holds also for monostable f . The second inclusion in Definition 2.2, with $c' := c'_Y$, follows from $\Lambda_u^0(t) \leq M$ (which holds for all large enough t by (4.14)) and (4.8). \square

6. PROOFS OF THEOREMS 2.4(I) AND 2.5(II)

As in Section 4, all constants will depend on f_0, K, ζ, η (but not on θ from (H), unless explicitly noted).

Note that the claim about global mean speed follows in both cases from the first claim in (2.8) as in the proof of Theorem 2.5(i). Since bounded width also follows from the first claim in (2.8), we are therefore left with proving (2.8) in both cases.

Let us start with the (easier to prove) analogous results for monostable reactions from Remark 4 after Theorem 2.4 and Remark 3 after Theorem 2.5.

Proof of Remark 4 after Theorem 2.4. We can assume without loss that $t_0 = 0$. The idea is to construct w_0 such that

$$\Delta w_0(\cdot) + f(\cdot, w_0(\cdot)) \geq 0 \tag{6.1}$$

and the solution w to (1.1) with $w(0, x) = w_0(x)$ satisfies $w(\tau, \cdot) \geq u_0(\cdot)$ and $u(\tau, \cdot) \geq w_0(\cdot)$ for some $\tau > 0$. Then u will satisfy

$$w(t - \tau, \cdot) \leq u(t, \cdot) \leq w(t + \tau, \cdot) \tag{6.2}$$

for $t \geq \tau$. Since $w_t \geq 0$, Theorem 4.3(ii) for w, u in place of u, v will now do the trick.

Let us first consider (2.15), and assume without loss $e = (1, 0, \dots, 0)$. Let $s_0 > 0$ be such that there is a smooth, even, $2s_0$ -periodic C^2 function $U : \mathbb{R} \rightarrow [0, \frac{1}{2}(1 + \theta_0)]$ satisfying $U'' + f_0(U) > 0$ on \mathbb{R} , $U(0) = \frac{1}{2}(1 + \theta_0)$, $U(s_0) = 0$, and $U' < 0$ on $(0, s_0)$ (then obviously

$U'(0) = U'(s_0) = 0$). Such U is obtained by perturbing the solution of $\tilde{U}'' + f_0(\tilde{U}) = 0$ with $\tilde{U}(0) = \frac{1}{2}(1 + \theta_0)$ and $\tilde{U}'(0) = 0$. The latter satisfies $\tilde{U}' < 0$ on some interval $(0, \tilde{s}_0]$ with $\tilde{U}(\tilde{s}_0) = 0$ because multiplying the ODE by \tilde{U}' and integrating yields

$$\tilde{U}'(s)^2 = \tilde{U}'(0)^2 + 2 \int_{\tilde{U}(s)}^{\tilde{U}(0)} f_0(u) du = 2 \int_{\tilde{U}(s)}^{(1+\theta_0)/2} f_0(u) du > 0$$

as long as $\tilde{U}(s) > 0$ (notice that $\tilde{U}''(0) < 0$). Thus we can perturb \tilde{U} to obtain the desired U , with s_0 near \tilde{s}_0 . Then U, s_0 depend only on f_0 , and we define

$$W(s) := \begin{cases} \frac{1}{2}(1 + \theta_0) & s \leq R_2, \\ U(s - R_2) & s \in (R_2, R_2 + s_0), \\ 0 & s \geq R_2 + s_0. \end{cases} \quad (6.3)$$

Note that

$$\inf_{s < R_2 + s_0} [W''(s) + f_0(W(s))] = \inf_{s \in \mathbb{R}} [U''(s) + f_0(U(s))] > 0.$$

Then $w_0(x) := W(x_1)$ satisfies (6.1), and $w(\tau, \cdot) \geq u_0(\cdot)$ follows for some (f_0, ε_2) -dependent τ , from $\varepsilon_2 > 0$ and the second claim in Lemma 3.1. Similarly, $u(\tau, \cdot) \geq w_0(\cdot)$ follows for some $(f_0, R_2 - R_1, \varepsilon_1)$ -dependent τ from the second claim in Lemma 3.1 (which holds with $\frac{1}{2}(1 + \theta_0)$ replaced by $\theta_0 + \varepsilon_1$ when $\varepsilon_1 > 0$, and with $R = R(f_0, \varepsilon_1)$ [2]).

Thus w satisfies (6.2) for all $t \geq \tau$. Let us increase τ so that

$$w(\tau, \cdot) \geq (1 - \varepsilon_0)\chi_{\{x \mid x \cdot e < R_2\}}(\cdot).$$

This makes τ also depend on K , and then Lemma 4.1(i) applied to w yields

$$\Lambda_w^0(\tau) \leq s_0 + c'_Y \tau \quad (6.4)$$

because $Y_y^0(0) \geq y_1 - (R_2 + s_0)$ if Y_y^0 is defined with respect to w . With $M_\varepsilon, \tau_\varepsilon$ from Theorem 4.3, let $\ell_\varepsilon := M_{\varepsilon/2} + 3c'_Y \tau$ and

$$T_\varepsilon := 2\tau + \tau_{\varepsilon/2} + \left(\frac{c_0}{2} - \sqrt{\zeta}\right)^{-1} (s_0 + c'_Y \tau). \quad (6.5)$$

Then Theorem 4.3(ii) with w, u in place of u, v gives $L_{u,\varepsilon}(t) \leq \ell_\varepsilon$ for $t \geq T_\varepsilon$. The proof in the case (2.15) is finished.

Let us now assume (2.14) as well as $x_0 = 0$ without loss, and first also assume that $\sup u_0 < 1$. We can also assume without loss that (with U as above)

$$R_2 \geq \frac{(d-1)\|U'\|_\infty}{\inf_{s \in \mathbb{R}} [U''(s) + f_0(U(s))]} \quad (6.6)$$

The result now holds for any $R_1 \geq R(f_0, \varepsilon_1)$, where the latter is from the argument above so that the conclusion of Lemma 3.1 still holds. Indeed, this time we let $w_0(x) := W(|x|)$, which also satisfies (6.1) due to (6.6). As above, we obtain (6.2) for some $\tau > 0$, and then again $L_{u,\varepsilon}(t) \leq \ell_\varepsilon$ for $t \geq T_\varepsilon$.

Finally, assume (2.14) with $x_0 = 0$, and $\sup u_0 = 1$. We let again (6.6) and $w_0(x) := W(|x|)$, but now $w(\tau, \cdot) \not\geq u_0(\cdot)$ for all $\tau \geq 0$. Solve (1.1), (1.2) and replace u_0 by $u(1, \cdot)$. It is obviously sufficient to prove this claim for the new u_0 . This now satisfies

$$u_0(x) \leq \min \left\{ 1 - \varepsilon_2, \frac{|B_1(0)|R_2^d}{(4\pi)^{d/2}} e^K e^{-\max\{|x|-R_2, 0\}^2/4} \right\},$$

by the comparison principle, for some (K, R_2) -dependent $\varepsilon_2 > 0$. Since $w(\tau, \cdot)$ converges locally uniformly to 1 as $\tau \rightarrow \infty$, $w_t \geq 0$, and $w(\tau, x) \sim e^{-|x|^2/4\tau}$ as $|x| \rightarrow \infty$ by the heat equation asymptotics, we again obtain $w(\tau, \cdot) \geq u_0(\cdot)$ for some τ . The rest of the argument is as before. \square

Proof of Remark 3 after Theorem 2.5. The first claim in (2.8) is proved as above, now with u playing the role of w because $u_t \geq 0$. Indeed, assume $t_0 = 0$ and let $\tau < \infty$ be such that $s := \sup_{y \in \mathbb{R}^d} (Z_y(\tau) - Y_y^0(\tau)) < \infty$ (which exists by the proof of Remark 2 after Theorem 2.5). With $M_\varepsilon, \tau_\varepsilon, \ell_\varepsilon$ from the previous proof, let T_ε be from (6.5) but with $s_0 + c'_Y \tau$ replaced by s . Then again $L_{v,\varepsilon}(t) \leq \ell_\varepsilon$ for $t \geq T_\varepsilon$ by Theorem 4.3(ii), as above.

The global mean speed claim is proved as in the proof of Remark 2 after Theorem 2.5, this time using $\Lambda_v^0(t) \leq \Lambda_u^0(t - \tau) + 2c'_Y \tau \leq M + 2c'_Y \tau$ (which again holds for all large t). \square

The proof of (2.8) in Theorem 2.4(i) resp. Theorem 2.5(ii) is similar but a little more involved. To show that ℓ_ε is independent of $R_1, R_2, \varepsilon_1, \varepsilon_2$ resp. τ , as well as to obtain the second claim in (2.8), we will need to use Theorem 2.11. In addition, the exponential tails of the initial data in Theorem 2.4(i) will be handled by constructing appropriate super-solutions and obtaining inequalities as in (4.16) instead of (6.2).

We will start with proving the result for general solutions u which (essentially) lie between two time-translates of a solution w with initial datum satisfying (6.1). The bounds in this result will, in fact, be independent of u, w for large t as long as the number $\Lambda_w^h(0)$, defined in (4.18), is finite for each small enough $h > 0$.

Theorem 6.1. *Let $d \leq 3$, let f_0, K be as in (H), and let $\eta > 0$ and $\zeta \in (0, c_0^2/4)$. For any $\varepsilon' \in (0, \frac{1}{2})$, there are $\ell_{\varepsilon'}, m_{\varepsilon'} \in (0, \infty)$ such that if $\theta > 0$, $\lambda : (0, \frac{1}{2}) \rightarrow (0, \infty)$ is left-continuous and non-increasing, $\tau < \infty$, and $\nu : (0, \infty) \rightarrow [0, \infty)$ satisfies $\lim_{t \rightarrow \infty} \nu(t) = 0$, then there is $T_{\varepsilon', \theta, \lambda, \tau, \nu} < \infty$ such that the following holds. If $f \in F(f_0, K, \theta, \zeta, \eta)$ and $u, w \in [0, 1]$ are solutions of (1.1) on $(0, \infty) \times \mathbb{R}^d$ with $w_0(\cdot) := w(0, \cdot)$ satisfying (6.1), with $\Lambda_w^h(0) \leq \lambda(h)$ for all $h \in (0, \min \{ \theta(c_0^2 - 4\zeta)(c_0^2 + 4\zeta)^{-1}, \frac{\eta}{4K}, \varepsilon_0 \})$, and with*

$$w(t - \tau, \cdot) - \nu(t) \leq u(t, \cdot) \leq w(t + \tau, \cdot) + \nu(t) \tag{6.7}$$

for each $t > \tau$, then

$$\sup_{t \geq T_{\varepsilon', \theta, \lambda, \tau, \nu}} L_{u, \varepsilon'}(t) \leq \ell_{\varepsilon'} \quad \text{and} \quad \inf_{\substack{t \geq T_{\varepsilon', \theta, \lambda, \tau, \nu} \\ u(t, x) \in [\varepsilon', 1 - \varepsilon']}} u_t(t, x) \geq m_{\varepsilon'}. \tag{6.8}$$

Remark. We stress that $\ell_{\varepsilon'}, m_{\varepsilon'}$ are independent of f, u as well as of $\theta, \lambda, \tau, \nu$.

Proof. Let $h_0 := \min \{ \theta(c_0^2 - 4\zeta)(c_0^2 + 4\zeta)^{-1}, \frac{\eta}{4K}, \varepsilon_0 \} > 0$. For $\varepsilon \in (0, 4h_0]$, and with $M_\delta, \tau_\varepsilon$ from Theorem 4.3, let $T(\varepsilon) \geq \tau + \tau_{\varepsilon/2}$ be such that $\sup_{t \geq T(\varepsilon)} \nu(t) \leq \frac{\varepsilon}{2}$, and define $L(\varepsilon) := M_{\varepsilon/4} + 3c'_Y \tau + \lambda(\frac{\varepsilon}{4})$. For $\varepsilon \in (4h_0, \frac{1}{2})$ let $T(\varepsilon) := T(4h_0)$ and $L(\varepsilon) := L(4h_0)$.

Then for any $\varepsilon \in (0, 4h_0]$, by Theorem 4.3(ii) with $h := \frac{\varepsilon}{4}$,

$$L_{u,\varepsilon}(t) \leq L(\varepsilon) \quad \text{for } t \geq T(\varepsilon) \quad (6.9)$$

(and this also holds for $\varepsilon \in (4h_0, \frac{1}{2})$ because $L_{u,\varepsilon}(t)$ is non-increasing in ε).

Let us now prove the first claim in (6.8), with $\ell_{\varepsilon'} := M_{\varepsilon'/2} + 1$. If there is no such $T_{\varepsilon',\theta,\lambda,\tau,\nu}$, then there is a sequence $(f_n, u_n, w_n, t_n, x_n)$ with f_n, u_n, w_n satisfying the hypotheses of the theorem, $\lim_{n \rightarrow \infty} t_n = \infty$, $u_n(t_n, x_n) \in [\varepsilon', 1 - \varepsilon']$ and

$$\inf_{u_n(t_n, y) \geq 1 - \varepsilon'} |y - x_n| > \ell_{\varepsilon'}. \quad (6.10)$$

After shifting f_n by $(-x_n, 0)$ and u_n by $(-t_n, -x_n)$, and then applying Lemma 3.3(ii) (with $-t_n + T(\varepsilon)$ in place of $t_n(\varepsilon)$, using (6.9)), we obtain new $(f, u) \in S_{-\infty, L}(f_0, K, \theta)$ such that $u(0, 0) \in [\varepsilon', 1 - \varepsilon']$ and $L_{u, \varepsilon'/2}(0) \geq \ell_{\varepsilon'} > M_{\varepsilon'/2}$. We also have $f \in F(f_0, K, \theta, \zeta, \eta)$ because that set is closed under locally uniform limits.

