DIFFUSION IN FLUID FLOW: DISSIPATION ENHANCEMENT BY FLOWS IN 2D

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Abstract. We consider the advection-diffusion equation

$$\phi_t + Au\cdot\nabla\phi = \Delta\phi, \qquad \phi(0,x)=\phi_0(x)$$

on $\mathbb{R}^2$, with u a periodic incompressible flow and $A \gg 1$ its amplitude. We provide a sharp characterization of all u that optimally enhance dissipation in the sense that for any initial datum $\phi_0 \in L^p(\mathbb{R}^2)$, $p<\infty$, and any $\tau>0$,

$$\|\phi(\cdot,\tau)\|_\infty \to 0 \qquad \text{as } A\to\infty.$$

Our characterization is expressed in terms of simple geometric and spectral conditions on the flow. Moreover, if the above convergence holds, it is uniform for ϕ_0 in the unit ball in $L^p(\mathbb{R}^2)$, and $\|\cdot\|_\infty$ can be replaced by any $\|\cdot\|_q$, $q>p$. Extensions to higher dimensions and applications to reaction-advection-diffusion equations are also considered.

1. Introduction and Backgroud

In the present paper we study the influence of fast incompressible advection on diffusion. We are mainly interested in the case of unbounded domains and consider the *passive scalar equation*

$$\phi^A_t + Au\cdot\nabla\phi^A = \Delta\phi^A, \qquad \phi^A(x,0)=\phi_0(x) \tag{1.1}$$

on $D=\mathbb{R}^n\times\mathbb{T}^m$, with initial datum $\phi_0\in L^p(D)$ for some $p<\infty$. Here u is a periodic divergence-free vector field (also called *flow*), the parameter $A\in\mathbb{R}$ accounts for its amplitude, and we will study the behavior of the solutions ϕ^A of (1.1) at fixed positive times $\tau>0$ in the regime of large A. Our goal is to identify those flows which are the most efficient in enhancing the dissipative effect of diffusion at arbitrarily small time scales, provided their amplitude is large enough. More precisely, we aim to characterize the class of flows which yield

$$\|\phi^A(\cdot,\tau)\|_{L^\infty} \to\ 0 \qquad \text{as } A\to\infty \tag{1.2}$$

for any $\tau>0$ and any $\phi_0\in L^p(D)$. We will call such flows *dissipation-enhancing* and our main result is their characterization in two dimensions in Theorem 2.1 below (see also Theorem 9.1 for the case of more dimensions). In addition to being of independent interest, our study of this class is also motivated by applications to *quenching* in reaction-diffusion equations, where good understanding of short-term dynamics for (1.1) is often an essential ingredient (see Theorem 8.3 and [3, 4, 5, 19, 25, 27, 28]).

The problem of diffusion of passive scalars in the presence of a flow is one with a long history. It has been studied in both mathematical and physical literature and we start with reviewing some related literature. The long time behavior of the solutions of (1.1) for a fixed A is by now well understood and, in particular, one has for each ϕ_0,

$$\|\phi^A(\cdot,t)\|_{L^\infty} \to 0 \qquad \text{as } t \to \infty. \tag{1.3}$$

The question of determining finer properties of the solutions has been addressed within the framework of homogenization theory, which identifies an effective diffusion equation that governs the long time–large space asymptotic behavior of solutions of (1.1). The dependence of the corresponding *effective diffusivity matrix* $\sigma(Au)$ on A has been investigated by many authors and classes of flows which enhance diffusion in arbitrary directions e the least ($e^T\sigma(Au)e$ bounded in A) or the most ($e^T\sigma(Au)e \sim A^2$) have been identified. The paper of Fannjiang and Papanicolaou [7] and the extensive review by Majda and Kramer [21] contain these results and provide many further references. However, flows efficient in enhancing diffusion on large time scales may have isolated pockets of stagnation in which solutions persist on short and intermediate time scales. Thus, in general, the knowledge of the asymptotic behavior of $\sigma(Au)$ is not sufficient to obtain satisfactory answers to the kind of questions about short-term dynamics we are interested in.

Closer in spirit is the Freidlin-Wentzell theory [10, 11, 12, 13] which addresses the question of short-term behavior of solutions of (1.1) with large A in the equivalent formulation of fixed flow and vanishing diffusion $-\varepsilon\Delta + u\cdot\nabla$ (on time scales $\sim \varepsilon^{-1} \gg 1$). It applies to a class of Hamiltonian flows in two dimensions, with the Hamiltonian (stream function) satisfying certain non-degeneracy and growth assumptions. By studying random perturbations of Hamiltonian systems via probabilistic methods, it shows the convergence of the solution of (1.1) to that of an effective diffusion equation on the Reeb graph of the Hamiltonian, which is obtained by collapsing each streamline of the flow to a point. However, the flows in this class have only closed streamlines and thus cannot be dissipation-enhancing, meaning that the Freidlin-Wentzell method is not applicable to the problem of our interest.

It turns out that this problem can instead be approached via methods pioneered in a recent work of Constantin, Kiselev, Ryzhik, and the author [4]. That paper has studied the question of influence of advection on diffusion in the simpler setting of bounded domains and compact Riemannian manifolds, and we now briefly review the literature most relevant to this setting.

Long time behavior of solutions of (1.1) on bounded domains D with Dirichlet boundary conditions at ∂D has been investigated in many works. It is well known (see, e.g. [14]) that the asymptotic rate of decay of the solution of (1.1) in this setting is given by the principal eigenvalue λ_0^A of the corresponding elliptic operator $-\Delta + Au\cdot\nabla$. More precisely, we have

$$t^{-1}\log\|\phi^A(\cdot,t)\|_{L^2} \to -\lambda_0^A \qquad \text{as } t \to \infty. \tag{1.4}$$

The question of dependence of λ_0^A on A has been addressed by Kifer [14, 15, 16, 17] in the small diffusion formulation $-\varepsilon\Delta + u\cdot\nabla$. Using probabilistic methods, he has obtained estimates on λ_0^A for large A under certain smoothness assumptions on u.

More recently and using PDE methods, Berestycki, Hamel, and Nadirashvili [2] have characterized those flows u for which $\lambda_0^A \to \infty$ as $A \to \infty$ (the limit $\lim_{A\to\infty}\lambda_0^A$ in the opposite case is also determined via a variational principle). These are those that have no non-zero first integrals (i.e., solutions of $u\cdot\nabla\psi \equiv 0$) in $H_0^1(D)$. Moreover, and particularly interesting to us, [2] shows that (1.2) with L^2 in place of L^∞ holds precisely for these flows. This can be shown to imply (1.2) using Lemma 5.4 below, thus answering our basic question in the setting of bounded domains with Dirichlet boundary conditions.

However, the situation is quite different in the case of unbounded domains, when the equivalent of λ_0^A, the bottom of the spectrum of $-\Delta + Au\cdot\nabla$, is always zero. In the light of this fact, the problem on bounded domains with Neumann boundary conditions or on compact manifolds (when the principal eigenvalue is also always zero) is more relevant to our investigation. This is precisely the focus of [4]. In this setting the average of the solution of (1.1) over D (which has a finite volume) stays constant and we are therefore interested in the enhancement of the speed of *relaxation* of ϕ^A to this average $\bar\phi_0$. That is, one wants to characterize the flows for which

$$\|\phi^A(\cdot,\tau) - \bar\phi_0\|_{L^\infty} \to\ 0 \qquad \text{as } A\to\infty \tag{1.5}$$

for each $\phi_0 \in L^p(D)$ and $\tau > 0$. It has been proved in [4] that these *relaxation-enhancing* flows are precisely those for which the operator $u\cdot\nabla$ has no non-constant eigenfunctions in $H^1(D)$. The method is based on a functional-analytic approach and spectral techniques (e.g., the RAGE theorem), and rests on an abstract result concerning evolution equations in a Hilbert space governed by the coupling of a dissipative evolution to a fast unitary evolution, with the latter having no "slowly dissipating" eigenfunctions (see Theorem 2.3 below). We also note that the property (1.5) has been showed to be equivalent to $\Re(\lambda_1^A)\to\infty$ as $A\to\infty$ by Franke et al. [9], where λ_1^A is an eigenvalue of $-\Delta + Au\cdot\nabla$ with the smallest positive real part (so $\Re(\lambda_1^A)$ is the spectral gap of $-\Delta + Au\cdot\nabla$).

Similar questions in the time-discrete setting have been studied by Fannjiang, Nonnenmacher, and Wolowski [6, 8]. In these a unitary evolution step, represented by a measure preserving automorphism of the torus, alternates with a dissipative step, represented by multiplication of Fourier coefficients by damping factors. Estimates on the dissipation rates for certain classes of toral automorphisms have been provided, along with results linking enhanced dissipation and absence of sufficiently regular eigenfunctions.

As mentioned before, our main goal is to extend the results of [4] to the non-compact setting of unbounded domains. Having [4] at hand, one may consider the following idea. Let ϕ_0 have a compact support and assume that a periodic flow u on $\mathbb{R}^n$ (let the period be 1 in all directions) is relaxation-enhancing on all scales. That is, u is relaxation-enhancing on each compact manifold $\mathcal{M}_k = (k\mathbb{T})^n$. Since the average of ϕ_0 over $\mathcal{M}_k$ decays to zero as $k\to\infty$, large k and A together with (1.5) will make $\|\phi^A(\cdot,\tau)\|_{L^\infty}$ as small as desired when ϕ^A solves (1.1) on $\mathcal{M}_k$. The comparison principle ensures that the solution on $\mathbb{R}^n$ is dominated by that on $\mathcal{M}_k$ and so (1.2) follows. Moreover, we prove in Lemma 5.3 that flows which are relaxation-enhancing on a single scale are also relaxation-enhancing on all other scales. Thus we obtain the result of Theorem 9.1 below that all flows which are relaxation-enhancing on their cell of periodicity are also dissipation-enhancing on $\mathbb{R}^n$ (Theorem 9.1 also covers the more general case of spacetime-periodic flows).