Thus $(f, u) \in S_L(f_0, K, \theta)$ since $u \not\equiv 0, 1$ (with S_L defined in Section 3). Then Theorem 2.11(ii) shows $u_t \geq 0$, because bounded width of u and Lemma 3.1 immediately show that u propagates with a positive global mean speed. But then $L_{u, \varepsilon'/2}(0) > M_{\varepsilon'/2}$ yields a contradiction with Corollary 4.4(ii,i) (with $h = 0$). The first claim in (6.8) is proved.

The second claim is proved similarly with $m_{\varepsilon'} := \frac{1}{2}\mu_{\varepsilon', M_{\varepsilon'/2}}$ for $\varepsilon \in (0, 2\varepsilon_0]$, where $\mu_{\varepsilon, \ell} > 0$ is the inf in (3.5) (then it also holds with $m_{\varepsilon'} := m_{2\varepsilon_0}$ for $\varepsilon' \in (2\varepsilon_0, \frac{1}{2})$). Non-existence of $T_{\varepsilon', \theta, \lambda, \tau, \nu}$ again yields a sequence $(f_n, u_n, w_n, t_n, x_n)$ with f_n, u_n, w_n satisfying the hypotheses of the theorem, $\lim_{n \rightarrow \infty} t_n = \infty$, $u_n(t_n, x_n) \in [\varepsilon', 1 - \varepsilon']$ and $(u_n)_t(t_n, x_n) < m_{\varepsilon'}$. We again obtain new $(f, u) \in S_L(f_0, K, \theta)$ such that $f \in F(f_0, K, \theta, \zeta, \eta)$, $u_t \geq 0$, as well as $u(0, 0) \in [\varepsilon', 1 - \varepsilon']$, and $u_t(0, 0) \leq m_{\varepsilon'} < \mu_{\varepsilon', M_{\varepsilon'/2}}$. This contradicts Corollary 4.4(ii,i) (with $h = 0$), and the second claim in (6.8) is also proved. \square

Recall that in the proof of Theorem 2.5(i) we obtained $\Lambda_u^h(t_0 + T) \leq c'_Y T + L_{u, h, \varepsilon'}(t_0)$ for all $h \in (0, \min \{ \theta(c_0^2 - 4\zeta)(c_0^2 + 4\zeta)^{-1}, \frac{\eta}{4K}, \varepsilon_0 \}]$, with $T = 0$ if $d = 3$ and some $T > 0$ if $d \leq 2$. If we thus let $\lambda(h) := c'_Y T + \inf_{h' \in (0, h)} L_{u, h', \varepsilon'}(t_0)$ (which is left-continuous) and $\nu \equiv 0$, then (2.8) in Theorem 2.5(ii) immediately follows from Theorem 6.1 with u, v in place of w, u and time shifted by $-(t_0 + T)$.

Hence we are left with proving (2.8) in Theorem 2.4(i). As in the proof of Remark 4 after Theorem 2.4, we will start with assuming (2.13), and also without loss that $t_0 = 0$, $e = (1, 0, \dots, 0)$, as well as $\varepsilon_2 \leq c_0/4$ (recall that $u \in [0, 1]$). We again let w solve (1.1) with $w(0, x) = W(x_1)$, where W is from (6.3). As before, $w_t \geq 0$ and we have $u(\tau, \cdot) \geq w(0, \cdot)$ provided τ is large enough (depending on $f_0, R_2 - R_1, \varepsilon_1$). This yields the first inequality in (6.7), with $\nu \equiv 0$.

To obtain the second inequality in (6.7), we define $\beta(t) := \tau - e^{-\varepsilon_2^2 t}$ and

$$v(t, x) := w(t + \beta(t), x) + e^{\varepsilon_2^2 t - \varepsilon_2(x_1 - R_2)} \quad (6.11)$$

for some large τ to be determined later. We then have for $t > 0$,

$$v_t - \Delta v - f(x, v) = f(x, w(t + \beta(t), x)) - f(x, v) + \varepsilon_2^2 e^{-\varepsilon_2^2 t} w_t(t + \beta(t), x), \quad (6.12)$$

where we extend f so that $f(x, u) \leq 0$ for $u \geq 1$ (cf. (2.20)).

We want to show that v is a super-solution of (1.1), that is, the right hand side of (6.12) is ≥ 0 for $t > 0$ and $x \in \mathbb{R}^d$. When $w(t + \beta(t), x) \geq 1 - \theta$, then $f(x, w(t + \beta(t), x)) \geq f(x, v(t, x))$ by the hypotheses on f and $w \leq 1$, so this is indeed the case.

Let $\ell_{\theta/4}, m_{\theta/2}$ be from Theorem 6.1 (i.e., with $\varepsilon' := \frac{\theta}{4}$ and $\varepsilon'' := \frac{\theta}{2}$). We now let τ be large enough so that $w(t + \tau - 1, x) \geq 1 - \theta$ whenever $t \geq 0$ and

$$x_1 \leq \frac{c_0}{2}t + \frac{1}{\varepsilon_2} \log \max \left\{ \frac{K}{\varepsilon_2^2 m_{\theta/2}}, \frac{2}{\theta} \right\} + R_2, \quad (6.13)$$

and also that

$$\sup_{t \geq 0} L_{w, \theta/4}(t + \tau - 1) \leq \ell_{\theta/4} \quad \text{and} \quad \inf_{\substack{t \geq 0 \\ w(t + \tau - 1, x) \in [\theta/2, 1 - \theta/2]}} w_t(t + \tau - 1, x) \geq m_{\theta/2}. \quad (6.14)$$

The former holds for all large τ due to the second claim in Lemma 3.1. The latter holds for all large τ due to Theorem 6.1 applied to $u = w$, $\nu \equiv 0$, and $\tau = 0$, but starting from some positive time for which $\Lambda_w^0 (\geq \Lambda_w^h$ for all $h > 0$) is finite (see (6.4)), instead from time 0. This τ then only depends on $f_0, K, \zeta, \eta, \varepsilon_2, \theta$.

When $w(t + \beta(t), x) < 1 - \theta$, then $w(t + \tau - 1, x) < 1 - \theta$ by $w_t \geq 0$, so

$$e^{\varepsilon_2^2 t - \varepsilon_2(x_1 - R_2)} \leq \min \left\{ \frac{\varepsilon_2^2 m_{\theta/2}}{K}, \frac{\theta}{2} \right\} e^{(\varepsilon_2^2 - c_0 \varepsilon_2/2)t} \leq \min \left\{ \frac{\varepsilon_2^2 m_{\theta/2}}{K} e^{(\varepsilon_2^2 - c_0 \varepsilon_2/2)t}, \frac{\theta}{2} \right\}$$

by the opposite inequality to (6.13). So either $w(t + \beta(t), x) \leq \frac{\theta}{2}$, in which case $v(t, x) \leq \theta$ and we have $f(x, w(t + \beta(t), x)) = f(x, v(t, x)) = 0$; or $w(t + \beta(t), x) \in (\frac{\theta}{2}, 1 - \theta)$, in which case the right hand side of (6.12) can be bounded below by

$$-K e^{\varepsilon_2^2 t - \varepsilon_2(x_1 - R_2)} + \varepsilon_2^2 e^{-\varepsilon_2^2 t} m_{\theta/2} \geq -\varepsilon_2^2 m_{\theta/2} e^{(\varepsilon_2^2 - c_0 \varepsilon_2/2)t} + \varepsilon_2^2 m_{\theta/2} e^{-\varepsilon_2^2 t} \geq 0$$

(using $\varepsilon_2 \leq c_0/4$ in the last inequality).

It follows that v is a super-solution of (1.1), with $v(0, \cdot) \geq u(0, \cdot)$ due to (2.13). Hence $v \geq u$, and the second inequality in (6.7) holds with

$$\nu(t) := \max \left\{ \sup_{x_1 \leq R_2 + c_0 t/2} [1 - w(t + \tau - 1, x)], e^{(\varepsilon_2^2 - c_0 \varepsilon_2/2)t} \right\}$$

because $u \leq 1$ and $w_t \geq 0$ (notice that ν depends only on τ, f_0, ε_2). Since $\lim_{t \rightarrow \infty} \nu(t) = 0$ due to Lemma 3.1 and $0 < \varepsilon_2 < c_0/2$, Theorem 2.4(i) for (2.13) follows from Theorem 6.1.

The proof of (2.8) in the (2.12) case of Theorem 2.4(i) is similar, with x_1 replaced by $|x - x_0|$ in the whole argument, and $\varepsilon_2(d-1)|x - x_0|^{-1} e^{\varepsilon_2^2 t - \varepsilon_2(|x - x_0| - R_2)}$ added to the right hand side of (6.12).

Remark. For later reference, in the proof of Theorem 2.9 below, we also construct a sub-solution of (1.1) with the same flavor. Let w be as in the above proof, solving (1.1) with $w(0, x) := W(x_1)$. We have $\inf_{w(0, x) \geq \theta/4} w_t(0, x) > 0$ by the construction of W (because

$U'' + f_0(U) > 0$), uniformly in all $f \geq f_0$. It follows from this and parabolic regularity that on some short time interval $[0, \tilde{t}]$, w_t is bounded away from zero uniformly in all (t, x) with $w(t, x) \geq \frac{\theta}{2}$ and in all $f \geq f_0$ with $f(x, 0) \equiv 0$ and Lipschitz constant K . This, $w_t \geq 0$, and the first claim in (6.14) now yield

$$m := \inf \left\{ w_t(t, x) \mid f \in F(f_0, K, \theta, \zeta, \eta), t \geq 0, \text{ and } w(t, x) \in \left[\frac{\theta}{2}, 1 - \frac{\theta}{2} \right] \right\} > 0, \quad (6.15)$$

provided we also assume (without loss) that $\theta \leq 4\varepsilon_0$. This is because an argument as in Lemma 3.3(iii) shows that otherwise there would be some $f \in F(f_0, K, \theta, \zeta, \eta)$ and a solution u of (1.1) on $(-\tilde{t}, \infty) \times \mathbb{R}^d$ with $\sup_{t \geq \tau} L_{u, \theta/4}(t) \leq \ell_{\theta/4}$ for τ from (6.14), $u(0, 0) \in [\frac{\theta}{2}, 1 - \frac{\theta}{2}]$, and $u_t \equiv 0$. Then Lemma 3.1 and $\frac{\theta}{4} \leq \varepsilon_0$ show $\lim_{t \rightarrow \infty} u(t, 0) = 1$, contradicting $u_t \equiv 0$.

Next, pick any $r \in \mathbb{R}$ and define

$$v(t, x) := w(t - 1 + e^{-\varepsilon_2^2 t}, x) - e^{\varepsilon_2^2 t - \varepsilon_2(x_1 - r)}, \quad (6.16)$$

so that for $t > 0$,

$$v_t - \Delta v - f(x, v) = f(x, w(t - 1 + e^{-\varepsilon_2^2 t}, x)) - f(x, v) - \varepsilon_2^2 e^{-\varepsilon_2^2 t} w_t(t + \beta(t), x). \quad (6.17)$$

Here we extend f so that $f(x, u) \geq 0$ for $u \leq 0$ (cf. (2.20)). We would like to show that the right hand side of (6.17) is ≤ 0 .

This is obviously true when $v(t, x) \geq 1 - \theta$ because then the hypotheses on f and $w \leq 1$ show $f(x, w(t - 1 + e^{-\varepsilon_2^2 t}, x)) \leq f(x, v(t, x))$.

Now consider $(t, x) \in (0, \infty) \times \mathbb{R}^d$ with

$$x_1 \geq \frac{c_0}{2}t + \frac{1}{\varepsilon_2} \log \max \left\{ \frac{K}{\varepsilon_2^2 m}, \frac{2}{\theta} \right\} + r. \quad (6.18)$$

Then

$$e^{\varepsilon_2^2 t - \varepsilon_2(x_1 - r)} \leq \min \left\{ \frac{\varepsilon_2^2 m}{K}, \frac{\theta}{2} \right\} e^{(\varepsilon_2^2 - c_0 \varepsilon_2 / 2)t} \leq \min \left\{ \frac{\varepsilon_2^2 m}{K} e^{(\varepsilon_2^2 - c_0 \varepsilon_2 / 2)t}, \frac{\theta}{2} \right\}.$$

This, $w \in [0, 1]$, and the hypotheses on f show that if $w(t - 1 + e^{-\varepsilon_2^2 t}, x) \notin (\theta, 1 - \frac{\theta}{2})$, then we have $f(x, w(t - 1 + e^{-\varepsilon_2^2 t}, x)) \leq f(x, v(t, x))$, so the right hand side of (6.17) is indeed ≤ 0 . If instead $w(t - 1 + e^{-\varepsilon_2^2 t}, x) \in (\theta, 1 - \frac{\theta}{2})$, then we conclude the same because the right hand side can be bounded above by

$$K e^{\varepsilon_2^2 t - \varepsilon_2(x_1 - r)} - \varepsilon_2^2 e^{-\varepsilon_2^2 t} m \leq \varepsilon_2^2 m e^{(\varepsilon_2^2 - c_0 \varepsilon_2 / 2)t} - \varepsilon_2^2 m e^{-\varepsilon_2^2 t} \leq 0.$$

We cannot, however, conclude this when the opposite of (6.18) holds and $v(t, x) < 1 - \theta$. Thus we have obtained that v is a sub-solution of (1.1) on the set of $(t, x) \in (0, \infty) \times \mathbb{R}^d$ such that either (6.18) holds or $v(t, x) \geq 1 - \theta$. This will turn out to be sufficient for our purposes because typical solutions u spread with speed $> c_0/2$. Hence for appropriate u we will have $u(t, x) \geq 1 - \theta$ when the opposite of (6.18) holds, and we will still be able to conclude $u \geq v$.

7. PROOF OF LEMMA 4.2 IN THE CASE $d = 3$ (CASE $u^+ \equiv 1$)

Recall the setup from the beginning of Section 4. In particular, all constants depend on f_0, K, ζ, η (but not on θ, h unless explicitly noted). Let us also assume, without loss, that $t_1 = 0$ and $y = 0$, and denote $Y_0^h = Y$, $Z_0 = Z$. Thus (4.12) becomes $Z(\tau) \leq Y(0)$. Finally, recall that $\alpha_f(x) = \alpha_f(x; \zeta)$ and ψ corresponds to ζ' in Lemma 3.5.

Let $\kappa \in (0, \frac{1}{2})$ be such that if $u(t, \tilde{x}) \in [\alpha_f(\tilde{x}), 1 - \varepsilon_0]$ for some $(t, \tilde{x}) \in [\frac{1}{2}, \infty) \times \mathbb{R}^3$, then

$$u(t, x) \geq \frac{\eta}{2K} \quad \text{and} \quad f(x, u(t, x)) \geq \kappa \quad \text{for any } x \in B_{\sqrt{3}\kappa}(\tilde{x}). \quad (7.1)$$

Note that κ exists and is independent of f, u due to $\inf_{x \in \mathbb{R}^3} \alpha_f(x) \geq \eta/K$, parabolic regularity, and $f \in F(f_0, K, 0, \zeta, \eta)$. Let also $Q \geq K$ be such that if $\mathcal{C} := [0, \kappa]^3$ and $\tilde{w} \geq 0$ solves

$$\tilde{w}_t = \Delta \tilde{w} + [\zeta' + Q\chi_{[1/2, 1]}(t)\chi_{\mathcal{C}}(x)] \tilde{w}$$

on $(\frac{1}{2}, \infty) \times \mathbb{R}^3$ with $\tilde{w}(\frac{1}{2}, \cdot) \geq \frac{\eta}{4K}\chi_{\mathcal{C}}(\cdot)$, then $\tilde{w}(t, \cdot) \geq \chi_{\mathcal{C}}(\cdot)$ for any $t \geq 1$ (which exists because $\zeta' > 0$).

Assume now that $v \in [0, 1]$ solves (3.6), and that $\mathcal{C}'_1, \mathcal{C}'_2, \dots$ are all (finitely or infinitely many) disjoint cubes such that \mathcal{C}'_n is a $\kappa\mathbb{Z}^d$ -translation of \mathcal{C} (i.e., by an integer multiple of κ in each coordinate) and $v(x'_n) \in (\alpha_f(x'_n), 1 - \varepsilon_0]$ for some $x'_n \in \mathcal{C}'_n$. Since (7.1) applies to v in place of $u(t, \cdot)$, its second claim and (3.7) show for each $x_0 \in \mathbb{R}^d$,

$$\sum_{n \geq 1} (1 + |x'_n - x_0|)^{-1} \leq \kappa^{-4}. \quad (7.2)$$

Let $T = T_Y > 0$, $R \geq T$, and $\tau = \tau_Y \geq T + 1$, all to be chosen later (but independent of θ, h, f, u). Also let $\mathcal{C}_1, \dots, \mathcal{C}_N$ be as above but such that $u(T, x) > \alpha_f(x; \zeta')$ for some $x \in \mathcal{C}_n \cap B_{Y(0)}(0)$. Let $t_n \in [0, T)$ be the last time such that $u(t, x) \leq \alpha_f(x; \zeta')$ for all $(t, x) \in [0, t_n] \times [\mathcal{C}_n \cap B_{Y(0)}(0)]$, let $I_n := [t_n, t_n + 1]$, and let $x_n \in \mathcal{C}_n \cap B_{Y(0)}(0)$ be any point such that $u(T, x_n) \geq \alpha_f(x_n; \zeta')$ ((t_n, x_n) will be fixed from now on). Then $u_t \geq 0$ and $Z(\tau) \leq Y(0)$ show that

$$u(t, x_n) \in [\alpha_f(x_n; \zeta'), 1 - \varepsilon_0] \quad \text{for } n = 1, \dots, N \text{ and } t \in [T, \tau]. \quad (7.3)$$

We now claim that if τ is large enough (depending only on T, R in addition to f_0, K, ζ, η), then we must have

$$\sum_{|x_n - x_0| \leq 2R+2} (1 + |x_n - x_0|)^{-1} < 2\kappa^{-4} \quad (7.4)$$

for each $x_0 \in \mathbb{R}^3$. This holds due to the same argument as in the case $d \leq 2$ of this lemma. Indeed, if such τ did not exist, we take a sequence of counter-examples $(f_\tau, u_\tau, x_0^\tau)$ to (7.4) for $\tau = T + 1, T + 2, \dots$ and shift each in space by (the negative of) the vector whose each coordinate is the largest multiple of κ smaller than the same coordinate of x_0^τ . Parabolic regularity then shows that there is a subsequence along which these shifted solutions converge locally uniformly to a solution of (1.1) (with some $f \in F(f_0, K, 0, \zeta, \eta)$), whose $t \rightarrow \infty$ limit $v \in [0, 1]$ satisfies (3.6). Moreover, by taking a further subsequence (along which those shifted $\mathcal{C}_1^\tau, \dots, \mathcal{C}_{N_\tau}^\tau$ for which $|x_n^\tau - x_0^\tau| \leq 2R + 2$ are all the same, and the corresponding shifted x_n^τ and as well as the shifted x_0^τ converge), one obtains existence of

$x'_0 \in \bar{\mathcal{C}}$ and of $\kappa\mathbb{Z}^d$ -translations $\mathcal{C}'_1, \dots, \mathcal{C}'_{N'} \subseteq B_{2R+4}(0)$ of \mathcal{C} and $x'_n \in \bar{\mathcal{C}}'_n \cap B_{2R+3}(0)$, such that $v(x'_n) \in (\alpha_f(x'_n), 1 - \varepsilon_0]$ (because (7.3) holds for each $(u_\tau, f_\tau, x_n^\tau)$, and $f_\tau(x_n^\tau, \alpha_{f_\tau}(x_n^\tau; \zeta')) = \zeta' \alpha_{f_\tau}(x_n^\tau; \zeta') \geq \zeta \alpha_{f_\tau}(x_n^\tau; \zeta') + (\zeta' - \zeta) \frac{\eta}{K}$) and

$$\sum_{n=1}^{N'} (1 + |x'_n - x'_0|)^{-1} \geq 2\kappa^{-4}.$$

This obviously contradicts (7.2), so there must be $\tau \geq T + 1$ such that (7.4) holds.

We now reorder the $(\mathcal{C}_n, t_n, x_n)$ so that $t_1 \leq \dots \leq t_N$. Define

$$A(t, x) := Q \sum_{n=1}^N \chi_{I_n}(t) \chi_{\mathcal{C}_n}(x)$$

and let w solve

$$w_t = \Delta w + [\zeta' + A(t, x)](w - h) \quad (7.5)$$

on $(0, \infty) \times \mathbb{R}^3$, with $w(0, x) = h + \psi(Y(0))^{-1} \psi(|x|)$ (so that $w(0, \cdot) \geq u(0, \cdot)$ by the definition of $Y(0)$). We will now show that $w \geq u$ on $[0, T] \times \mathbb{R}^3$.

Since the time-independent function $h + \psi(Y(0))^{-1} \psi(|x|)$ is a sub-solution of (7.5), we have $w(t, x) \geq 1 \geq u(t, x)$ for $(t, x) \in [0, T] \times (\mathbb{R}^3 \setminus B_{Y(0)}(0))$. Also, $w \geq h$ and (4.2) show

$$[\zeta' + A(t, x)](w(t, x) - h) \geq f(x, w(t, x))$$

for $(t, x) \in [0, T] \times (B_{Y(0)}(0) \setminus \bigcup_{n=1}^N \mathcal{C}_n)$, as well as for any n and any $(t, x) \in [0, t_n] \times \mathcal{C}_n$. The same is true for $(t, x) \in I_n \times \mathcal{C}_n$ because $f \in F(f_0, K, \theta, \zeta, \eta)$, $h \leq \theta$, and $Q \geq K$. Since $w(0, \cdot) \geq u(0, \cdot)$ and $u \leq 1$ solves (1.1), the comparison principle yields $w \geq u$ on $[0, t_1 + 1] \times \mathbb{R}^3$.

From $Z(\tau) \leq Y(0)$, the definition of $t_1 \in [0, \tau - 1]$, and the first claim in (7.1) we have $u(t_1 + \frac{1}{2}, \cdot) \geq \frac{\eta}{2K} \chi_{\mathcal{C}_1}(\cdot)$. Since $w(t_1 + \frac{1}{2}, \cdot) \geq u(t_1 + \frac{1}{2}, \cdot)$ and $h \leq \frac{\eta}{4K}$, the function $\tilde{w}(t, x) := w(t - t_1, x) - h$ satisfies $\tilde{w}(\frac{1}{2}, \cdot) \geq \frac{\eta}{4K} \chi_{\mathcal{C}_1}(\cdot)$. Our choice of Q then shows $w(t, x) \geq 1$ ($\geq u(t, x)$) for $(t, x) \in [t_1 + 1, T] \times \mathcal{C}_1$, so the comparison principle now yields $w \geq u$ on $[0, t_2 + 1] \times \mathbb{R}^3$.

Using the argument from the previous paragraph $n - 1$ more times (with t_2, \dots, t_n in place of t_1) ultimately indeed gives $w \geq u$ on $[0, T] \times \mathbb{R}^3$. It therefore suffices to show

$$w(T, \cdot) - h \leq \psi(Y(0) - c_Y T)^{-1} \psi(|\cdot|) \quad (7.6)$$

to conclude the proof. This will be achieved by using (7.4), for appropriately chosen T, R .

Let

$$a(t, x) := e^{-2\zeta' t} \psi(Y(0)) \psi(|x|)^{-1} (w(t, x) - h),$$

so that we have $a(0, x) \equiv 1$ and

$$a_t = \Delta a + \frac{2x\psi'(|x|)}{|x|\psi(|x|)} \cdot \nabla a + A(t, x)a.$$

Thus (7.6) will follow if we prove

$$\|a(T, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{e^{-2\zeta' T} \psi(Y(0))}{\psi(Y(0) - c_Y T)} \quad (7.7)$$

Let

$$\delta := 2\zeta' \frac{c_Y - 2\sqrt{\zeta'}}{c_Y + 2\sqrt{\zeta'}} > 0.$$

Since $\frac{d}{dr}[\ln \psi(r)] \geq 4\zeta'(c_Y + 2\sqrt{\zeta'})^{-1} = (2\zeta' + \delta)c_Y^{-1}$ for $r \geq r_Y$ and we assume $Y(0) - c_Y T \geq r_Y$, it follows that $\psi(Y(0))\psi(Y(0) - c_Y T)^{-1} \geq e^{(2\zeta' + \delta)T}$. Hence it suffices to prove

$$\|a(T, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq e^{\delta T}. \quad (7.8)$$

We now choose $R \geq T$ to be such that if B_t is the standard Brownian motion in \mathbb{R}^3 (defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$), then

$$\mathbb{P}\left(\sup_{t \in [0, T]} |B_t| \geq R - 2\|\psi'\psi^{-1}\|_\infty T\right) \leq \frac{1}{3}e^{-QT} \quad (7.9)$$

(so R depends on T in addition to f_0, K, ζ, η). For any $|x| \leq Y(0)$, we let X_t^x be the stochastic process given by $X_0^x = x$ and

$$dX_t^x = b(X_t^x)dt + dB_t := \frac{2X_t^x \psi'(|X_t^x|)}{|X_t^x| \psi(|X_t^x|)} dt + dB_t.$$

Then the well-known Feynman-Kac formula gives

$$a(T, x) = \mathbb{E}\left(e^{\int_0^T A(T-t, X_t^x) dt}\right) \quad (7.10)$$

We will now show that this is $\leq e^{\delta T}$ for $x = 0$ (the general case is identical). Denote $X_t^0 = X_t$ and note that $|b| \leq 2\|\psi'\psi^{-1}\|_\infty$ yields

$$\mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq R\right) \leq \mathbb{P}\left(\sup_{t \in [0, T]} |B_t| \geq R - 2\|\psi'\psi^{-1}\|_\infty T\right) \leq \frac{1}{3}e^{-QT}. \quad (7.11)$$

Since $A \leq Q$, this means that the contribution to (7.10) from those paths which leave $B_R(x=0)$ before time T is at most $\frac{1}{3}e^{-QT}e^{QT} = \frac{1}{3}$.

Next reorder again the $(\mathcal{C}_n, t_n, x_n)$, now so that $\mathcal{C}_n \cap B_R(0) \neq \emptyset$ precisely when $n \leq N'$ (for some N'). Since (7.4) holds for any $x_0 \in \mathbb{R}^3$, we have

$$\sum_{n=1}^{N'} (1 + |x_n - x_0|)^{-1} < 2\kappa^{-4} \quad (7.12)$$

for any $x_0 \in B_{R+1}(0)$. Then $x_n \in B_{R+1}(0)$ implies that (7.12) holds for any $x_0 \in \mathbb{R}^3$.