It turns out, however, that these flows are only some of the periodic dissipation-enhancing ones on $\mathbb{R}^n$. For instance, it has been showed in [3, 19] that all generic *shear* (i.e., unidirectional) flows satisfy (1.2) but no shear flow is relaxation-enhancing on $\mathbb{T}^n$. The requirement

(1.5) is quite strong and forces the flow to have certain mixing properties which are not possessed by shear flows, whose streamlines are straight lines. The issue here is that when k is large, a flow need not make ϕ^A "evenly distributed" over all of $\mathcal{M}_k$ in order to make $\|\phi^A\|_{L^\infty}$ small.

Nevertheless, we are still able to provide a sharp characterization of the dissipation-enhancing flows in two dimesions in Theorem 2.1. The proof uses, among others, a generalization of the abovementioned abstract Hilbert space result from [4], which also allows the existence of slowly dissipating eigenfunctions of the fast unitary evolution (see Theorem 2.4 and its time-periodic version, Theorem 4.1). It turns out that this characterization can again be stated in terms of a simple condition concerning $H^1(\mathcal{C})$ eigenfunctions of the operator $u\cdot\nabla$, with $\mathcal{C}$ the cell of periodicity of u, plus the requirement that no open bounded set be invariant under u. This time, however, the condition excludes only the existence of $H^1(\mathcal{C})$ eigenfunctions of $u\cdot\nabla$ with *non-zero* eigenvalues, placing no restrictions on the first integrals of u.

We note that an extension of this characterization to more than two dimensions will not be trivial. Limitations of the proof include the use of the stream function of the flow (e.g., in the proof that flows with bounded invariant domains are not dissipation-enhancing) as well as the use of continuity of $H^1(\mathcal{C})$ eigenfunctions of $u\cdot\nabla$ where $u(x)\neq 0$ (because streamlines have co-dimension 1). The descriptions of some of the main ideas of the proof at the beginnings of Sections 5 and 6 provide a few more details in this direction. As a result, in more than two dimensions we are only able to obtain a sufficient condition for dissipation-enhancement in Theorem 9.1 mentioned above. A characterization of (periodic incompressible) dissipation-enhancing flows in three and more dimensions thus remains open.

As mentioned above, our study of dissipation enhancement by flows on fixed time scales is motivated in part by applications to quenching in reaction-diffusion equations. In this case we consider (1.1) in two dimensions with an *ignition-type* non-negative non-linear reaction term added to the right-hand side (8.1), and the question is which flows are able to extinguish (quench) any initially compactly supported reaction, provided their amplitude is large enough. Our main result here is Theorem 8.2 (and its extension to some strictly positive non-linearities, Theorem 8.3), which shows that outside of the class of flows that do have $H^1(\mathcal{C})$ eigenfunctions other than the first integrals but none of them belongs to $C^{1,1}(\mathcal{C})$, these *strongly quenching* flows are precisely the dissipation-enhancing ones.

The rest of the paper is organized as follows. In Section 2 we state our main result, Theorem 2.1, as well as the abstract Hilbert space result, Theorem 2.4, which is an important step in the proof. In Section 3 we prove Theorem 2.4 and in Section 4 its time-periodic version, Theorem 4.1. Sections 5 and 6 contain the proof of Theorem 2.1, and Section 7 extends it to the case of the strip $\mathbb{R}\times(0,1)$ with Dirichlet or Neumann boundary conditions, along with providing some examples. In Section 8 we apply of our main result to quenching in reaction-diffusion equations, and in Section 9 we provide sufficient conditions for dissipation-enhancement and quenching by space- and time-periodic flows in all dimensions.

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2. Statements of the Main Results

Let u be a periodic, incompressible, Lipschitz flow on the domain $D = \mathbb{R}^n \times \mathbb{T}^m$, and let $\mathcal{C}$ be its cell of periodicity with each couple of opposite $(n+m-1)$-dimensional "faces" identified, so that $\mathcal{C}$ is a blown-up $(m+n)$-dimensional torus. Then u defines a unitary evolution $\{U_t\}_{t\in\mathbb{R}}$ on $L^2(\mathcal{C})$ (and also on $L^2(D)$) in the following manner. For each $x \in \mathcal{C}$ there is a unique solution $X(x,t)$ to the ODE

$$\frac{d}{dt}X(x,t) = u(X(x,t)), \qquad X(x,0) = x. \tag{2.1}$$

We then let

$$(U_t\psi)(x) \equiv \psi(X(x,-t))$$

for any $\psi \in L^2(\mathcal{C})$. Incompressibility of u implies that the group $\{U_t\}_{t\in\mathbb{R}}$ is unitary, and its generator is the operator $-iu\cdot\nabla$. It is self-adjoint on $L^2(\mathcal{C})$ and for each $\psi \in H^1(\mathcal{C})$ we have

$$i\frac{d}{dt}(U_t\psi) = -iu\cdot\nabla(U_t\psi). \tag{2.2}$$

If $\psi \in L^2(\mathcal{C})$ is an eigenfunction of the anti-self-adjoint operator $u\cdot\nabla$ (i.e., $u\cdot\nabla\psi \equiv i\lambda\psi$ for some $\lambda \in \mathbb{R}$), and therefore also an eigenfunction of each $U_t = e^{-(u\cdot\nabla)t}$, we say that ψ is an *eigenfunction of the flow* u on $\mathcal{C}$. The eigenfunctions ψ of u that correspond to eigenvalue zero (i.e., $u\cdot\nabla\psi \equiv 0$) are called the *first integrals* of u. We also say that a set $V \subseteq D$ is *invariant under the flow* u if and only if $V \neq \emptyset$ and $X(x,t) \in V$ for all $x \in V$ and $t \in \mathbb{R}$. Finally, if v is an incompressible Lipschitz flow on D, we let $\mathcal{P}_t(v)$ be the solution operator for

$$\psi_t + v\cdot\nabla\psi = \Delta\psi, \qquad \psi(0) = \psi_0 \tag{2.3}$$

on D. That is, $\mathcal{P}_t(v)\psi_0 = \psi(\cdot,t)$ when ψ solves (2.3).

We can now state our main result.

Theorem 2.1. *Let u be a periodic, incompressible, Lipschitz flow on $D = \mathbb{R}^2$ or $D = \mathbb{R}\times\mathbb{T}$ with a cell of periodicity $\mathcal{C}$, and let ϕ^A solve* (1.1) *on D. Then the following are equivalent.*

(i) *For some $1 \le p \le q \le \infty$ and each $\tau > 0$, $\phi_0 \in L^p(D)$,*

$$\|\phi^A(\cdot,\tau)\|_{L^q(D)} \to 0 \qquad \text{as } A \to \infty. \tag{2.4}$$

(ii) *For any $1 \le p \le q \le \infty$ such that $p < \infty$ and $q > 1$, and each $\tau > 0$, $\phi_0 \in L^p(D)$,*

$$\|\phi^A(\cdot,\tau)\|_{L^q(D)} \to 0 \qquad \text{as } A \to \infty.$$

(iii) *For any $1 \le p < q \le \infty$ and each $\tau > 0$,*

$$\|\mathcal{P}_\tau(Au)\|_{L^p(D)\to L^q(D)} \to 0 \qquad \text{as } A \to \infty. \tag{2.5}$$

(iv) *No bounded open subset of D is invariant under u and any eigenfunction of u on $\mathcal{C}$ that belongs to $H^1(\mathcal{C})$ is a first integral of u.*

Remarks. 1. The couples p, q in the theorem are the only ones for which the corresponding claims can possibly hold. The conclusion of (ii) and (iii) cannot hold for $p > q$ because then $\mathcal{P}_\tau(v)$ does not map L^p to L^q. As for $p = q$, note that since $\frac{d}{dt}\int_D \psi\, dx \equiv 0$ for solutions of (2.3) when v is incompressible, the L^1 norm of non-negative solutions does not decay. This and the fact that $\psi \equiv 1$ is a constant solution mean that (ii) cannot hold for $p = q \in \{1, \infty\}$. The above arguments and the maximum principle give

$$\|\mathcal{P}_\tau(v)\|_{L^p \to L^p} = 1 \tag{2.6}$$

for $p \in \{1, \infty\}$. Finally, the estimate $0 \le k_v(x - \bar{v}t, y, t) \le Ct^{-1}e^{-|x-y|^2/Ct}$ on the *heat kernel* $k_v(x, y, t)$ for the flow v on D from [24] (with $\bar{v}$ the mean of v and $C = C(v)$) and $\int_D k_v(x, y, t)dy = 1$, applied to slowly varying initial data ϕ_0, gives $\|\mathcal{P}_\tau(v)\|_{L^p \to L^p} \ge 1$ for any $p \ge 1$. Interpolation then extends (2.6) to all p and so (iii) cannot hold for any $p = q$.