Consider now the paths which stay in $B_R(0)$ until time T . These have non-zero $A(T-t, X_t)$ only at those times $t \in [0, T]$ for which $X_t \in \mathcal{C}_n$ for some $n \leq N'$, and also $T-t \in I_n$. Since $|I_n| = 1$, the contribution to (7.10) from those of these paths which hit fewer than $\delta(2Q)^{-1}T$ of the cubes $\mathcal{C}_1, \dots, \mathcal{C}_{N'}$ before time T is at most $\exp(Q\delta(2Q)^{-1}T) \leq \frac{1}{3}e^{\delta T}$, provided we choose $T \geq \delta^{-1} \ln 9$.

Finally, the contribution to (7.10) from those paths which stay in $B_R(0)$ until time T and hit at least $\delta(2Q)^{-1}T$ of the cubes $\mathcal{C}_1, \dots, \mathcal{C}_{N'}$ before time T is at most $e^{-2QT}e^{QT} \leq \frac{1}{3}$ by

$A \leq Q$ and Lemma 7.1 below, provided we let $p := \delta(2Q)^{-1}$, $P := 2 \max\{\kappa^{-4}, Q, \|\psi'/\psi\|_\infty\}$, and choose T large enough (which then depends on f_0, K, ζ, η).

Thus $a(T, x = 0) \leq e^{\delta T}$, and the general case $|x| \leq Y(0)$ is identical. Hence (7.8) follows for any such T and the proof will be finished once we prove the following lemma.

Lemma 7.1. *If $p, P > 0$, then for any large enough $T > 0$ (depending only on d, p, P) the following holds. If $N \leq \infty$, the points $x_n \in \mathbb{R}^d$ satisfy*

$$\sum_{n=1}^N (1 + |x_n - x|)^{-1} \leq P \quad (7.13)$$

for any $x \in \mathbb{R}^d$, and the process X_t satisfies $X_0 = 0$ and $dX_t = b(X_t)dt + dB_t$ with $\|b\|_\infty \leq P$ and B_t the standard Brownian motion in \mathbb{R}^d , then

$$\mathbb{P}(X_t \text{ hits at least } pT \text{ of the balls } B_1(x_n) \text{ before time } T) \leq e^{-PT}. \quad (7.14)$$

Remark. The point here is that if X_t hits at least pT of these balls, the sum of the $[pT]$ displacements it undergoes in-between hits will be bounded below by a quantity super-linear in T because of (7.13). The same will then hold for B_t because b is bounded, but the probability of this decreases super-exponentially in T due to the nature of the Gaussian.

Proof. Define the stopping times $t_0 := 0$,

$$t'_j := \inf\{t \geq 0 \mid X_s \text{ hits at least } j \text{ of the balls } B_1(x_n) \text{ before time } t\},$$

and $t_j := \min\{t'_j, T\}$ for $j = 1, \dots, [pT]$. Let $h_j := \sum_{k=1}^j |X_{t_k} - X_{t_{k-1}}|$ and let

$$j_t := \max\{j \leq [pT] \mid t_j < t\}$$

be the smaller of $[pT]$ and the number of the balls hit by X_s before time $t \in (0, T]$ (if $t > T$, then $j_t = [pT]$). Of course, these are all measurable functions of $\omega \in \Omega$. Finally, let $\Omega' := \{\omega \in \Omega \mid j_T(\omega) = [pT]\}$ be the set of those ω for which at least $[pT]$ balls are hit by $X_s(\omega)$ before time T . Thus we need to show $\mathbb{P}(\Omega') \leq e^{-PT}$.

We now claim that there is $\gamma(T) \rightarrow \infty$ as $T \rightarrow \infty$ (also depending on p, P but nothing else) such that (cf. the Remark above)

$$h_{[pT]} \geq \gamma(T)T \quad \text{for any } \omega \in \Omega'. \quad (7.15)$$

Indeed, let $\omega \in \Omega'$ and $H = H(\omega) := h_{[pT]}(\omega)$. For any $x \in \mathbb{R}^d$ we have by (7.13),

$$\sum_{j=1}^{[pT]} (2 + |X_{t_j} - x|)^{-1} \leq P.$$

If we take $x := rX_{t_k} + (1-r)X_{t_{k-1}}$ for some $k = 1, \dots, [pT]$ and $r \in [0, 1)$, then this gives

$$\sum_{j=1}^{[pT]} (2 + |rh_k + (1-r)h_{k-1} - h_j|)^{-1} \leq P. \quad (7.16)$$

For each $q \in [0, H)$ we let $(r_q, k_q) \in [0, 1) \times \{1, \dots, \lceil pT \rceil\}$ be the unique couple such that $q = r_q h_{k_q} + (1 - r_q) h_{k_q - 1}$. Integrating (7.16) over q with $(r, k) = (r_q, k_q)$ yields

$$\sum_{j=1}^{\lceil pT \rceil} \int_0^H (2 + |q - h_j|)^{-1} dq \leq PH.$$

Since $h_j \in [0, H]$, we obtain

$$\lceil pT \rceil \ln \frac{2 + H}{2} \leq PH,$$

yielding (7.15) with $\gamma(T) := \frac{p}{2} \ln T$ for $T \geq e^{2P/p}$. Then $|b| \leq P$ implies also

$$\sum_{k=1}^{\lceil pT \rceil} |B_{t_k} - B_{t_{k-1}}| \geq (\gamma(T) - P)T \quad \text{for any } \omega \in \Omega'. \quad (7.17)$$

Let now $\{e^{(1)}, \dots, e^{(d)}\}$ be the standard basis in \mathbb{R}^d and let

$$E := \{e \in \mathbb{R}^d \mid e \cdot e^{(l)} \in \{1, -1\} \text{ for each } l = 1, \dots, d\}$$

be the set of the 2^d reflections of $(1, \dots, 1)$ across subspaces generated by all 2^d subsets of the standard basis. Notice that E is a group if endowed with coordinate-wise multiplication. For any $\hat{e} = (e_0, \dots, e_{\lceil pT \rceil}) \in E^{\lceil pT \rceil + 1}$, define

$$B_t^{\hat{e}} := \sum_{j=1}^{j_t} (B_{t_j} - B_{t_{j-1}}) \prod_{k=0}^{j-1} e_k + (B_t - B_{t_{j_t}}) \prod_{k=0}^{j_t} e_k,$$

with all multiplications coordinate-wise. That is, $B_t^{\hat{e}}$ is obtained from B_t after $\lceil pT \rceil + 1$ reflections corresponding to $e_0, \dots, e_{\lceil pT \rceil}$ at stopping times $t_0 = 0, t_1, \dots, t_{\lceil pT \rceil}$. (Note that what gets reflected according to e_j is the displacement $B_t - B_{t_j}$ for any $t > t_j$. So in particular, $B_t - B_{t_{j_t}}$ gets reflected $j_t + 1$ times — according to e_0, e_1, \dots, e_{j_t} .)

Since t_j are stopping times, each $B_t^{\hat{e}}$ is also a standard Brownian motion. For any $\omega \in \Omega$, there is $\hat{e} \in E^{\lceil pT \rceil + 1}$ such that for $j = 1, \dots, \lceil pT \rceil$ (and with \cdot the usual dot product in \mathbb{R}^d),

$$\left[(B_{t_j} - B_{t_{j-1}}) \prod_{k=0}^{j-1} e_k \right] \cdot (1, \dots, 1) \geq |B_{t_j} - B_{t_{j-1}}|.$$

Indeed, one only needs to choose e_j successively so that $(B_{t_j} - B_{t_{j-1}}) \prod_{k=0}^{j-1} e_k$ has all d coordinates non-negative. So by (7.17), for each $\omega \in \Omega'$, there is $\hat{e} \in E^{\lceil pT \rceil + 1}$ such that

$$B_{t_{\lceil pT \rceil}}^{\hat{e}} \cdot (1, \dots, 1) \geq (\gamma(T) - P)T$$

Since $t_{\lceil pT \rceil} \leq T$, we obtain

$$\mathbb{P}(\Omega') \leq 2^{d(\lceil pT \rceil + 1)} \mathbb{P}(B_t \cdot (1, \dots, 1) \geq (\gamma(T) - P)T \text{ for some } t \leq T).$$

Given any $C > 0$, the last probability is less than e^{-CT} for all large enough T because $\lim_{T \rightarrow \infty} \gamma(T) = \infty$. Taking $C := 2dp \ln 2 + P$ yields $\mathbb{P}(\Omega') \leq e^{-PT}$ for all large enough T , finishing the proof of Lemma 7.1. \square

8. PROOFS OF THEOREMS 2.7 AND 2.9

These proofs follow the same lines as those of Theorems 2.4(i) and 2.5. The only differences in the proof of Theorem 2.7 will be in the proofs of Lemma 3.1 and Lemma 4.2, where (2.23) will play a central role. In what follows, let us consider the setting of Theorem 2.7 (in particular, \mathcal{F} is fixed) but for now also only $(f, u^+) \in \mathcal{F}$ and $\theta \geq 0$. All constants will now depend on \mathcal{F} instead of f_0, K, ζ, η (but not on θ from (H) unless explicitly noted).

Before we start, let us note that since $\inf_{x \in \mathbb{R}^d} u^+(x) > \theta_0$ and $f(x, u) \geq f_0(u) > 0$ for $u \in (\theta_0, \theta_1)$, it follows from elliptic regularity and the maximum principle that in fact $\inf_{x \in \mathbb{R}^d} u^+(x) \geq \theta_1$.

Lemma 8.1. *There is $\varepsilon_0 = \varepsilon_0(\mathcal{F}) > 0$ such that for each $c < c_0$ and $\varepsilon > 0$ there is $\tau = \tau(\mathcal{F}, c, \varepsilon)$ such that the following holds. If $u \in [0, u^+]$ solves (1.1), (1.2) with $(f, u^+) \in \mathcal{F}$, and $u(t_1, x) \geq u^+(x) - \varepsilon_0$ for some $(t_1, x) \in [t_0 + 1, \infty) \times \mathbb{R}^d$, then for each $t \geq t_1 + \tau$,*

$$\sup_{|y-x| \leq c(t-t_1)} [u^+(x) - u(t, y)] \leq \varepsilon. \quad (8.1)$$

The same result holds if the hypothesis $u(t_1, x) \geq u^+(x) - \varepsilon_0$ is replaced by

$$u(t_1, \cdot) \geq \frac{\theta_1 + \theta_0}{2} \chi_{B_R(x)}(\cdot) \quad (8.2)$$

for some $(t_1, x) \in [t_0, \infty) \times \mathbb{R}^d$ and a large enough $R = R(f_0) > 0$.

Proof. Without loss we can assume $t_0 = 0$ and $x = 0$. As in the argument in the proof of Lemma 3.1, one shows that $u(t_1, x) \geq u^+(x) - \varepsilon_0$ (with $t_1 \geq 1$ and $u^+ \geq \theta_1$) yields (8.2), provided $\varepsilon_0 > 0$ is sufficiently small. So we only need to prove the second claim.

Let us therefore assume (8.2). The result from [2] used in Lemma 3.1 also applies to bistable f_0 , and together with the comparison principle gives for any $c' < c_0$ and $\varepsilon' > 0$ existence of τ' such that,

$$\inf_{|y| \leq c't} u(t, y) \geq \theta_1 - \varepsilon' \quad (8.3)$$

when $t \geq \tau'$. To upgrade this to (8.1), we will use (2.23) along with $\sup_{x \in \mathbb{R}^d} \alpha_f(x) < \theta_1$. The latter holds because otherwise $f_0(u) \leq \zeta u$ for all $u \in [0, \theta_1]$, which contradicts $c_0 > 2\sqrt{\zeta}$.

If (8.1) does not hold for some $c < c_0$ and $\varepsilon > 0$, we let u_n be a counterexample with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and $|y_n| \leq ct_n$, corresponding to some $(f_n, u_n^+) \in \mathcal{F}$. We can assume $(u_n)_t \geq 0$, because (8.3) with $c' \in (c, c_0)$ and a small enough $\varepsilon' \in (0, \varepsilon)$ lets us find a time-increasing solution w_n of (1.1) between 0 and u_n , defined for $t \geq t'$ with some $t' \geq \tau'$, which still spreads with speed $\geq c'$ in the sense of (8.3). Indeed, similarly to (6.3), we let $w_n(t', x) := W(|x|)$, where

$$W(s) := \begin{cases} \theta_1 - \varepsilon' & s \leq R', \\ U(s - R') & s \in (R', R' + s_0], \\ 0 & s > R' + s_0 \end{cases} \quad (8.4)$$

and U, s_0 are obtained as for (6.3) but with the current f_0 and $U(0) = \theta_1 - \varepsilon'$. Here we need $\varepsilon' > 0$ to be small enough (such that $\int_0^{\theta_1 - \varepsilon'} f_0(u) du > 0$) and R' larger than the right-hand side of (6.6) (so that $W(|x|)$ satisfies (6.1)). So each w_n is time-increasing, and also satisfies (8.3) for large t if $\varepsilon' \leq \frac{1}{2}(\theta_1 - \theta_0)$ and $R' \geq R$, with R from (8.2). Then we only need $t' \geq \max\{\tau', (R' + s_0)/c'\}$ to get $w_n \leq u_n$ for all n (note that $\varepsilon', c', \tau', U, s_0, R', t'$ are independent of n).