2. The claim (iii) means that for $p < q$, the decay in (ii) is uniform for $\|\phi_0\|_{L^p} \le 1$. In particular, taking $p = 1$ and $q = \infty$ yields a characterization of the (periodic incompressible Lipschitz) flows for which the corresponding heat kernel $k_{Au}(x, y, t)$ on D satisfies

$$\|k_{Au}(\cdot, \cdot, \tau)\|_{L^\infty} \to 0 \qquad \text{as } A \to \infty$$

for each $\tau > 0$. Namely, these are the flows from Theorem 2.1(iv). One direction of our proof — (iv)⇒(i),(ii),(iii) in Section 5 — will actually only concentrate on the case $(p, q) = (1, \infty)$, since the others will follow by (2.6) and interpolation.

3. In the case of the strip $D = \mathbb{R} \times (0, 1)$ we only consider periodic boundary conditions here. The result remains unchanged (with $\mathcal{C}$ the surface of a cylinder rather than a torus) if Dirichlet or Neumann boundary conditions are assumed and $u(x) \cdot (0, 1) = 0$ for $x \in \partial D$. See Section 7 below, which also provides examples demonstrating that the two conditions in (iv) are independent in general.

4. Notice that for some u (e.g., vertical shear flows), the first condition in (iv) is satisfied when $D = \mathbb{R}^2$ but not when $D = \mathbb{R} \times \mathbb{T}$.

Definition 2.2. We will call the flows that satisfy (2.5) *dissipation-enhancing* on D.

It is natural to ask what makes the first integrals different from eigenfunctions corresponding to a non-zero eigenvalue. The answer is essentially the fact that the existence of a single H^1 eigenfunction corresponding to eigenvalue $\lambda \ne 0$ implies the existence of infinitely many eigenspaces of u with H^1 eigenfunctions — those corresponding to all integer multiples of λ. This can be seen from the proof of Lemma 5.3 below.

We will see from the proof of Theorem 2.1 that condition (iv) essentially tells us that the flow Au quickly "stretches" any initial datum ϕ_0 and exposes it to diffusion (by making $\|\nabla\phi^A\|_{L^2}$ large), thus enhancing the dissipation rate as much as desired when A is large. One might therefore think that a sufficient condition for a flow to not be dissipation-enhancing could be the existence of a stable solution of (2.1). If (2.1) on $\mathcal{C}$ has no dense orbits, then this is indeed the case (see Theorem 7.3). However, the claim is not true in general. We will not provide all details here, but a counterexample can be obtained in the following manner. One first constructs a 1-periodic flow u whose spectrum is $\{n + m\alpha \,|\, n, m \in \mathbb{Z}\}$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and such that all the eigenfunctions of u except of the constant function belong to $C(\mathbb{T}^2) \setminus H^1(\mathbb{T}^2)$. This can be done using Example 2 in Section 6 of [4], with the obtained flow smoothly isomorphic to a reparametrization of the constant flow $(\alpha, 1)$ (and therefore no bounded subset of D is invariant under the flow). The construction goes back to Kolmogorov's work [20] and at its core is the problem of small divisors — it is based on the existence of a smooth function $Q : \mathbb{T} \to \mathbb{T}$ with $\int_{\mathbb{T}} Q(\xi)\, d\xi = 1$ and $\alpha \in \mathbb{T}$ such that the homology equation (6.2) in [4]

$$R(\xi + \alpha) - R(\xi) = Q(\xi) - 1, \tag{2.7}$$

has a solution $R \in H^{\frac{1}{2}+\varepsilon}(\mathbb{T}) \setminus H^1(\mathbb{T}) \subseteq C(\mathbb{T}) \setminus H^1(\mathbb{T})$. This is possible when α is Liouvillean, that is, well approximated by rationals. The (continuous, non-H^1) eigenfunctons of the constructed flow have absolute value one and u is non-zero everywhere, so our analysis in Section 6 below can be used to show that all solutions of (2.1) are stable. Nevertheless, Theorem 2.1 shows that u is dissipation-enhancing!

As we have mentioned in the Introduction, the proof of Theorem 2.1 crucially uses an abstract result concerning dissipative evolution in a Hilbert space — Theorem 2.4. This is a generalization of Theorem 1.4 in [4] and we now state both results.

Let Γ be a self-adjoint, non-negative, unbounded operator with a discrete spectrum on a separable Hilbert space $\mathcal{H}$. Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of Γ (so that $\lambda_1 \geq 0$ and $\lambda_n \to \infty$) and let κ_j be the corresponding orthonormal eigenvectors forming a basis in $\mathcal{H}$. The Sobolev space $H^m(\Gamma)$ associated with Γ is formed by all vectors $\psi = \sum_j c_j \kappa_j$ such that

$$\|\psi\|_{\dot{H}^m(\Gamma)} \equiv \left(\sum_j \lambda_j^m |c_j|^2 \right)^{1/2} < \infty.$$

This is the homogeneous Sobolev semi-norm (which is a norm if $\lambda_1 > 0$), and the Sobolev norm is defined by $\|\cdot\|^2_{H^m(\Gamma)} = \|\cdot\|^2_{\dot{H}^m(\Gamma)} + \|\cdot\|^2_{\mathcal{H}}$. Note that the domain of Γ is $H^2(\Gamma)$.

Let L be a self-adjoint operator on $\mathcal{H}$ such that, for any $\psi \in H^1(\Gamma)$ and $t > 0$ we have

$$\|L\psi\|_{\mathcal{H}} \leq C\|\psi\|_{H^1(\Gamma)} \qquad \text{and} \qquad \|e^{iLt}\psi\|_{H^1(\Gamma)} \leq B(t)\|\psi\|_{H^1(\Gamma)} \tag{2.8}$$

where the constant $C < \infty$ and the function $B(t) \in L^2_{\mathrm{loc}}[0, \infty)$ are independent of ψ. Here e^{iLt} is the unitary evolution group generated by the self-adjoint operator L. It has been shown in [4] that the two conditions in (2.8) are independent in general.

Finally, let $\phi^A(t)$ be a solution of the Bochner differential equation

$$\frac{d}{dt}\phi^A(t) = iAL\phi^A(t) - \Gamma\phi^A(t), \qquad \phi^A(0) = \phi_0. \tag{2.9}$$

Then we have the following result from [4].

Theorem 2.3 ([4]). *Let Γ be a self-adjoint, positive, unbounded operator with a discrete spectrum and let a self-adjoint operator L satisfy* (2.8). *Then the following are equivalent.*

(i) *For any $\tau, \delta > 0$ and $\phi_0 \in \mathcal{H}$ there exists $A_0(\tau, \delta, \phi_0)$ such that for any $A > A_0(\tau, \delta, \phi_0)$, the solution $\phi^A(t)$ of* (2.9) *satisfies $\|\phi^A(\tau)\|^2_{\mathcal{H}} < \delta$.*

(ii) *For any $\tau, \delta > 0$ there exists $A_0(\tau, \delta)$ such that for any $A > A_0(\tau, \delta)$ and any $\phi_0 \in \mathcal{H}$ with $\|\phi_0\|_{\mathcal{H}} \leq 1$, the solution $\phi^A(t)$ of* (2.9) *satisfies $\|\phi^A(\tau)\|^2_{\mathcal{H}} < \delta$.*

(iii) *The operator L has no eigenfunctions belonging to $H^1(\Gamma)$.*

Remark. Note that the theorem says that if A_0 above exists, it is independent of ϕ_0 inside the unit ball in $\mathcal{H}$. This is the same as the equivalence of Theorem 2.1(ii) and (iii).

If one takes $\Gamma \equiv -\Delta$ and $L \equiv iu\cdot\nabla$, both restricted to the space of mean-zero L^2 functions, then (2.9) is (1.1) and this result can be applied to the study of fast relaxation by flows on compact manifolds or on bounded domains D (where $\Gamma > 0$ has a discrete spectrum). It obviously only provides $L^2 \to L^2$ estimates, but after coupling these with Lemma 5.4 below, one can obtain the characterization of relaxation-enhancing flows on D from [4] mentioned in the Introduction.

In the light of our earlier observation that on unbounded domains not all periodic flows satisfying (1.2) are relaxation-enhancing on their cell of periodicity, a natural next question is what happens to the dissipative dynamics (2.9) when some eigenfunctions of L do lie in $H^1(\Gamma)$. We denote by $P_h : \mathcal{H} \to \mathcal{H}$ the projection onto the closed subspace $P_h\mathcal{H} \subseteq \mathcal{H}$ generated by all such eigenfunctions. That is, $P_h\mathcal{H}$ is the closure in $\mathcal{H}$ of the set of all linear combinations of those eigenfunctions of L which lie in $H^1(\Gamma)$. Notice that $P_h\mathcal{H}$ need not be contained in $H^1(\Gamma)$ since the latter is not closed. Now we can provide the following answer.

Theorem 2.4. *Let Γ be a self-adjoint, non-negative, unbounded operator with a discrete spectrum and let a self-adjoint operator L satisfy conditions* (2.8)*. Then for any $\tau, \delta > 0$ there exists $A_0(\tau,\delta)$ such that for any $A > A_0(\tau,\delta)$ and any $\phi_0 \in \mathcal{H}$ with $\|\phi_0\|_{\mathcal{H}} \le 1$, the Lebesgue measure of the set of times t for which the solution $\phi^A(t)$ of* (2.9) *satisfies*

$$\|(I - P_h)\phi^A(t)\|_{\mathcal{H}}^2 \ge \delta \tag{2.10}$$

is smaller than τ. Moreover, if $\dim(P_h\mathcal{H}) < \infty$, then $\|(I - P_h)\phi^A(t)\|_{\mathcal{H}}^2 < \delta$ for all $t > \tau$.