Let therefore $(u_n)_t \geq 0$ be such counterexamples, with $t_n \rightarrow \infty$ and $|y_n| \leq ct_n$ such that $u_n(t_n, y_n) \leq u_n^+(y_n) - \varepsilon$. After shifting u_n by $(-\frac{c'+c}{2c'}t_n, -y_n)$ (and f_n, u_n^+ by y_n) and passing to a subsequence, we recover an entire time-increasing solution u of (1.1) with new $(f, u^+) \in \mathcal{F}$, such that $u \in [\theta_1, u^+]$ (due to (8.3) for each u_n and all $\varepsilon' > 0$) and $\lim_{t \rightarrow \infty} u(t, 0) \leq u^+(0) - \varepsilon$. As before, the function $p(x) := \lim_{t \rightarrow \infty} u(t, x)$ is an equilibrium solution of (1.1) with $p(0) \leq u^+(0) - \varepsilon$. Since $p \in [\theta_1, u^+]$, the strong maximum principle yields $p < u^+$, and we also have $p \geq \theta_1 > \alpha_f$ on \mathbb{R}^d . So the sum in (2.23) equals ∞ , contradicting $(f, u^+) \in \mathcal{F}$. \square

Proof of Theorem 2.7. This is essentially identical to the proofs of Theorem 2.4(i) and Theorem 2.5 for $d = 3$, but with $u \in [0, u^+]$ instead of $u \in [0, 1]$ and using Lemma 8.1 instead of Lemma 3.1. We will again only assume $(f, u^+) \in \mathcal{F}$ and $\theta \geq 0$ in most of the proof.

Lemmas 3.2 and 3.3 are unchanged, with the sets $S_{t_0, \varepsilon, \ell}, S_{t_0, L}, S_L$ containing triples (f, u^+, u) and restricted to $(f, u^+) \in \mathcal{F}$ and $u \in [0, u^+]$, and with $1 - \varepsilon$ replaced by $u^+(x) - \varepsilon$ in (3.5). In Section 4 we take

$$Z_y(t) := \inf_{u(t, x) \geq u^+(x) - \varepsilon_0} |x - y|$$

and keep $Y_y^h(t)$ as before because $u^+ \leq 1$. Lemma 4.1 is unchanged and Lemma 4.2 is proved as in the case $d = 3$. We cannot use Lemma 3.4 here but (2.23) will do the job. Indeed, we let $\kappa \in (0, d^{-1/2})$ be such that if $u(t, \tilde{x}) \geq \eta$ for some $(t, \tilde{x}) \in [\frac{1}{2}, \infty) \times \mathbb{R}^d$, then

$$u(t, x) \geq \frac{\eta}{2K} \quad \text{for any } x \in B_{\sqrt{d}\kappa}(\tilde{x}), \quad (8.5)$$

which replaces (7.1). We then still conclude (7.2) using (2.23), although with the right hand side being $[\kappa^{-1}]^d \eta^{-1}$ instead of κ^{-4} . The rest of the proof is unchanged, as is Theorem 4.3 and Corollary 4.4, except for $1 - \varepsilon$ being replaced by $u^+(x) - \varepsilon$ in (4.21). Section 5 is also unchanged, using only the arguments in Case $d = 3$. This proves Theorem 2.7(ii) for u .

The proofs of the remarks at the beginning of Section 6 remain the same, with W from (8.4) instead of (6.3) and $R' := R_2$. Theorem 6.1 is also unchanged (note that here we need $\theta > 0$ because we employ Theorem 2.11(ii)) and so is the proof of Theorem 2.5(ii). This proves Theorem 2.7(ii) for v .

Finally, since we have (2.20), the argument after Theorem 6.1 which proves Theorem 2.4(i) also remains the same, with each “1−” is replaced by “ $u^+(x)−$ ”. \square

Proof of Theorem 2.9. Let us define $f_0(u) = 0$ for $u < 0$, and for $\gamma \leq 0$ let c_γ be the front/spreading speed for f_0 but corresponding to fronts connecting γ and θ_1 resp. to sufficiently large $u_0 \in [\gamma, \theta_1]$ converging to γ as $|x| \rightarrow \infty$. It is well known (using phase-plane analysis) that $c_\gamma \in (0, c_0)$ for any $\gamma < 0$ as well as $\lim_{\gamma \rightarrow 0^-} c_\gamma = c_0$.

(i) To prove this we will need to construct an appropriate (non-positive) sub-solution, in addition to the super-solution constructed previously. We will use here the remark at the end of Section 6. Let us first assume (2.26), and let $\gamma := \inf_{x \in \mathbb{R}^d} u_0(x) \leq 0$ (then $\gamma = \inf_{(t,x) \in [t_0, \infty) \mathbb{R}^d} u(t, x)$ by (2.20)). Without loss, we also assume $t_0 = 0$, $e = e_1$, and $\varepsilon_2 \leq c_\gamma/4$. The latter can be done because (2.26) continues to hold if we replace ε_2 by $\min\{\varepsilon_2, c_\gamma/4\}$ and R_2 by $R_2 + 4c_\gamma^{-1} \ln_+ \|u_0\|_\infty$.

First, we claim that

$$\limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^d} [u(t, x) - u^+(x)] \leq 0 \quad \text{and} \quad \liminf_{t \rightarrow \infty} \inf_{x \in \mathbb{R}^d} u(t, x) \geq 0, \quad (8.6)$$

where the rate of these decays depends on the same parameters as T_ε in (C) does, except of ε (by ‘‘rate’’ we mean a function $\tilde{T} : (0, \infty) \rightarrow (0, \infty)$ such that $\sup_{x \in \mathbb{R}^d} [u(t, x) - u^+(x)] \leq \delta$ and $\inf_{x \in \mathbb{R}^d} u(t, x) \geq -\delta$ for all $t \geq \tilde{T}(\delta)$).

For the first claim in (8.6), let $v(t, x) := u(t, x) - u^+(x)$. Then $v_t \leq \Delta v + g(v)$, with

$$g(s) := \sup_{(f, u^+, x) \in \mathcal{F} \times \mathbb{R}^d} [f(x, u^+(x) + s) - f(x, u^+(x))].$$

We have $g(s) < 0$ for all $s > 0$ because otherwise translation invariance of \mathcal{F} and its closure under locally uniform convergence would yield $(f, u^+) \in \mathcal{F}$ with $f(x, u^+(x) + s) = f(x, u^+(x))$ for some $s > 0$, contradicting the extra hypothesis in the case of (2.26). Obviously $\sup_{x \in \mathbb{R}^d} v(t, x) \leq \kappa(t)$, where $\kappa(0) := \sup_{x \in \mathbb{R}^d} v(0, x) < \infty$ and $\kappa' = g(\kappa)$. Thus $\lim_{t \rightarrow \infty} \kappa(t) = 0$, and the first claim in (8.6) follows. (If we only have $g \leq 0$ but assume $\limsup_{x_1 \rightarrow -\infty} v(0, x) \leq 0$, the claim is immediate from this and (2.26).)

We now turn to the second claim in (8.6). The result from [2] (see the proof of Lemma 8.1) for (1.1) with $u_0 \geq (\theta_0 + \varepsilon_1 + |\gamma|)\chi_{\{x | x_1 < R_1\}} - |\gamma|$ shows

$$\inf_{x_1 \leq R_1 + c't} u(t, x) \geq \theta_1 - \varepsilon' \quad (8.7)$$

for any $c' < c_\gamma$, $\varepsilon' > 0$, and $t \geq \tau'$ (with τ' depending only on $f_0, \gamma, \varepsilon_1, c', \varepsilon'$). The comparison principle and (2.20) yield $u(t, x) \geq -e^{-\varepsilon_2(x_1 - \varepsilon_2 t - R_2)}$ because the latter solves the heat equation. But this, (8.7) with $c' := c_\gamma/2$ and $\varepsilon' := \theta_1$, and $\varepsilon_2 \leq c_\gamma/4$ show $\inf_{x \in \mathbb{R}^d} u(t, x) \geq -e^{-\varepsilon_2(c_\gamma t/4 + R_1 - R_2)}$ for $t \geq \tau'$. The second claim in (8.6) follows.

(8.6) shows that for any $\varepsilon > 0$ and large enough t , the sets $\Omega_{u, \varepsilon}(t)$ and $\Omega_{u, 1-\varepsilon}(t)$ from (2.21) and (2.22) are the same as those in Definition 2.8. Hence we will use (2.21) and (2.22).

We next claim that because of (8.6) and the parabolic Harnack inequality, it suffices to prove the result with $L_{u, \varepsilon}(t)$ from (1.6) instead of (2.24). First, there is $(K, \|u\|_\infty)$ -dependent $M \geq 1$ such that $\sup_{t \geq t_0+1} \|u_t\|_\infty \leq M$ for any solution of (1.1) on $(t_0, \infty) \times \mathbb{R}^d$ with f Lipschitz with constant K and satisfying $f(\cdot, 0) \equiv 0$. Given any $\varepsilon \in (0, \frac{1}{2})$, consider some small $\varepsilon' > 0$ and let $v := \varepsilon' + u^+ - u$ and $T' < \infty$ be such that $\inf_{x \in \mathbb{R}^d} v(t, x) \geq 0$ for all $t \geq T'$ (which exists by (8.6)). Then we have $v_t - \Delta v - \lambda(t, x)v \geq 0$ for $t \geq T'$, with

$$\lambda(t, x) := v(t, x)^{-1} \min\{f(x, u^+(x)) - f(x, u(t, x)), 0\}$$

which satisfies $\lambda(t, x) \in [-K, 0]$ due to (2.20). So by the Harnack inequality, there is $C \geq 1$ (depending on ε, K) such that if $v(t, x) \leq 2\varepsilon'$ for $(t, x) \in [T' + 1, \infty) \times \mathbb{R}^d$, then

$\sup_{|y-x| \leq 1/\varepsilon} v(t - \frac{\varepsilon}{2M}, y) \leq C\varepsilon'$. If we now let $\varepsilon' := \frac{\varepsilon}{2C}$, then this means $u(t, y) \geq u^+(y) - \varepsilon$ for all $y \in B_{1/\varepsilon}(x)$ whenever $u(t, x) \geq u^+(x) - \varepsilon'$ and $t \geq T' + 1$. Therefore $\ell_{\varepsilon'}$ for $L_{u, \varepsilon'}(t)$ from (1.6) works as ℓ_{ε} for $L_{u, \varepsilon}(t)$ from (2.24), provided we can obtain (C) for the former.

So we can consider $L_{u, \varepsilon}(t)$ from (1.6), with $\Omega_{u, \varepsilon}(t)$ and $\Omega_{u, 1-\varepsilon}(t)$ from (2.21) and (2.22), which is what was done in the proofs of Theorems 2.4(i) and 2.7(i).

Next, notice that (8.7) and $u \geq \gamma$ can be upgraded to

$$\sup_{x_1 \leq R_1 + ct} [u^+(x) - u(t, x)] \leq \varepsilon \quad (8.8)$$

for any $c < c_\gamma$, $\varepsilon > 0$, and $t \geq \tilde{\tau}$ (with $\tilde{\tau}$ depending only on $\mathcal{F}, c, \varepsilon, \gamma, \varepsilon_1$). Indeed, this is done using (8.7) in the same way (8.3) is used to prove (8.1), but with $U(s_0) = -\gamma$ in the definition of W (we still have $\int_{-\gamma}^{\theta_1 - \varepsilon'} f_0(u) du > 0$). In fact, (8.7) and then (8.8) hold for any $c < c_0$ because of the second claim in (8.6) and $\lim_{\gamma \rightarrow 0^-} c_\gamma = c_0$.

Let us assume, without loss, that $\theta > 0$ is small enough so that $\theta \leq \frac{1}{2}(\theta_1 - \theta_0)$ and $\int_0^{\theta_1 - \theta} f_0(u) du > 0$. We now use (8.8) with $c := c_\gamma/2$, $\varepsilon := \theta$, and the corresponding $\tilde{\tau}$, together with $u(t, x) \geq -e^{-\varepsilon_2(x_1 - \varepsilon_2 t - R_2)}$ (shown above) and $\varepsilon_2 \leq c_\gamma/4$, to obtain for $t \geq \tilde{\tau}$,

$$u(t, x) \geq (u^+(x) - \theta)\chi_{\{x | x_1 \leq R_1 + c_\gamma t/2\}}(x) - e^{-\varepsilon_2(x_1 - R_2 - c_\gamma t/4)}\chi_{\{x | x_1 > R_1 + c_\gamma t/2\}}(x).$$

Of course, (2.26) and $f(u) \leq Ku$ for $u \geq 0$ also give

$$u(t, x) \leq e^{(\varepsilon_2^2 + K)t - \varepsilon_2(x_1 - R_2)}.$$

Now consider W from (8.4), with $\varepsilon' := \theta$ and $R' := R_2$. Consider also w solving (1.1) with $w(0, x) := W(x_1)$. As in the remark at the end of Section 6, we obtain

$$m := \inf \left\{ w_t(t, x) \mid (f, u^+) \in \mathcal{F}_\theta, t \geq 0, \text{ and } w(t, x) \in \left[\frac{\theta}{2}, u^+(x) - \frac{\theta}{2} \right] \right\} > 0. \quad (8.9)$$

With s_0 from (8.4), we let

$$r := R_2 + s_0 - \frac{1}{\varepsilon_2} \log \max \left\{ \frac{K}{\varepsilon_2^2 m}, \frac{2}{\theta} \right\},$$

and shift u by $(-T, -R)$, and f, u^+ by $-R$ in space, where

$$T := \max \left\{ \tilde{\tau}, 4c_\gamma^{-1} (2R_2 - R_1 + s_0 - r) \right\},$$

$$R := \frac{c_\gamma T}{4} + R_2 - r.$$

Since $\frac{1}{2}c_\gamma T \geq R_2 - R_1 + s_0 + R$, the above estimates on the original $u(t, x)$ for $t \geq T$ ($\geq \tilde{\tau}$) now give for the shifted u, u^+ and $t \geq 0$,

$$u(t, x) \geq (u^+(x) - \theta)\chi_{\{x | x_1 \leq R_2 + s_0 + c_\gamma t/2\}}(x) - e^{-\varepsilon_2(x_1 - r - c_\gamma t/4)}\chi_{\{x | x_1 > R_2 + s_0 + c_\gamma t/2\}}(x), \quad (8.10)$$

$$u(0, x) \leq e^{-\varepsilon_2(x_1 - R_2 + R - \varepsilon_2 T - KT/\varepsilon_2)}. \quad (8.11)$$

The crucial point here is that $u(t, x) \geq u^+(x) - \theta$ when

$$x_1 \leq \frac{c_\gamma t}{2} + \frac{1}{\varepsilon_2} \log \max \left\{ \frac{K}{\varepsilon_2^2 m}, \frac{2}{\theta} \right\} + r. \quad (8.12)$$

Hence v from (6.16), which by the argument from the remark at the end of Section 6 is a subsolution of (1.1) on the set of $(t, x) \in (0, \infty) \times \mathbb{R}^d$ such that either the opposite of (8.12) holds or $v(t, x) \geq u^+(x) - \theta$, will stay below u as long as $v(0, \cdot) \leq u(0, \cdot)$. But this holds due to (8.10) for $t = 0$ because $w(0, \cdot) \leq \theta_1 - \theta \leq u^+(\cdot) - \theta$ and $w(0, \cdot)$ vanishes for $x_1 \geq R_2 + s_0$.

Thus (6.16) shows that the first inequality in (6.7) holds with $\tau = 1$ and

$$\nu(t) := \max \left\{ \sup_{x_1 \leq r + c_\gamma t/2} [u^+(x) - u(t, x)], e^{(\varepsilon_2^2 - c_\gamma \varepsilon_2/2)t} \right\}$$

because $w \leq u^+$ and $w_t \geq 0$. Moreover, $\lim_{t \rightarrow \infty} \nu(t) = 0$ due to $0 < \varepsilon_2 < c_\gamma/2$ and (8.8) for some $c > c_\gamma/2$ and any $\varepsilon > 0$.

On the other hand, as in Section 6, we also have a super-solution of (1.1) on $(0, \infty) \times \mathbb{R}^d$ from (6.11), with some large τ and R_2 replaced by $R_2 - R + \varepsilon_2 T + KT/\varepsilon_2$. This then stays above u due to (8.11). As in Section 6, we obtain the second inequality in (6.7) with some $\nu(t) \rightarrow 0$ as $t \rightarrow \infty$. The (H') version of Theorem 6.1 now finishes the proof.

The proof in the case (2.25) is similar, with x_1 replaced by $|x - x_0|$ and sufficiently large R_1 to guarantee (8.7) with x_1 replaced by $|x - x_0|$. Notice that the first claim in (8.6) follows from (2.25), even though now we only have $g \leq 0$.

(ii) Similarly to (i), the extra hypotheses in (ii) imply both claims in (8.6). So again we can consider $L_{u,\varepsilon}(t)$ from (1.6), with $\Omega_{u,\varepsilon}(t)$ and $\Omega_{u,1-\varepsilon}(t)$ from (2.21) and (2.22). Moreover (2.16) shows $u_t \geq 0$ so we must have $u \leq u^+$.

Assume again $t_0 = 0$ without loss and notice that the hypotheses continue to hold if we replace u_0 by $u(t, \cdot)$ for any $t \geq 0$. Indeed, for all small enough $\varepsilon > 0$ and all $y \in \mathbb{R}^d$ we have $Z_y(0) \leq Y_y^{\varepsilon/2}(0) + L_{u,\varepsilon/2,1-\varepsilon_0}(0)$ as in the case $d = 3$ of the proof of Theorem 2.5(i). Since $u_t \geq 0$, the (H') version of Lemma 4.1(i) now gives

$$Z_y(t) \leq Z_y(0) \leq Y_y^{\varepsilon/2}(0) + L_{u,\varepsilon/2,1-\varepsilon_0}(0) \leq Y_y^{\varepsilon/2}(t) + c'_Y t + r_Y + L_{u,\varepsilon/2,1-\varepsilon_0}(0).$$

If $u(t, y) \geq \varepsilon$, then $Y_y^{\varepsilon/2}(t) \leq \psi^{-1}(\frac{\varepsilon}{2})$, so this yields

$$L_{u,\varepsilon,1-\varepsilon_0}(t) \leq \psi^{-1}(\frac{\varepsilon}{2}) + c'_Y t + r_Y + L_{u,\varepsilon/2,1-\varepsilon_0}(0) < \infty.$$

This and (8.6) mean that we can assume without loss that $\gamma := \min\{\inf_{x \in \mathbb{R}^d} u_0(x), 0\} = \min\{\inf_{(t,x) \in [0,\infty) \times \mathbb{R}^d} u(t,x), 0\}$ is such that $c_\gamma > c_Z$ from (4.6). But then Lemma 8.1 shows that the (H') version of Lemma 4.1(iii) continues to hold because now $u \in [\gamma, u^+]$. The rest of the proof of Theorem 2.5 (or rather Theorem 2.7(ii)) then carries over to the case $u \in [\gamma, u^+]$, with c_0 replaced by c_γ and the obvious (minimal) changes (notice also that the second claim in (8.6), which holds for any $(f, u^+) \in \mathcal{F}$ and bounded u_0 satisfying (2.16), precludes existence of equilibrium solutions p of (1.1) with $\gamma < p < u^+$ and $\inf_{x \in \mathbb{R}^d} p(x) < 0$). Since we can take γ arbitrarily close to 0, by replacing u_0 with $u(t, \cdot)$ for a large enough t , we finally also obtain global mean speed in $[c_0, c_1]$. The proof is thus finished. \square

9. PROOF OF THEOREM 2.11

Let $K \geq 1$ be a Lipschitz constant for f and pick a non-increasing and continuous function $L : (0, \frac{1}{2}) \rightarrow (0, \infty)$ such that $(\sup_{t \in \mathbb{R}} L_{u,\varepsilon}(t) =) L^{u,\varepsilon} \leq L(\varepsilon)$ for all $\varepsilon \in (0, \frac{1}{2})$. We can do so because $L^{u,\varepsilon}$ is finite and non-increasing in ε .

(i) Let $m_0 := \inf_{(t,x) \in \mathbb{R}^{d+1}} u(t,x)$ and $m_1 := \sup_{(t,x) \in \mathbb{R}^{d+1}} [u(t,x) - u^+(x)]$. It is easy to see that Lemma 3.3(i,ii) extends to the case when $S_{t_0,\varepsilon,\ell} = S_{t_0,\varepsilon,\ell}(K, m_0, m_1)$ is defined to be the set of all triples (f, u^+, u) such that f is Lipschitz with constant K , the functions $u^- \equiv 0$ and u^+ satisfy (2.18) and are equilibrium solutions of (1.1), and u with $m_0 \leq u \leq u^+ + m_1$ solves (1.1) on $(t_0, \infty) \times \mathbb{R}^d$ and satisfies $L_{u,\varepsilon'}(t) \leq \ell$ for all $\varepsilon' \in (\varepsilon, \frac{1}{2})$ and all $t > t_0$ (with $L_{u,\varepsilon}(t)$ from Definition 2.8).

Assume first that $m_1 > 0$. Take (t_n, x_n) such that $u(t_n, x_n) - u^+(x_n) \rightarrow m_1$ and define $f_n(x, u) := f(x - x_n, u)$, $u_n^+(x) := u^+(x - x_n)$, and $u_n(t, x) := u(t - t_n, x - x_n)$. Since (f_n, u_n^+, u_n) belongs to the corresponding $S_{-\infty,L} = S_{-\infty,L}(K, m_0, m_1)$, we obtain as in the proof of Lemma 3.3 some new $(f, u^+, u) \in S_{-\infty,L}$ (and thus also $u \not\equiv u^+ + m_1$) such that $u \leq u^+ + m_1$ and $u(0, 0) = u^+(0) + m_1$. The function $u^+ + m_1$ is a super-solution of (1.1) due to (2.20), so the strong maximum principle yields a contradiction with $u \not\equiv u^+ + m_1$. Thus $m_1 \leq 0$ and the strong maximum principle also shows $u < u^+$.

The case $m_0 < 0$ is identical, this time using that the constant m_0 is a sub-solution of (1.1) due to (2.20). We obtain $m_0 \geq 0$ and then also $u > 0$.

(ii) By the discussion following Definition 2.8 (see also the proof of Theorem 2.9(i) above), (i) shows that it is equivalent to use (1.6) instead of (2.24) in what follows. We will do so in (ii) and (iii), including in the definition of $S_{t_0,\varepsilon,\ell} = S_{t_0,\varepsilon,\ell}(K, 0, 0)$ from (i) (and thus also in $S_{t_0,L}, S_L$, with the condition $u \not\equiv 0, u^+$ for S_L). In addition, (i) and $u^- \equiv 0$ show that we have (2.21) and (2.22).

Since u propagates with a positive global mean speed, (2.6) shows that u is a transition solution connecting $u^- \equiv 0$ and u^+ . Indeed, $u \not\equiv u^+$ gives $\Omega_{u,1-\varepsilon}(0) \neq \mathbb{R}^d$ for all small $\varepsilon > 0$. Thus the first inclusion in (2.6), with $t \rightarrow -\infty$ and $\tau := -t$, shows the $t \rightarrow -\infty$ limit in (2.27). Also, $u \not\equiv 0$ gives $\Omega_{u,\varepsilon}(0) \neq \emptyset$ for all small $\varepsilon > 0$. Thus the first inclusion in (2.6), with $t = 0$ and $\tau \rightarrow \infty$, shows the $t \rightarrow \infty$ limit in (2.27).

Assume now that $\theta > 0$ is as in (ii), take $\varepsilon_0 := \theta/2 > 0$, and let

$$u^s(t, x) := u(t + s, x)$$

be a time shift of u . It is then sufficient to show that $u^s \geq u$ for any $s \geq 0$. Indeed, we then obtain $u_t \geq 0$, and the strict inequality follows from the strong maximum principle for u_t (which satisfies a linear equation and is not identically 0 because u is a transition solution).

Lemma 9.1. *There is s_0 such that $u^s \geq u$ whenever $s \geq s_0$.*

Proof. Since u propagates with a positive global mean speed (with some $c > 0$ and $\tau_{\varepsilon,\delta} < \infty$ in Definition 2.2), we have $\Omega_{u,\varepsilon_0}(t) \subseteq \Omega_{u,1-\varepsilon_0}(t + s)$ for all $t \in \mathbb{R}$ and $s \geq s_0 := \tau_{\varepsilon_0,c/2}$. Thus for $s \geq s_0$ we have $u^s + \varepsilon_0 \geq u$ as well as

$$u^s(t, x) \geq u(t, x) \text{ whenever } u(t, x) \in [\varepsilon_0, u^+(x) - \varepsilon_0]. \quad (9.1)$$

Next take any $s \geq s_0$ and let $\varepsilon \geq 0$ be the smallest number such that $u^s + \varepsilon \geq u$. Obviously, ε exists and $\varepsilon \leq \varepsilon_0$. We need to show that $\varepsilon = 0$, so assume that $\varepsilon > 0$ and let (t_n, x_n) satisfy

$$\lim_{n \rightarrow \infty} [u^s(t_n, x_n) + \varepsilon - u(t_n, x_n)] = 0.$$

Then (9.1) shows that $u(t_n, x_n) \notin [\varepsilon_0, u^+(x_n) - \varepsilon_0]$ for large enough n , so either $u(t_n, x_n) \in [\varepsilon, \varepsilon_0]$ or $u(t_n, x_n) \in [u^+(x_n) - \varepsilon_0, u^+(x_n)]$. Apply the $u^+ \not\equiv 1$ version of Lemma 3.3(ii) with $t_n(\varepsilon) = -\infty$, $f_n(x, u) := f(x - x_n, u)$, $u_n^+(x) := u^+(x - x_n)$, and $u_n(t, x) := u(t - t_n, x - x_n)$. We obtain $(\tilde{f}, \tilde{u}^+, \tilde{u}) \in S_{-\infty, L}(K, 0, 0)$ such that $\tilde{u} \in [0, \tilde{u}^+]$, $\tilde{u}^s + \varepsilon \geq \tilde{u}$,

$$\tilde{u}^s(t, x) \geq \tilde{u}(t, x) \text{ whenever } \tilde{u}(t, x) \in [\varepsilon_0, \tilde{u}^+(x) - \varepsilon_0] \quad (9.2)$$

by (9.1), and

$$\tilde{u}^s(0, 0) + \varepsilon = \tilde{u}(0, 0) \quad (\in [\varepsilon, \varepsilon_0] \cup [\tilde{u}^+(0) - \varepsilon_0, \tilde{u}^+(0)]).$$

Moreover, \tilde{f} is non-increasing in u on $[0, \theta]$ and on $[\tilde{u}^+(x) - \theta, \tilde{u}^+(x)]$ because f, u^+ have the same property.