Remarks. 1. That is, if A is large, any solution starting in the unit ball in $\mathcal{H}$ will spend a lot of time $\sqrt{\delta}$-close to the subspace $P_h\mathcal{H}$. We will actually show that this is even true for some (τ,δ)-dependent finite-dimensional subspace of $P_h\mathcal{H} \cap H^1(\Gamma)$ (see the proof).

2. It follows from (3.9) below that $P_h\mathcal{H}$ is the smallest closed subspace (and its unit ball the smallest closed subset) of $\mathcal{H}$ for which a result like this holds. In this sense, Theorem 2.4 is not only natural but also optimal.

3. It remains an open problem whether for large A the evolution stays close to $P_h\mathcal{H}$ for all $t > \tau$ when $\dim(P_h\mathcal{H}) = \infty$ (note that, e.g., $\dim(P_h\mathcal{H}) = \infty$ when u in Theorem 2.1 has an H^1 eigenfunction with non-zero eigenvalue, due to Lemma 5.3). We conjecture that this is the case. A related interesting problem is to find the $A \to \infty$ asymptotics of the evolution (2.9) and determine whether one recovers an effective evolution equation on the subspace $P_h\mathcal{H}$ in this way.

4. We allow here $\Gamma \ge 0$ rather than $\Gamma > 0$ (Theorem 2.3 can also be extended to this case). In the proof of Theorem 2.1 we will take $\Gamma \equiv -\Delta$ and $L \equiv iu \cdot \nabla$ on $\mathcal{H} \equiv L^2(\mathcal{M}_k)$ (with $\mathcal{M}_k$ from the Introduction), rather than just the mean-zero L^2 functions.

3. The Abstract Result

In this section we prove Theorem 2.4. As in [4], we reformulate (2.9) as a small diffusion–long time problem. By setting $\varepsilon = A^{-1}$ and rescaling time by the factor $1/\varepsilon$, we pass from considering (2.9) to

$$\frac{d}{dt}\phi^\varepsilon(t) = (iL - \varepsilon\Gamma)\phi^\varepsilon(t), \qquad \phi^\varepsilon(0) = \phi_0. \tag{3.1}$$

We now want to show that for all small enough $\varepsilon > 0$ the measure of times for which (2.10) holds (with A replaced by ε) is smaller than τ/ε.

We will be comparing this dissipative dynamics to the "free" one given by

$$\frac{d}{dt}\phi^0(t) = iL\phi^0(t), \qquad \phi^0(0) = \phi_0, \tag{3.2}$$

so that $\phi^0(t) = e^{iLt}\phi_0$. Notice that if $L \equiv iu\cdot\nabla$, then this is just

$$\phi^0_t + u\cdot\nabla\phi^0 = 0, \qquad \phi^0(x,0) = \phi_0(x), \tag{3.3}$$

that is, (2.2) with $\phi_0 \equiv \psi$ and $\phi^0(x,t) \equiv (U_t\psi)(x) = \phi_0(X(x,-t))$.

For the sake of convenience, in the remainder of this section we will denote the norm $\|\cdot\|_{\mathcal{H}}$ by $\|\cdot\|$, the space $H^m(\Gamma)$ by H^m, and the semi-norm $\|\cdot\|_{\dot H^m(\Gamma)}$ by $\|\cdot\|_m$.

We begin with collecting some preliminary results from [4] in the following lemma.

Lemma 3.1 ([4])**.** *Assume that conditions* (2.8) *hold.*

(i) *For $\varepsilon \geq 0$ and $\phi_0 \in H^1$ there exists a unique solution $\phi^\varepsilon(t)$ of* (3.1) *on* $[0,\infty)$*. If $\varepsilon > 0$, then for any $T < \infty$,*

$$\phi^\varepsilon(t) \in L^2([0,T],H^2)\cap C([0,T],H^1), \qquad \frac{d}{dt}\phi^\varepsilon(t) \in L^2([0,T],\mathcal{H}). \tag{3.4}$$

If $\varepsilon = 0$, then for any $T < \infty$,

$$\phi^0(t) \in L^2([0,T],H^1)\cap C([0,T],\mathcal{H}), \qquad \frac{d}{dt}\phi^0(t) \in L^2([0,T],\mathcal{H})$$

(ii) *We have*

$$\frac{d}{dt}\|\phi^\varepsilon\|^2 = -2\varepsilon\|\phi^\varepsilon\|_1^2 \tag{3.5}$$

for a.e. t, and hence

$$\|\phi^\varepsilon(t)\|^2 \leq \|\phi_0\|^2 \qquad \text{and} \qquad \int_0^\infty \|\phi^\epsilon(t)\|_1^2 dt \leq \frac{\|\phi_0\|^2}{2\varepsilon}. \tag{3.6}$$

(iii) *If ϕ^ε and ϕ^0 solve* (3.1) *and* (3.2)*, respectively, with $\phi_0 \in H^1$, then*

$$\frac{d}{dt}\|\phi^\varepsilon(t) - \phi^0(t)\|^2 \leq \frac{\varepsilon}{2}\|\phi^0(t)\|_1^2 \leq \frac{\varepsilon}{2}B(t)^2(\|\phi_0\|_1^2 + \|\phi_0\|^2) \tag{3.7}$$

for a.e. t. In particular,

$$\|\phi^\varepsilon(t) - \phi^0(t)\|^2 \leq \frac{\varepsilon}{2}(\|\phi_0\|_1^2 + \|\phi_0\|^2)\int_0^{T_0} B(t)^2\,dt \tag{3.8}$$

for any ϕ_0 *and* $t \le T_0$. *Moreover, if* $b = b(\phi_0, T_0) < \infty$ *is such that* $\|\phi^0(t)\|_1 \le b\|\phi_0\|$ *for any* $t \in [0, T_0]$, *then for all such* t,

$$\|\phi^\varepsilon(t) - \phi^0(t)\|^2 \le \frac{b^2 \varepsilon t}{2} \|\phi_0\|^2. \tag{3.9}$$

Remarks. 1. We consider here ϕ^ε to be a solution of (3.1) if it is continuous and satisfies (3.1) for a.e. t.

2. The solution ϕ^ε also exists for any $\phi_0 \in \mathcal{H}$, but then it may be rougher on time intervals containing 0 when $\varepsilon > 0$, and everywhere when $\varepsilon = 0$. We will only need to consider $\phi_0 \in H^1$ in the proof of Theorem 2.4. This is because H^1 is dense in $\mathcal{H}$, and the norm of the difference of two solutions of (3.1) with the same ε cannot increase due to (3.6).

Notice that according to (3.5), the rate of decrease of $\|\phi^\varepsilon\|^2$ is proportional to $\|\phi^\varepsilon\|_1^2$. This illuminates the following result from which Theorem 2.4 will follow.

Theorem 3.2. *Consider the setting of Theorem 2.4. Then for any* $\tau, \delta > 0$ *there exists* $\varepsilon_0(\tau, \delta) > 0$ *and* $T_0 = T_0(\tau, \delta)$ *such that for any* $\varepsilon \in (0, \varepsilon_0(\tau, \delta))$, *any* $\phi_0 \in H^1$ *with* $\|\phi_0\| \le 1$, *and any* $t \ge 0$, *the solution* ϕ^ε *of* (3.1) *satisfies at least one of the following:*

(a)

$$\|\phi^\varepsilon(t)\|_1^2 > \frac{1}{\tau}; \tag{3.10}$$

(b)

$$\int_t^{T_0+t} \|\phi^\varepsilon(s)\|_1^2 ds \ge \frac{T_0}{\tau}; \tag{3.11}$$

(c)

$$\|(I - P_h)\phi^\varepsilon(t)\|^2 < \delta \quad \textit{and neither } \text{(a)} \textit{ nor } \text{(b)} \textit{ holds.} \tag{3.12}$$

Remark. This result in the absence of H^1-eigenfunctions was the cornerstone of the proof of Theorem 1.4 in [4].

Proof of Theorem 2.4 given Theorem 3.2. Let $t_1 \ge 0$ be the first time such that Theorem 3.2(b) holds for $t = t_1$, let $t_2 \ge t_1 + T_0$ be first such time after $t_1 + T_0$, etc. Thus we obtain a sequence of times t_j such that $t_{j+1} \ge t_j + T_0$ and (b) holds for $t = t_j$. If $J_1 = \bigcup_j [t_j, t_j + T_0]$, then (b) does not hold for any $t \in \mathbb{R}^+ \setminus J_1$ by construction. Let J_2 be the set of all $t \in \mathbb{R}^+ \setminus J_1$ for which (a) holds and let $J_0 \equiv J_1 \cup J_2$, so that neither (a) nor (b) holds for $t \in \mathbb{R}^+ \setminus J_0$. Therefore (c) holds for these t, and so (2.10) (with ε in place of A) can only hold for $t \in J_0$. From the definition of J_0 we have that

$$\int_{J_0} \|\phi^\varepsilon(t)\|_1^2 dt \ge \frac{|J_0|}{\tau}.$$

From (3.6) we obtain $|J_0| \le \tau/2\varepsilon < \tau/\varepsilon$. This proves the first claim in Theorem 2.4 when $\phi_0 \in H^1$ (after rescaling time by a factor of ε to pass from (3.1) to (2.9)). As explained above, the case $\phi_0 \in \mathcal{H}$ is immediate from the density of H^1 in $\mathcal{H}$.

Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $P_h\mathcal{H}$ with each ϕ_n an H^1 eigenfunction of L. Notice that each $\phi^\varepsilon(t)$ satisfying Theorem 3.2(c) belongs to

$$K \equiv \left\{\phi \,\middle|\, \|\phi\|^2 \le 1,\ \|\phi\|_1^2 \le \frac{1}{\tau},\ \text{and}\ \|(I-P_h)\phi\|^2 \le \delta\right\}.$$

This set is compact and hence so is $P_hK \subseteq P_h\mathcal{H}$. Each element of K is $\sqrt{\delta}$-close to P_hK, and compactness shows that there is $n_0 = n_0(\tau,\delta) < \infty$ such that each element of P_hK is $\sqrt{\delta}$-close to the subspace with basis $\{\phi_n\}_{n=1}^{n_0}$. Replacing δ by $\delta/4$ proves the claim of Remark 1 after Theorem 2.4.

Finally, assume $\dim(P_h\mathcal{H}) < \infty$. Then $P_h\mathcal{H} \subseteq H^1$ and there must be $b < \infty$ such that $\|\phi\|_1 \le b\|\phi\|$ for all $\phi \in P_h\mathcal{H}$. By (3.9), there is $\tau_1 \equiv 2\delta/b^2$ such that for all $\varepsilon > 0$ and all $\phi_1 \in P_h\mathcal{H}$ with $\|\phi_1\| \le 1$, the solution of (3.1) with initial condition ϕ_1 stays $\sqrt{\delta}$-close to $P_h\mathcal{H}$ on the time interval $[0, \tau_1/\varepsilon]$.

Now change τ to $\min\{\tau,\tau_1\}$, and change $\varepsilon_0(\tau,\delta)$ accordingly. The first claim of Theorem 2.4 says that for any ϕ_0 with $\|\phi_0\| \le 1$ and any $t > \tau/\varepsilon$ there is $t_0 \in [t-\tau/\varepsilon, t]$ and $\phi_1 \in P_h\mathcal{H}$ with $\|\phi_1\| \le 1$ such that $\|\phi^\varepsilon(t_0) - \phi_1\| < \sqrt{\delta}$. But then from (3.6) and $t - t_0 \le \tau/\varepsilon \le \tau_1/\varepsilon$,

$$\operatorname{dist}(\phi^\varepsilon(t), P_h\mathcal{H}) \le \|\phi^\varepsilon(t) - \phi_1^\varepsilon(t-t_0)\| + \operatorname{dist}(\phi_1^\varepsilon(t-t_0), P_h\mathcal{H}) < 2\sqrt{\delta},$$

where ϕ_1^ε is the solution of (3.1) with initial condition ϕ_1. Again, replacing δ by $\delta/4$ gives the second claim of Theorem 2.4. $\square$

We devote the rest of this section to the proof of Theorem 3.2.

Proof of Theorem 3.2. We let P_c and P_p be the spectral projections in $\mathcal{H}$ onto the continuous and pure point spectral subspaces of L, respectively. We also let e_j be the eigenvalues of L and P_j the projection onto the eigenspace corresponding to e_j. Finally, let Q_N be the projection onto the subspace generated by the eigenfunctions $\kappa_1, \dots, \kappa_N$ corresponding to the first N eigenvalues of Γ.

Take $\varepsilon > 0$ and let us assume that $\|\phi^\varepsilon(t_0)\|_1^2 \le 1/\tau$ and $\|(I-P_h)\phi^\varepsilon(t_0)\|^2 \ge \delta$. We then need to show that (b) holds with $t = t_0$ provided $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\tau,\delta)$ and $T_0 = T_0(\tau,\delta)$ to be determined later. To simplify notation we rename $\phi^\varepsilon(t_0)$ to ϕ_0 and $\phi^\varepsilon(t_0+t)$ to $\phi^\varepsilon(t)$ so that $\phi^\varepsilon(t)$ solves (3.1) and we have

$$\|\phi_0\|^2 \le 1, \quad \|\phi_0\|_1^2 \le \frac{1}{\tau}, \quad \text{and} \quad \|(I-P_h)\phi_0\|^2 \ge \delta. \tag{3.13}$$

We now need to show

$$\int_0^{T_0} \|\phi^\varepsilon(t)\|_1^2 dt \ge \frac{T_0}{\tau} \tag{3.14}$$

in order to conclude (b), which is what we will do.

The idea, partly borrowed from [4], is as follows. We let $\phi^0(t) \equiv e^{iLt}\phi_0$ solve (3.2) and note that (3.8) guarantees $\phi^\varepsilon(t)$ to be close to $\phi^0(t)$ for all $t \le T_0$ as long as ε is sufficiently small. As a result we will be left with studying the free dynamics $\phi^0(t)$. We will show, in an averaged sense over $[0, T_0]$, that its pure point part $P_p\phi^0(t)$ will "live" in low and intermediate modes of Γ (i.e., in $Q_N\mathcal{H}$ for some $N < \infty$) with a large H^1 norm there if $\|(P_p - P_h)\phi_0\|^2 \ge \delta/2$.

On the other hand the continuous part $P_c\phi^0(t)$ will live in high modes (i.e., in $(I-Q_N)\mathcal{H}$), and thus also have large H^1 norm if $\|P_c\phi_0\|^2 \ge \delta/2$. Since $I-P_h = P_c + (P_p - P_h)$, (3.13) will ensure (3.14) for both the free and dissipative dynamics. The key to these conclusions will be the compactness of the set of ϕ_0 satisfying (3.13).

The main point is that the pure point and continuous parts of the free dynamics effectively "decouple" into different modes of Γ and therefore do not cancel out each other's contribution to the H^1 norm. We note that in [4], the possibility of $P_p\phi^0(t)$ leaking into the high modes has not been excluded, resulting in the limitation mentioned in the remark after Theorem 3.2.

The next three lemmas make the above heuristic rigorous.

Lemma 3.3 ([4])**.** *Let $K \subset \mathcal{H}$ be a compact set. For any $N, \omega > 0$, there exists $T_c(N,K,\omega)$ such that for all $T \ge T_c(N,K,\omega)$ and any $\phi \in K$, we have*

$$\frac{1}{T}\int_0^T \|Q_N e^{iLt} P_c\phi\|^2\,dt < \omega. \tag{3.15}$$

This is the "uniform RAGE theorem" for a compact set of vectors [4]. It says that if we wait long enough, the continuous part of the free dynamics starting in K escapes into the high modes of Γ in a time average. The next lemma shows that the pure point dynamics stays in low and intermediate modes.

Lemma 3.4. *Let $K \subset \mathcal{H}$ be a compact set. For any $\omega > 0$, there exists $N_p(K,\omega)$ such that for any $N \ge N_p(K,\omega)$, $\phi \in K$, and $t \in \mathbb{R}$, we have*

$$\|(I-Q_N)e^{iLt}P_p\phi\|^2 < \omega. \tag{3.16}$$

Proof. Let $\{\phi_n\}$ be an orthonormal basis of $P_p\mathcal{H}$ such that each ϕ_n is an eigenfunction of L with eigenvalue $e_{j(n)}$. Since P_pK is compact, it has a finite $1/k$ net for any $k \in \mathbb{N}$. Moreover, this net can be chosen so that each its element is a finite linear combination of the ϕ_n, since these are dense in $P_p\mathcal{H}$ (of course, it may happen that some elements of this net are not in P_pK). Let the net be $\{\sum_{n=1}^{n_0}\alpha_{m,n}\phi_n\}_{m=1}^{m_0}$. Since e^{iLt} is unitary, $R \equiv \bigcup_{t\in\mathbb{R}}\{\sum_{n=1}^{n_0} e^{ie_{j(n)}t}\alpha_{m,n}\phi_n\}_{m=1}^{m_0}$ is a $1/k$ net for $K' \equiv \bigcup_{t\in\mathbb{R}} e^{iLt}P_pK$. Let $\alpha \ge \sup|\alpha_{m,n}|$ be an integer and $S \equiv \frac{2\pi}{4n\alpha k}\{1,2,\dots,4n\alpha k\}$. Then

$$\bigcup_{q_{m,n}\in S}\left\{\sum_{n=1}^{n_0} e^{iq_{m,n}}\alpha_{m,n}\phi_n\right\}_{m=1}^{m_0}$$

is a finite $1/k$ net for R, and thus a $2/k$ net for K'. Since k was arbitrary, K' must be compact. We have that $I-Q_N$ converges strongly to zero as $N\to\infty$, and so there must be N such that $\|(I-Q_N)\phi\|^2 < \omega$ for all $\phi \in K'$. $\square$

Finally, we show that the H^1 norm of the pure point part of the free dynamics will become large provided $P_p\phi_0$ is sufficiently "rough". Recall that P_j are the projections onto the eigenspaces of L.

Lemma 3.5. *Let $K \subset \mathcal{H}$ be a compact set and $\Omega < \infty$ be such that each $\phi \in K$ satisfies $\sum_j \|P_j\phi\|_1^2 \ge 3\Omega$ (the sum may be equal to ∞). Then there exists $N_1(K,\Omega)$ and $T_1(K,\Omega)$*

such that for all $N \geq N_1(K,\Omega)$ *and* $T \geq T_1(K,\Omega)$ *we have*

$$\frac{1}{T}\int_0^T \|Q_N e^{iLt} P_p \phi\|_1^2 \, dt > \Omega. \tag{3.17}$$

This is almost identical to Lemma 3.3 in [4] (which only treats the case $\sum_j \|P_j\phi\|_1^2 = \infty$) and the proof carries over. Namely, one first shows using compactness that there is N such that for all $\phi \in K$ we have $\sum_j \|Q_N P_j \phi\|_1^2 \geq 2\Omega$. Then one shows

$$\left| \frac{1}{T}\int_0^T \|Q_N e^{iLt} P_p \phi\|_1^2 \, dt - \sum_j \|Q_N P_j \phi\|_1^2 \right| \to 0$$

as $T \to \infty$, with the convergence being uniform on compacts. We refer to [4] for details.

We are now ready to finish the proof of Theorem 3.2. Recall that we need to show (3.14), assuming (3.13). Let

$$\begin{aligned}
K_0 &\equiv \left\{ \phi \,\middle|\, \|\phi\|^2 \leq 1 \text{ and } \|\phi\|_1^2 \leq \frac{1}{\tau} \right\}, \\
K_1 &\equiv K_0 \cap \left\{ \phi \,\middle|\, \|(P_p - P_h)\phi\|^2 \geq \frac{\delta}{2} \right\}, && (3.18)\\
N_p &= N_p\left(K_0, \frac{\delta}{12}\right), \\
N_1 &= N_1\left(K_1, \frac{10}{\tau}\right), && (3.19)\\
N_2 &\equiv \min\left\{ N \,\middle|\, \lambda_N \geq \frac{48}{\tau\delta} \right\}, \\
N_c &\equiv \max\left\{ N_p, N_1, N_2 \right\}, \\
\omega &\equiv \min\left\{ \frac{\delta}{48}, \frac{1}{\tau \lambda_{N_1}} \right\}, \\
T_0 &= T_0(\tau,\delta) \geq \max\left\{ T_1\left(K_1, \frac{10}{\tau}\right), T_c\big(N_c, K_0, \omega\big) \right\}, && (3.20)\\
\varepsilon &< \varepsilon_0(\tau,\delta) \equiv \frac{2\tau}{1+\tau}\left(\int_0^{T_0} B(t)^2 \, dt \right)^{-1} \omega.
\end{aligned}$$

Note that K_0 is compact and hence so is K_1. Also, N_1 and T_1 are well defined because if $\phi \in K_1$, then $P_n\phi \notin P_h\mathcal{H}$ for some n by the definition of K_1. Since $P_n\phi$ is an eigenfunction of L, we have $P_n\phi \notin H^1$, and so $\sum_j \|P_j\phi\|_1^2 = \infty$ (note that we do not claim $P_p\phi \notin H^1$). This suggests that we could have used the version of Lemma 3.5 from [4] (with 3Ω replaced by ∞). We shall see later that the current form will be necessary in the proof of Theorem 2.1.

From (3.13) we know that either $\|(P_p - P_h)\phi_0\|^2 \geq \delta/2$ or $\|P_c\phi_0\|^2 \geq \delta/2$. Assume the former. Then $\phi_0 \in K_1$ and so by Lemma 3.5,

$$\frac{1}{T_0}\int_0^{T_0} \|Q_{N_1}e^{iLt}P_p\phi_0\|_1^2\,dt \geq \frac{10}{\tau}.$$

We also know from Lemma 3.3 and (3.20) that

$$\frac{1}{T_0}\int_0^{T_0} \|Q_{N_1}e^{iLt}P_c\phi_0\|^2\,dt \leq \frac{1}{T_0}\int_0^{T_0} \|Q_{N_c}e^{iLt}P_c\phi_0\|^2\,dt \leq \omega \leq \frac{1}{\tau\lambda_{N_1}}$$

and so

$$\frac{1}{T_0}\int_0^{T_0} \|Q_{N_1}e^{iLt}P_c\phi_0\|_1^2\,dt \leq \frac{1}{\tau}.$$

It follows using the triangle inequality for $\|\cdot\|_1$ and $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ that

$$\frac{1}{T_0}\int_0^{T_0} \|Q_{N_1}\phi^0(t)\|_1^2\,dt \geq \frac{4}{\tau}.$$

From (3.8) and (3.13) we know that

$$\|\phi^\varepsilon(t) - \phi^0(t)\|^2 \leq \frac{\varepsilon}{2}\left(\frac{1}{\tau}+1\right)\int_0^{T_0} B(t)^2\,dt \leq \omega \tag{3.21}$$

for $t \leq T_0$, and so

$$\frac{1}{T_0}\int_0^{T_0} \|Q_{N_1}(\phi^\varepsilon(t) - \phi^0(t))\|_1^2\,dt \leq \lambda_{N_1}\omega \leq \frac{1}{\tau}.$$

Using again $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ yields

$$\frac{1}{T_0}\int_0^{T_0} \|Q_{N_1}\phi^\varepsilon(t)\|_1^2\,dt \geq \frac{1}{\tau}$$

and (3.14) follows.

Next we assume (3.13) and $\|P_c\phi_0\|^2 \geq \delta/2$. Since $\phi_0 \in K_0$, Lemma 3.3 gives

$$\frac{1}{T_0}\int_0^{T_0} \|(I - Q_{N_c})e^{iLt}P_c\phi_0\|^2\,dt > \frac{\delta}{2} - \omega > \frac{\delta}{3}.$$

Lemma 3.4 and $N_c \geq N_p$ give

$$\frac{1}{T_0}\int_0^{T_0} \|(I - Q_{N_c})e^{iLt}P_p\phi_0\|^2\,dt < \frac{\delta}{12},$$

so we obtain

$$\frac{1}{T_0}\int_0^{T_0} \|(I - Q_{N_c})\phi^0(t)\|^2\,dt > \frac{\delta}{12}.$$

Applying (3.21) yields

$$\frac{1}{T_0}\int_0^{T_0} \|(I - Q_{N_c})\phi^\varepsilon(t)\|^2\,dt > \frac{\delta}{24} - \omega \geq \frac{\delta}{48}$$

and so

$$\frac{1}{T_0}\int_0^{T_0}\|(I-Q_{N_c})\phi^\varepsilon(t)\|_1^2\,dt > \frac{\delta}{48}\lambda_{N_c} \ge \frac{\delta}{48}\lambda_{N_2} \ge \frac{1}{\tau}.$$

Again (3.14) follows and the proof of Theorem 3.2 is complete. □

4. The Time-Periodic Case

Theorem 2.3 has a natural extension to the case of time-periodic family of operators L_t in place of L [18]. We provide here the corresponding extension of Theorem 2.4.

Let Γ be as before and let L_t be a periodic family of self-adjoint operators on $\mathcal{H}$ such that for some $C<\infty$, all $\psi \in H^1(\Gamma)$, and all $t\in\mathbb{R}$,

$$\|L_t\psi\|_{\mathcal{H}} \le C\|\psi\|_{H^1(\Gamma)}. \tag{4.1}$$

Without loss of generality assume L_t has period 1. Let $\{U_t\}_{t\in\mathbb{R}}$ be a strongly continuous family of unitary operators on $\mathcal{H}$ such that for any $\phi_0 \in H^1(\Gamma)$, the function $\phi^0(t) \equiv U_t\phi_0$ satisfies

$$\frac{d}{dt}\phi^0(t) = iL_t\phi^0(t) \tag{4.2}$$

for almost every t. Notice that if $L_t \equiv L$ is constant, then $U_t = e^{iLt}$. Finally, assume there is a locally bounded function $B(t)$ such that for any $\psi\in H^1(\Gamma)$ and $t\in\mathbb{R}$,

$$\|U_t\psi\|_{H^1(\Gamma)} \le B(t)\|\psi\|_{H^1(\Gamma)}. \tag{4.3}$$

We denote by P_h the projection onto the closed subspace $P_h\mathcal{H}\subseteq\mathcal{H}$ generated by all $H^1(\Gamma)$ eigenfunctions of U_1 (these coincide with those of L when $L_t\equiv L$) and we let $\phi^A(t)$ be the solution of

$$\frac{d}{dt}\phi^A(t) = iAL_{At}\phi^A(t) - \Gamma\phi^A(t), \qquad \phi^A(0)=\phi_0. \tag{4.4}$$

Note that this is the right choice of the fast dissipative evolution to consider since the orbits of the fast free evolution $\frac{d}{dt}\phi(t) = iAL_{At}\phi(t)$ coincide with those of (4.2).

Theorem 4.1. *Let Γ be a self-adjoint, non-negative, unbounded operator with a discrete spectrum and let L_t and U_t satisfy conditions* (4.1)–(4.3)*. Then for any $\tau,\delta>0$ there exists $A_0(\tau,\delta)$ such that for any $A>A_0(\tau,\delta)$ and any $\phi_0\in\mathcal{H}$ with $\|\phi_0\|_{\mathcal{H}}\le 1$, the Lebesgue measure of the set of times $t\ge 0$ for which the solution $\phi^A(t)$ of* (4.4) *satisfies*

$$\|(I-P_h)U^*_{At}\phi^A(t)\|^2_{\mathcal{H}} \ge \delta \tag{4.5}$$

is smaller than τ.

Remark. Let $U_{s,t}\equiv U_tU_s^*$ with U_s^* the adjoint of U_s. Then $B(t), B(-t)<\infty$ and periodicity of L_t guarantee that $U_t = U_{0,t}$ maps H^1 eigenfunctions of $U_1 = U_{0,1}$ onto those of $U_{t,t+1}$, and that $U_{t,0} = U_t^*$ maps H^1 eigenfunctions of $U_{t,t+1}$ onto those of U_1. Hence $P_{t,h}\equiv U_tP_h$ is the projection on the subspace of $\mathcal{H}$ generated by all H^1 eigenfunctions of $U_{t,t+1}$, and so

$$\|(I-P_h)U_t^*\phi\| = \|(I-P_{t,h})\phi\|.$$

This illuminates (4.5). Notice also that $P_{t+1,h} = P_{t,h}$ by definition.