Let now $v := \tilde{u}^s + \varepsilon - \tilde{u} \geq 0$, so that $v_t = \Delta v + \tilde{f}(x, \tilde{u}^s) - \tilde{f}(x, \tilde{u})$. We then have

$$v_t \geq \Delta v - K v.$$

Indeed, this obviously holds when $\tilde{u}^s(t, x) \geq \tilde{u}(t, x)$. When $\tilde{u}^s(t, x) < \tilde{u}(t, x)$, then (9.2) and $\varepsilon \leq \varepsilon_0$ show $\tilde{u}^s(t, x), \tilde{u}(t, x) \in [0, \varepsilon_0] \cup [\tilde{u}^+(0) - 2\varepsilon_0, \tilde{u}^+(0)]$, so $\tilde{f}(x, \tilde{u}^s(t, x)) \geq \tilde{f}(x, \tilde{u}(t, x))$ by $2\varepsilon_0 = \theta$. Now we obtain $v \equiv 0$ on $(-\infty, 0] \times \mathbb{R}$ by $v \geq 0$, $v(0, 0) = 0$, and the strong maximum principle. But then $\tilde{u}^{-sn} \equiv \tilde{u} + n\varepsilon$ on $(-\infty, 0] \times \mathbb{R}$ for $n \in \mathbb{N}$, a contradiction with boundedness of \tilde{u} . Thus $\varepsilon = 0$ for any $s \geq s_0$ and the proof is finished. \square

Lemma 9.2. *We have $u^s \geq u$ for any $s \geq 0$.*

Proof. Let $s_1 \geq 0$ be the smallest number such that $u^s \geq u$ for any $s \geq s_1$ (which obviously exists), and assume that $s_1 > 0$. We first claim that

$$m := \min \left\{ \varepsilon_0, \inf_{u(t, x) \in [\varepsilon_0, u^+(x) - \varepsilon_0]} [u^{s_1}(t, x) - u(t, x)] \right\} > 0. \quad (9.3)$$

Indeed, if $m = 0$, then let (t_n, x_n) be such that $u(t_n, x_n) \in [\varepsilon_0, u^+(x_n) - \varepsilon_0]$ and

$$\lim_{n \rightarrow \infty} [u^{s_1}(t_n, x_n) - u(t_n, x_n)] = 0.$$

The $u^+ \not\equiv 1$ version of Lemma 3.3(ii) with $t_n(\varepsilon) := -\infty$, $f_n(x, u) := f(x - x_n, u)$, $u_n^+(x) := u^+(x - x_n)$, and $u_n(t, x) := u(t - t_n, x - x_n)$, again yields $(\tilde{f}, \tilde{u}^+, \tilde{u}) \in S_{-\infty, L}(K, 0, 0)$ such that

$$\tilde{u}^{s_1} \geq \tilde{u} \quad \text{and} \quad \tilde{u}^{s_1}(0, 0) = \tilde{u}(0, 0) \in [\varepsilon_0, \tilde{u}^+(0) - \varepsilon_0].$$

This contradicts the strong maximum principle for $v := \tilde{u}^{s_1} - \tilde{u} \geq 0$, which satisfies a linear equation $v_t = \Delta v + \lambda(t, x)v$ with $\|\lambda\|_\infty \leq K$, because $v(0, 0) = 0$ and $v \not\equiv 0$. The latter holds because otherwise \tilde{u} would be time-periodic, contradicting (2.27) for \tilde{u} (which propagates with a positive global mean speed because the same is true for u_n , with n -independent constants in Definition 2.2, so (2.27) holds for \tilde{u} by the first claim in (ii)).

So $m > 0$ and since u_t is uniformly bounded by parabolic regularity, there is $s_2 \in (0, s_1)$ such that $u^s \geq u^{s_1} - m$ for any $s \in [s_2, s_1]$. Thus (9.1) as well as $u^s + \varepsilon_0 \geq u$ hold for any

$s \in [s_2, s_1]$. Fix any such s and let $\varepsilon \in [0, \varepsilon_0]$ be the smallest number such that $u^s + \varepsilon \geq u$. The argument in the proof of Lemma 9.1 now shows that $\varepsilon = 0$. But since $s \in [s_2, s_1]$ was arbitrary, this means that we can decrease s_1 to s_2 , a contradiction. It follows that $s_1 = 0$. \square

This finishes the proof of (ii).

(iii) From bounded width of u and Lemma 8.1 it follows that u propagates with a positive global mean speed. Thus (ii) yields $u_t > 0$, and then the (H') version of Corollary 4.4(ii) from the proof of Theorem 2.7 gives the result. Note that we did not use Theorem 2.11 in the proof of Corollary 4.4.

10. PROOF OF THEOREM 2.4(II)

We will prove this by constructing an example of an ignition f which prevents most solutions from having a bounded width (an almost identical construction can be made with a monostable f). The idea is to find f such that there is an equilibrium solution $p \in (0, 1)$ of (1.1), independent of x_1 , with the transition $0 \rightarrow p$ propagating faster in the direction $e_1 = (1, 0, \dots, 0)$ than the transition $p \rightarrow 1$. Then as $t \rightarrow \infty$, the solution u will be close to p on a slab $I_t \times \mathbb{R}^{d-1}$ (to the left of which $u \sim 1$ and to the right of which $u \sim 0$), with I_t an interval of linearly growing length.

Let $\tilde{p} : \mathbb{R}^{d-1} \rightarrow (0, 1)$ be C^∞ , radially symmetric, radially decreasing, with

$$\tilde{p}(\tilde{x}) = 3^{d-4}|\tilde{x}|^{3-d} \text{ for } |\tilde{x}| \geq 3, \quad \Delta\tilde{p} < 0 \text{ on } B_3(0), \quad \text{and} \quad \tilde{p}(B_1(0)) \subseteq \left(\frac{2}{3}, \frac{3}{4}\right).$$

Let $f_0 : [0, 1] \rightarrow [0, \infty)$ be a C^∞ ignition reaction with $f_0(\tilde{p}(\tilde{x})) = -\Delta\tilde{p}(\tilde{x})$ for $\tilde{x} \in \mathbb{R}^{d-1}$ (so ignition temperature is $\theta_0 := \frac{1}{3}$) and decreasing on $[\frac{3}{4}, 1]$. Then $p(x) := p(x_2, \dots, x_d) \in (0, \frac{3}{4})$ satisfies on \mathbb{R}^d ,

$$\Delta p + f_0(p) = 0.$$

Consider f that satisfies (H) with this f_0 , some $K \geq 1$ and $\theta := \frac{1}{4}$, as well as $f(x, u) = f_0(u)$ for all $x \in \mathbb{R}^d$ and $u \in [0, \frac{1}{2}] \cup [p(x), 1]$, and $f(x, u) \geq f_0(u)$ for all $x \in \mathbb{R}^d$ and $u \in (\frac{1}{2}, p(x))$ (provided this interval is non-empty). Then obviously $f(x, u) = 0$ for $(x, u) \in \mathbb{R}^d \times [0, \theta_0]$.

Lemma 10.1. *Let f be as above and $c := \max\{2\sqrt{\|f'_0\|_\infty}, 1\} > 0$. If u solves (1.1), (1.2) with $t_0 = 0$ and $u_0(x) \leq p(x) + e^{-c(x_1-z)/2}$ for all $x \in \mathbb{R}^d$ and some $z \in \mathbb{R}$, then*

$$u(t, x) \leq p(x) + e^{-c(x_1-z-ct)/2}. \quad (10.1)$$

Proof. Let v be the right-hand side of (10.1). Then

$$v_t - \Delta v - f(x, v) = f_0(p(x)) - f_0(v) + \frac{c^2}{4}e^{-c(x_1-z-ct)/2} \geq 0$$

by $c^2/4 \geq \|f'_0\|_\infty$. So v is a super-solution for (1.1) with $v(0, \cdot) \geq u_0$, and we are done. \square

That is, transition $p \rightarrow 1$ is propagating in the direction e_1 with speed at most c , which is independent of K, f . We now make f sufficiently large for $u \in (\frac{1}{2}, p(x))$ so that the transition $0 \rightarrow p$ will be propagating faster than speed c .

Let f^M be as f above, with $f^M(x, u) := f_0(u) + M(u - \frac{1}{2})(p(x) - u)$ for $u \in (\frac{1}{2}, p(x))$. Let $t_0 := 0$ and fix a radially symmetric and radially non-increasing v_0 such that

$$\frac{2}{3}\chi_{B_{1/2}(0)} \leq v_0 \leq \frac{2}{3}\chi_{B_1(0)} \quad (\leq p) \quad (10.2)$$

and $\Delta v_0(x) + f^{M_0}(x, v_0(x)) \geq 0$ for some $M_0 \gg 1$. Then the solution v^M of (1.1), with f^M instead of f and $v^M(0, \cdot) = v_0(\cdot)$, is non-decreasing in both $t > 0$ and $M \geq M_0$. Therefore

$$w(t, x) := \lim_{M \rightarrow \infty} v^M(t, x) \quad (\leq p(x))$$

satisfies $w_t \geq 0$.

We claim that $w(t, x) \notin (\frac{1}{2}, p(x))$ for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$. Otherwise there is $M_1 \geq M_0$ such that $v^{M_1}(t, x) \in (\frac{1}{2}, p(x))$, and then there is $\varepsilon > 0$ such that $v^{M_1}(t - \varepsilon, y) \in (\frac{1}{2}, p(y))$ for all $y \in B_\varepsilon(x)$. Since $v^M(t - \varepsilon, \cdot) \geq v^{M_1}(t - \varepsilon, \cdot)$ for all $M \geq M_1$, it follows from the definition of f^M that $w(s, y) \geq p(y)$ for all $s > t - \varepsilon$ and $y \in B_\varepsilon(x)$. This is a contradiction with the hypothesis $w(t, x) \in (\frac{1}{2}, p(x))$. This, (10.2), and $v_t^M \geq 0$ thus show

$$w(t, x) = p(x) \quad \text{for } (t, x) \in (0, \infty) \times B_{1/2}(0). \quad (10.3)$$

Pick some $0 < \tau \ll 1$. It is easy to show using the properties of the Gaussian that if τ is sufficiently small, then any super-solution of the heat equation on $D := \mathbb{R}^d \setminus B_{1/2 - \tau^{2/3}}(0)$ with initial condition $u(\tau, x) \geq 0$ for $x \in D$ and boundary condition $u(t, x) \geq \frac{3}{5}$ on $(\tau, \infty) \times \partial D$ satisfies $u(2\tau, x) > \frac{1}{2}$ for all $x \in B_{1/2 + \tau^{2/3}}(0)$. (10.3) shows that there is M such that v^M satisfies these conditions, and it follows that

$$w(t, x) = p(x) \quad \text{for } (t, x) \in (2\tau, \infty) \times B_{1/2 + \tau^{2/3}}(0). \quad (10.4)$$

We can repeat this argument with (10.4) as a starting point instead of (10.3) and eventually obtain for all integers $n \geq \tau^{-2/3}$,

$$w(t, x) = p(x) \quad \text{for } (t, x) \in (2n\tau, \infty) \times A_{n\tau^{2/3} - 1/2}, \quad (10.5)$$

where $A_a := \bigcup_{b \in [-a, a]} B_1(b, 0, \dots, 0) \subseteq \mathbb{R}^d$ (we need $A_{n\tau^{2/3} - 1/2}$ instead of $B_{n\tau^{2/3} + 1/2}(0)$ because $p(x) > \frac{1}{2}$ holds only when $|(x_2, \dots, x_d)| < C$, for some $C > 1$). One can in fact show that $w(t, \cdot) = p(\cdot)$ for all $t > 0$ but we will not need this. If we now choose $\tau > 0$ so that there exists an integer $n \in ((2c + 1)\tau^{-2/3}, (2\tau)^{-1})$, then (10.5) yields

$$v^M(1, \cdot) \geq \frac{2}{3}\chi_{A_{2c}}(\cdot),$$

for some $M \geq M_0$. Iterating this, we obtain for $m \in \mathbb{N}$,

$$v^M(m, \cdot) \geq \frac{2}{3}\chi_{A_{2cm}}(\cdot). \quad (10.6)$$

So let us take $f := f^M$ and any u_0 as in Theorem 2.4(ii) (without loss let $t_0 = 0$). It follows from $c \geq 1$, $p \leq \frac{3}{4}$, and Lemma 10.1 with $z = 0$ that

$$\sup_{t > 0, x_1 > ct + 4} u(t, x) \leq \frac{9}{10}. \quad (10.7)$$

If $u \rightarrow 1$ locally uniformly on \mathbb{R}^d as $t \rightarrow \infty$, then $u(t_1, \cdot) \geq \frac{2}{3}\chi_{B_{1/2}(0)}(\cdot)$ for some $t_1 > 0$, and so $u(t_1 + m, 2cm) \geq \frac{2}{3}$ for $m \in \mathbb{N}$ by (10.6). It follows from this and (10.7) that all claims in (C) are false for u . This also holds when $u \not\rightarrow 1$ locally uniformly on \mathbb{R}^d as $t \rightarrow \infty$, because then Lemma 3.1 shows $\sup_{(t,x) \in (1,\infty) \times \mathbb{R}^d} u(t, x) < 1$ (and $u \not\rightarrow 0$ uniformly by the hypothesis).