We will now sketch the proof, which follows the lines of the proof of Theorem 2.4. The point is to obtain Theorem 3.2 with (c) replaced by

$$\|(I-P_h)U_t^*\phi^\varepsilon(t)\|^2<\delta \quad \text{and neither (a) nor (b) holds} \tag{4.6}$$

(from which Theorem 4.1 follows immediately). This is done in two steps.

First we fix any $\gamma\in[0,1)$. We then obtain Theorem 3.2 with (b) replaced by

$$\sum_{n=0}^{T_0-2}\|\phi^\varepsilon(\lceil t\rceil+\gamma+n)\|_1^2>\frac{2T_0}{\tau} \tag{4.7}$$

and (c) by (4.6). Here $\lceil t\rceil$ is the least integer not smaller than t (and we let $\beta\equiv\lceil t\rceil-t$), and the obtained T_0,ε_0 additionally depend on γ. The proof extends directly with the following changes. After renaming $\phi^\varepsilon(t)$ to ϕ_0, we replace all integrals $\int_0^{T_0}\dots dt$ in the proof by the sums $\sum_{n=0}^{T_0-2}$, with the argument t inside the integrals replaced by the argument $\beta+\gamma+n$ inside the sums. The role of ϕ_0 is then played by $\phi^0(\beta+\gamma)$, and that of e^{iLt} by $U^n_{\gamma,\gamma+1}$ (since $\phi^0(\beta+\gamma+n)=U^n_{\gamma,\gamma+1}\phi^0(\beta+\gamma)$). The assumption (3.13) now reads

$$\|\phi_0\|^2\le1,\quad \|\phi_0\|_1^2\le\frac{1}{\tau},\quad\text{and}\quad \|(I-P_h)U_t^*\phi_0\|^2\ge\delta,$$

which together with

$$\|(I-P_{\gamma,h})\phi^0(\beta+\gamma)\|=\|U^*_{\lceil t\rceil}(I-P_h)U^*_\gamma\phi^0(\beta+\gamma)\|=\|(I-P_h)U^*_{t+\beta+\gamma}\phi^0(\beta+\gamma)\|=\|(I-P_h)U_t^*\phi_0\|$$

guarantees

$$\|\phi^0(\beta+\gamma)\|^2\le1,\quad \|\phi^0(\beta+\gamma)\|_1^2\le\frac{b}{\tau},\quad\text{and}\quad \|(I-P_{\gamma,h})\phi^0(\beta+\gamma)\|^2\ge\delta,$$

where $b\equiv\sup_{t\in[0,2]}B(t)$. From this (4.7) follows as in Section 3, with the definitions of K_0 and K_1 involving $\|\phi\|_1^2\le b/\tau$ and $\|(P_{\gamma,p}-P_{\gamma,h}))\phi\|^2\ge\delta/2$, respectively, and with τ replaced by $\tau/2$ in order to account for the extra factor of two in (4.7).

Next we notice that we can actually pick T_0,ε_0 uniformly for all γ inside a set G of measure $\frac12$. This is because the maximum in (3.20) is finite for each γ, and so the same T_0 (and hence the same ε_0) can be chosen for all γ outside of a set of a small measure. Integrating (4.7) over G now gives Theorem 3.2 with (a) and (b) the same as in (3.10) and (3.11), and (c) replaced by (4.6). This finishes the proof.

5. Proof of Theorem 2.1: Part I

We devote the next two sections to the proof of Theorem 2.1. We will consider $D=\mathbb{R}\times\mathbb{T}$ and since the case $D=\mathbb{R}^2$ is almost identical, we will just indicate along the way where adjustments for this setting are required. We will also assume that u has period one in each coordinate, that is, $\mathcal{C}=\mathbb{T}^2$. The general case is again identical.

In this section we prove that if Theorem 2.1(iv) holds, then so do parts (i)–(iii). Let us therefore assume that the 1-periodic incompressible Lipschitz flow u leaves no open bounded subset of D invariant and has no $H^1(\mathbb{T}^2)$ eigenfunctions except possibly with eigenvalue zero (i.e., first integrals). We will then show that Theorem 2.1(i)–(iii) hold.

Let us start with a description of the main idea. Fix any $\tau, \delta > 0$ and let $\|\phi_0\|_{L^2} \le 1$ (we will actually take $\|\phi_0\|_{L^1} \le 1$ to obtain the desired $L^1 \to L^\infty$ bounds). As mentioned in the Introduction, we periodize the domain and consider the solution ϕ^A of (1.1) on $\mathcal{M} \equiv k\mathbb{T} \times \mathbb{T}$ with $k \gg 1$ depending on τ, δ (we use $\mathcal{M} \equiv (k\mathbb{T})^2$ when $D = \mathbb{R}^2$). Here $k\mathbb{T}$ for $k \in \mathbb{N}$ is the interval $[0, k]$ with 0 and k identified, and the ϕ^A on $\mathcal{M}$ will majorize the $|\phi^A|$ on D. We will show that on $\mathcal{M}$ the flow u also cannot have H^1 eigenfunctions other than the first integrals (i.e., the operator $u \cdot \nabla$ on $\mathcal{M}$ can only have $H^1(\mathcal{M})$ eigenfunctions with eigenvalue zero). We will then show that if k and A are large enough, $\|\phi^A(\tau)\|_{L^\infty}$ will be small.

To this end we notice that we are now in the setting of our main abstract result because the Laplacian on $\mathcal{M}$ has a discrete spectrum. Similarly to (3.5), we now have

$$\frac{d}{dt}\|\phi^A\|_{L^2}^2 = -2\|\phi^A\|_{H^1}^2, \tag{5.1}$$

and so $\|\phi^A\|_{L^2}$ will decay quickly as long as $\|\phi^A\|_{H^1}$ stays large. Theorem 3.2 for ϕ^A says that if A is large, then the latter can only be prevented by ϕ^A becoming close to an H^1 first integral ψ of u.

If $\|\psi\|_{L^\infty}$ is small, $\|\phi^A\|_{L^\infty}$ will also become small after a short time interval during which Lemma 5.4 below takes care of $\phi^A - \psi$ (which is small). In this case we will be done.

If, on the other hand, $\|\psi\|_{L^\infty}$ is large, then we will show that ψ has to be large on a long streamline of u. More precisely, we will show using $\dim(\mathcal{M}) = 2$ that under our hypotheses ψ has to be continuous, constant on the streamlines of u on $\mathcal{M}$, and that long streamlines must be dense. As a result, we will obtain that $\|\psi\|_{H^1}$ is large (again using $\dim(\mathcal{M}) = 2$). We would like to conclude that $\|\phi^A\|_{H^1}$ must also be large but we only know that $\|\phi^A - \psi\|_{L^2}$ is small, which does not guarantee this. Instead, we will need to slightly adjust the proof of Theorem 2.4 to obtain that $\|\phi^A\|_{H^1}$ will be large on average during a short time interval (essentially showing that the fast flow quickly aligns ϕ^A with ψ). Thus the fast decay of $\|\phi^A\|_{L^2}$ can only be stopped by $\|\phi^A\|_{L^\infty}$ becoming small. Since this fast decay can only be sustained for a short time due to $\|\phi_0\|_{L^2} \le 1$, we will indeed obtain that $\|\phi^A(\tau)\|_{L^\infty}$ is small. Lemma 5.4 and interpolation will take care of the rest.

In what follows we make this heuristic rigorous. A *stream function* for u is a function $H \in C^1(D)$ with values in $\mathbb{R}$ if $D = \mathbb{R}^2$ and in $a\mathbb{T}$ for some $a > 0$ if $D = \mathbb{R} \times \mathbb{T}$ such that

$$u(x_1, x_2) = \big(u_1(x_1, x_2), u_2(x_1, x_2)\big) = \nabla^\perp H(x_1, x_2) \equiv \bigg(-\frac{\partial H}{\partial x_2}(x_1, x_2), \frac{\partial H}{\partial x_1}(x_1, x_2)\bigg). \tag{5.2}$$

If $D = \mathbb{R}^2$, then we can take

$$H(x_1, x_2) \equiv \int_0^{x_1} u_2(s, 0)\, ds - \int_0^{x_2} u_1(x_1, s)\, ds,$$

which satisfies (5.2) because u is incompressible and so

$$\int_0^{x_1} u_2(s, 0)\, ds - \int_0^{x_2} u_1(x_1, s)\, ds = -\int_0^{x_2} u_1(0, s)\, ds + \int_0^{x_1} u_2(s, x_2)\, ds.$$

For the same reason and from periodicity of u we also have that $\tilde{a} \equiv H(x_1, x_2 + 1) - H(x_1, x_2)$ is independent of (x_1, x_2). Let $a \equiv |\tilde{a}|$ if $\tilde{a} \neq 0$ and let a be any positive number otherwise.

Changing H to $(H \mod a\mathbb{Z})$ gives a C^1 stream function with values in $a\mathbb{T}$ which is 1-periodic in x_2, that is, a stream function on $\mathbb{R}\times\mathbb{T}$. Without loss of generality we will assume $a\equiv 1$, as this can be achieved by changing u to $a^{-1}u$. Note also that

$$u\cdot\nabla H\equiv 0 \tag{5.3}$$

by (5.2), so that H is constant on the streamlines of u.