11. PROOF OF THEOREM 2.12

(i) Having Remark 2 after Theorem 2.7, this is now rather standard. Let U, s_0 , and small $\varepsilon' > 0$ be as in (8.4), with R' large enough so that $w_0(x) := W(|x|)$ satisfies (6.1) and the solution w of (1.1) with f_0 in place of f and $w(0, x) := w_0(x)$ spreads in the sense $w \rightarrow \theta_1$ locally uniformly as $t \rightarrow \infty$ [2]. If now $u_n \in [0, u^+]$ is the solution of (1.1) on $(0, \infty) \times \mathbb{R}^d$ with $u_n(0, x) = w_0(x - ne_1)$, then $(u_n)_t \geq 0$; and the proof of Lemma 8.1, along with $f \geq f_0$ and the comparison principle, shows that $u_n \rightarrow u^+$ locally uniformly as $t \rightarrow \infty$.

If t_n is the first time such that $u_n(t_n, 0) = \theta_0$, shift u_n in time by $-t_n$ so that now it solves (1.1) on $(-t_n, \infty) \times \mathbb{R}^d$ and $u_n(0, 0) = \theta_0$. Obviously $\lim_{n \rightarrow \infty} t_n = \infty$ because $u \leq u^+ \leq 1$, $f(x, u) \leq Ku$, and the comparison principle yield on $(-t_n, \infty) \times \mathbb{R}^d$,

$$u_n(t, x) \leq e^{-\sqrt{K}(x_1 + n - s_0 - R' - 2\sqrt{K}(t + t_n))}.$$

So by parabolic regularity, there is a sub-sequence along which u_n and their spatio-temporal first and spatial second derivatives converge locally uniformly to some solution u of (1.1) on $\mathbb{R} \times \mathbb{R}^d$. Obviously $0 \leq u \leq u^+$ and $u_t \geq 0$ (then all three inequalities are strict due to the strong maximum principle), and since all the u_n satisfy the Remark after Theorem 2.7 with the same $\ell_\varepsilon, T_\varepsilon$ (and $-t_n$ in place of t_0), u has a bounded width. Theorem 2.11(ii) now shows that u is a transition solution because bounded width and Lemma 8.1 yield a positive global mean speed of u , finishing the proof.

(ii) We will only *sketch* the proof, since the mechanics of the workings of the counterexample which we construct are more important than the detailed proof. The latter would only add tedious technical details, obscuring the main ideas. Let us also only consider the case $d = 2$ because the general case is identical, with annuli below replaced by shells.

To find f such that there is no transition solution with doubly-bounded width for (1.1) (and thus also no transition front), it is sufficient to take some ignition f_0 and let f be equal to $\beta f_0(u)$ outside the union of the discs $B_n := B_n(n^3 e_1)$ (for some $\beta \gg 1$), and $f(x, u) = f_0(u)$ inside each $B_{n-1}(n^3 e_1)$ (with a smooth transition between the two on $B_n(n^3 e_1) \setminus B_{n-1}(n^3 e_1)$). If u is a transition solution for (1.1) with a bounded width, let t_n be the first time when $\sup_{x \in B_n} u(t_n, x) = \frac{1}{10}$ (i.e., when the reaction zone of u “reaches” B_n). Since $\beta \gg 1$, the reaction will spread all over $A_n := B_{2n}(n^3 e_1) \setminus B_n(n^3 e_1)$ before it spreads to $B_{n/2}(x_n)$, as described in the introduction (see below for more details). So at the (later) time s_n when $\inf_{x \in B_n} u(s_n, x) = \frac{1}{2}$, we will also have $\inf_{x \in A_n} u(s_n, x) \geq \frac{1}{2}$. It follows that $L^{u, \varepsilon} \geq n$ for all n and $\varepsilon \in (\frac{1}{2}, 1)$. Hence u does not have a doubly-bounded width.

We will need to use a more involved construction to obtain $\inf_{x \in \mathbb{R}^2} u(t, x) > 0$ for any $t \in \mathbb{R}$ and any u from (ii). Let $f_0(u) = (2u - 1)(1 - u)\chi_{[1/2, 1]}(u)$ and let R be such that if $u_t = \Delta u + f_0(u)$ on $(0, \infty) \times \mathbb{R}^2$ and $u(0, \cdot) \geq \frac{3}{4}\chi_{B_R(0)}(\cdot)$, then $u \rightarrow 1$ locally uniformly as

$t \rightarrow \infty$. By Lemma 3.1, such u also satisfies

$$\lim_{t \rightarrow \infty} \inf_{|x| \leq ct} u(t, x) = 1 \quad (11.1)$$

for any $c < c_0$, with $c_0 > 0$ the spreading speed for f_0 (and we have $c_0 \leq 2\sqrt{\|f_0(u)/u\|_\infty} \leq 2$).

Let $\beta > 1$ be such that if $u_t = \Delta u + \beta f_0(u)$ on $(0, \infty) \times \mathbb{R} \times [-2, 2]$ with Dirichlet boundary conditions and $u(0, \cdot) \geq \frac{3}{4}\chi_{B_1(0)}(\cdot)$, then

$$\lim_{t \rightarrow \infty} \inf_{x \in [-100t, 100t] \times [-1, 1]} u(t, x) \geq \frac{4}{5}. \quad (11.2)$$

That is, the reaction with strength β spreads along a strip with a cold boundary at speed at least 100. It is not difficult to show that this holds for large enough β .

Next let $f(x, u) = a(|x|)f_0(u)$, where $a : [0, \infty) \rightarrow [1, \beta]$ is smooth, Lipschitz with the constant β , with $a(r) = \beta$ if $|r - 2^n| \leq 3$ for some $n \geq 3$, and with $a(r) = 1$ if $|r - 2^n| \geq 4$ for each $n \geq 3$. That is, the reaction is large on a sequence of annuli with uniformly bounded widths and exponentially growing radii, and small elsewhere. We obviously have $f \in F(f_0, \beta, \frac{1}{4}, \zeta, \eta)$ for any $\zeta, \eta > 0$ because $\alpha_f(\cdot; \zeta) > \theta_0$ for any $\zeta > 0$.

Then pick $\varepsilon_0 \in (0, \frac{1}{2})$ such that if $u \in [0, 1]$ solves (1.1), (1.2), then $\inf_{y \in B_R(x)} u(t, x) \geq \frac{3}{4}$ whenever $t \geq t_0 + 1$ and $u(t, x) \geq 1 - \varepsilon_0$ (which exists by parabolic regularity) and also such that the unique traveling front for $u_t = u_{xx} + f_0(u)$ connecting ε_0 and 1 has speed $c_{\varepsilon_0} < 1.1c_0$ (which is possible because $\lim_{\varepsilon \rightarrow 0} c_\varepsilon = c_0$).

Assume now that $u \not\equiv 0, 1$ is a bounded entire solution for (1.1) with bounded width. By Theorem 2.11(i) we have $u \in (0, 1)$, so Lemma 3.1 yields a positive global mean speed of u . Then Theorem 2.11(ii) shows that u is a transition solution with $u_t > 0$.

Let t_0 be the first time such that $u(t_0, 0) = \frac{1}{2}$ and for any large n , let t_n be the first time such that $\sup_{x \in B_{2^n}(0)} u(t_n, x) = \varepsilon_0$. Then the maximum principle shows that there is $x_n \in \partial B_{2^n}(0)$ with $u(t_n, x_n) = \varepsilon_0$. Since $L^{u, \varepsilon_0} < \infty$, our choice of ε_0 and R shows that there is $T > 0$ such that $\inf_{x \in B_1(x_n)} u(t_n + T, x) \geq \frac{3}{4}$ for each n . It then follows from (11.2) and $100 > 20\pi$ that

$$\inf_{x \in B_{2^{n+1}}(0) \setminus B_{2^{n-1}}(0)} u\left(t_n + T + \frac{2^n}{20}, x\right) \geq \frac{3}{4} \quad (11.3)$$

for all large n . From this and (11.1) it follows that for all large n ,

$$\inf_{x \in B_{2^n}(0) \setminus B_{2^{n-1}}(0)} u\left(t_n + T + \frac{2^n}{20} + \frac{2^{n-1}}{0.9c_0}, x\right) \geq \frac{3}{4}. \quad (11.4)$$

At the same time, $\sup_{x \in B_{2^n}(0)} u(t_n, x) = \varepsilon_0$ and $c_{\varepsilon_0} < 1.1c_0$ show that $u(t, 0) < \frac{1}{2}$ for $t \leq t_n + 2^n(1.1c_0)^{-1}$ if n is large, because the reaction can propagate radially no faster than at speed c_{ε_0} on any wide annulus where $a(|x|) = 1$, provided $u \leq \varepsilon_0$ initially (this is similar to the upper bound on the propagation speed in Lemma 3.2, and also uses the fact that the annuli on which $a(|x|) > 1$ have widths ≤ 4 , so they shorten the time to reach the origin only by an amount proportional to n). So $t_n + 2^n(1.1c_0)^{-1} \leq t_0$ for all large n , and if we let

$$s_n := t_n + T + \frac{2^n}{20} + \frac{2^{n-1}}{0.9c_0} \quad \left(\leq t_n + \frac{2^n}{1.1c_0} \text{ if } n \text{ is large because } c_0 \leq 2 \right),$$

then we obtain $s_n \leq t_0$ for all large n . But then (11.4) and $u_t > 0$ show for all large n ,

$$\inf_{x \in B_{2n}(0) \setminus B_{2n-1}(0)} u(t_0, x) \geq \frac{3}{4}.$$

The result now follows from Lemma 3.1.

Remark. It is an interesting question whether for the reaction in (ii), all entire solutions $u \in (0, 1)$ satisfy $\lim_{t \rightarrow \infty} \inf_{x \in \mathbb{R}^2} u(t, x) = 1$.

A rough sketch of the proof that the claim involving (1.8) is false. We construct here an example involving front-like solutions in \mathbb{R}^2 (essentially the same idea works for spark-like solutions as well as for all dimensions $d \geq 2$). The full proof of it working as claimed would be quite technical, but the following clearly illustrates the main idea.

For some rapidly growing $b_n \rightarrow \infty$, define

$$A := \bigcup_{n \geq 1} A_n := \bigcup_{n \geq 1} [\{x \mid 0 \leq x_1 \leq b_n \text{ and } |x_2| = b_n\} \cup \{x \mid x_1 = b_n \text{ and } |x_2| \leq b_n\}] \subseteq \mathbb{R}^2.$$

We then let $f(x, u) = a(d(x, A))f_0(u)$, where f_0 is the ignition reaction from part (ii) of the above proof and $a : [0, \infty) \rightarrow [1, \beta]$ is smooth, with $a(s) = \beta$ for $s \leq 1$ and $a(x) = 1$ for $s \geq 2$. Here $\beta \gg 1$ will be chosen later.

We also let $s_0 \gg 1$ and $w : \mathbb{R} \rightarrow [0, 1]$ be smooth and such that $w(s) = 0$ for $s \geq 1$ and $w(s) = 1$ for $s \leq 0$. We then define $v_0(x) := w(x_1)$ and let $u_0(x) \in [w(x_1), w(x_1 - 2s_0)]$ be smooth and such that $u_0(x) = w(x_1)$ for $x_2 \leq -1$ and $u_0(x) = w(x_1 - 2s_0)$ for $x_2 \geq 1$. Finally, let u, v solve (1.1) on $(0, \infty) \times \mathbb{R}^2$ with $u(0, \cdot) = u_0$ and $v(0, \cdot) = v_0$, and let t_n be the first time such that $u(t_n, b_n, 0) = \frac{1}{2}$.

It is obvious that u, v satisfy the hypothesis of the claim involving (1.8) because $u_0 \geq v_0$ and also $u_0 \leq v(T, \cdot)$ for some T . So if (1.8) holds, we must have

$$\lim_{n \rightarrow \infty} [u(t_n, b_n, r) - u(t_n, b_n, -r)] = 0 \quad \text{for any } r \in \mathbb{R} \quad (11.5)$$

because v is obviously even in x_2 . However, if we take $\beta \gg 1$ and sufficiently rapidly growing b_n , then the reaction zone of u spreads towards $(b_n, 0)$ along the two ‘‘arms’’ of A_n much faster than through anywhere else, and that propagation is virtually unaffected by the other ‘‘arms’’. This and the definition of u_0 means that the reaction zone moving towards $(b_n, 0)$ along the upper arm of A_n is distance $\sim 2s_0$ ahead of the one arriving along the lower arm. This means that if s_0 is chosen sufficiently large, depending on β (but not on b_n), then $\liminf_{n \rightarrow \infty} u(t_n, b_n, s_0) \geq \frac{3}{4}$ and $\limsup_{n \rightarrow \infty} u(t_n, b_n, -s_0) \leq \frac{1}{4}$. But this means $\liminf_{n \rightarrow \infty} [u(t_n, b_n, s_0) - u(t_n, b_n, -s_0)] > 0$, a contradiction with (11.5). \square

Remarks. 1. This example can easily be adjusted to v being a transition solution with a bounded width such that $v(t, x) = V(x_1 - c_0 t)$ for $t \ll -1$, where c_0 is the front/spreading speed and V the traveling front profile for f_0 .

2. If u, v are not required to be front-like (or spark-like), counter-examples to (1.8) can be constructed even for homogeneous reactions and dimensions $d \geq 2$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706, USA
EMAIL: zlatos@math.wisc.edu