Lemma 5.1. *Let u be a 1-periodic incompressible Lipschitz flow. Then u leaves no open bounded subset of D invariant if and only if the union of unbounded streamlines of u is a dense subset of D.*

Proof. If the unbounded streamlines of u are dense in D, clearly no open bounded subsets of D are left invariant by the flow.

Assume now that the unbounded streamlines of u are not dense in D and let $Y\subset D$ be open bounded and such that all streamlines intersecting Y are bounded. If $u\equiv 0$ on Y, then Y is an open bounded set invariant under u.

Otherwise take $x_0\in Y$ such that $u(x_0)\neq 0$. This means that $0\neq\nabla H(x_0)\perp u(x_0)$, and since $H\in C^1$, there is a neighborhood $V\subseteq Y$ of x_0 such that for each $y_0\in V$ the set

$$\big\{x\in V \,\big|\, H(x)=H(y_0)\big\}$$

is precisely the intersection of V with the streamline passing through y_0. Pick V small enough so that there is $t_0>0$ such that $X(V,t_0)\cap V=\emptyset$, with X the solution of (2.1) on D. This is possible because u is continuous. Finally, we let

$$W_0\equiv\big\{x\in V \,\big|\, |X(x,t)|\le M \text{ for all } t\in\mathbb{R}\big\}$$

with M large enough so that $|W_0|>0$. This is possible because all streamlines intersecting V are bounded.

Let $W_j\equiv X(W_0,jt_0)$, so that $W_j\subseteq B(0,M)$ and incompressibility of u gives $|W_j|=|W_0|>0$. Hence there must be $j<k$ with $W_j\cap W_k\neq\emptyset$, which in turn gives existence of $y_0\in W_0\cap W_m$ for $m\equiv k-j>0$ (and then obviously we must have $m\ge 2$). So there is $y\in W_0$ such that $X(t,y)=y_0$ for some $t\in[(m-1)t_0,(m+1)t_0]$. But then $H(y)=H(y_0)$, and so y must lie on the streamline through y_0. It follows that this non-trivial streamline $\mathcal{S}$ is closed, that is, $X(y,\tau)=y$ for some $\tau\ge(m-1)t_0>0$.

$\mathcal{S}$ is a simple closed curve in D and therefore it is either homotopic to a point (which is always the case if $D=\mathbb{R}^2$) or to $\{0\}\times\mathbb{T}$ (i.e., it winds around $D=\mathbb{R}\times\mathbb{T}$). In the first case the interior of $\mathcal{S}$ is an open bounded set invariant under u because streamlines of u cannot intersect. In the second case we let $\mathcal{S}'\equiv\mathcal{S}+(1,0)$. Then $\mathcal{S}'$ is also a streamline of u and $\mathcal{S}'\neq\mathcal{S}$, due to periodicity of u and boundedness of $\mathcal{S}$. Then the open bounded domain between $\mathcal{S}$ and $\mathcal{S}'$, homotopic to a cylinder, is invariant under u. □

Lemma 5.2. *Let u be an incompressible Lipschitz flow on $\mathcal{M}\equiv k\mathbb{T}\times l\mathbb{T}$ and let $\psi\in H^1(\mathcal{M})$ satisfy $u\cdot\nabla\psi\equiv 0$. Then ψ is constant on each streamline of u and continuous at each $x\in\mathcal{M}$ for which $u(x)\neq 0$. Moreover, if for some $\varepsilon>0$ the union of streamlines of u of diameter at least ε is dense in $\mathcal{M}$, then ψ is continuous.*

Proof. Let $x_0 \in \mathcal{M}$ be such that $u(x_0) \neq 0$, let $v \perp u(x_0)$ have length 1, and set $x_s \equiv x_0 + sv$ for $s \in \mathbb{R}$. Define $g(t,s) \equiv X(x_s,t)$ with X from (2.1). Since u is Lipschitz, g is a bilipschitz diffeomorphism between some neighborhoods of $0 \in \mathbb{R}^2$ and x_0. This means that the H^1 function $\omega(\cdot) \equiv \psi(g(\cdot))$ satisfies $(1,0) \cdot \nabla\omega \equiv 0$. That is, $\omega(t,s) = \tilde{\omega}(s)$ almost everywhere, with $\tilde{\omega}$ an H^1 function of a single variable and so continuous on a neighborhood of 0. We conclude that ψ is continuous on a neighborhood of x_0 (after possibly changing it on a measure-zero set). This means that ψ is (equivalent to a function) continuous at each x such that $u(x) \neq 0$. This and the dependency of ω on s only means that ψ is constant on all non-trivial streamlines. It is obviously constant on the trivial ones, too.

Next assume that the union of streamlines of diameter at least $\varepsilon > 0$ is dense in $\mathcal{M}$. It is sufficient to consider $\varepsilon = 1$, the general case is identical. The open set R of all x with $u(x) \neq 0$ is dense in D. We will now show that $\psi|_R$ can be continuously extended to $\mathcal{M}$. Assume the contrary, that is, there is $x_0 \in \mathcal{M}$ and $x_n, z_n \in R$ with $\lim x_n = \lim z_n = x_0$ such that either $\lim |\psi(x_n)| = \infty$ or $\lim \psi(x_n) \neq \lim \psi(z_n)$. We can assume without loss of generality that $x_n, z_n \in S$, the union of streamlines with diameter at least 1, because S is dense in R and ψ is continuous on R.

If $\lim |\psi(x_n)| = \infty$, then for each $M < \infty$ there is a curve joining the inner and outer perimeter of the annulus $B_2 \equiv B(x_0, \frac{1}{2}) \setminus B(x_0, \frac{1}{4})$ on which $|\psi|$ is continuous and larger than M. Namely, it is a part of the streamline going through $x_n \in B(x_0, \frac{1}{4})$ (which cannot be completely contained inside $B(x_0, \frac{1}{2})$, and on which ψ is constant). On the other hand, we have $|J| \geq \frac{1}{8}$ where $J \subseteq [\frac{1}{4}, \frac{1}{2}]$ is the set of all r for which the measure of all $\theta \in [0, 2\pi]$ such that $|\psi(x_0 + re^{i\theta})| \leq 4\|\psi\|_{L^2}$ is positive. But then

$$\|\psi\|_{H^1}^2 \geq \int_{B_2} |\nabla\psi|^2 dx \geq \int_J \int_0^{2\pi} r \left| \frac{1}{r} \frac{\partial\psi}{\partial\theta} \right|^2 d\theta dr \geq \int_J \frac{1}{2\pi r} \left(\int_0^{2\pi} \left| \frac{\partial\psi}{\partial\theta} \right| d\theta \right)^2 dr \geq \frac{(M - 4\|\psi\|_{L^2})^2}{8\pi}$$

using the Schwartz inequality in the third step. Since the rightmost expression diverges as $M \to \infty$, we have a contradiction.

If on the other hand $\lim \psi(x_n) = L_1 \neq L_2 = \lim \psi(z_n)$, then for each $n \in \mathbb{N}$ there must be two curves joining the inner and outer perimeters of the annulus $B_n \equiv B(x_0, \frac{1}{2}) \setminus B(x_0, 2^{-n})$, on which ψ is continuous and has constant values a_n and b_n, respectively, with $|a_n - b_n| \geq \frac{1}{2}|L_1 - L_2|$. A similar argument as above gives

$$\|\psi\|_{H^1}^2 \geq \int_{B_n} |\nabla\psi|^2 dx \geq \int_{2^{-n}}^{1/2} \frac{1}{r} \int_0^{2\pi} \left| \frac{\partial\psi}{\partial\theta} \right|^2 d\theta dr \geq \left| \frac{L_1 - L_2}{2} \right|^2 \int_{2^{-n}}^{1/2} \frac{1}{2\pi r} dr,$$

with a contradiction when $n \to \infty$.

Hence $\psi|_R$ has a continuous extension ω to $\mathcal{M}$ and it remains to show $\psi = \omega$ almost everywhere. Assume this is not the case and let $x_0 \in \mathcal{M}$ be a Lebesgue point of the set P_ε of all $x \in \mathbb{T}^2$ such that $|\psi(x) - \omega(x)| > 2\varepsilon$ (by the hypothesis, $|P_\varepsilon| > 0$ for some $\varepsilon > 0$). Then for some $r > 0$ and all $x \in B(x_0, r)$ and $z \in B(x_0, r) \cap P_\varepsilon$ we have

$$|\psi(z) - \omega(x)| > \varepsilon \tag{5.4}$$

because ω is continuous. Since x_0 is a Lebesgue point of P_ε, we have

$$|B(x_0,r)\cap P_\varepsilon|\cdot|B(x_0,r)|^{-1}\to 1$$

as $r\to 0$. Hence for any $\delta>0$ and a small enough δ-dependent r_0, there is a set J with $|J|\ge(1-\delta)r_0$ of $r\in[0,r_0]$ such that $|\{\theta\,|\,x_0+re^{i\theta}\in P_\varepsilon\}|>0$. Again we can find a curve joining the inner and outer perimeter of the annulus $B\equiv B(x_0,r_0)\setminus B(x_0,\delta r_0)$ on which ψ is continuous and equal to ω, and an argument as above together with (5.4) gives

$$\|\psi\|_{H^1}^2\ge\int_B|\nabla\psi|^2dx\ge\int_{J\cap[\delta r_0,r_0]}\frac{1}{r}\int_0^{2\pi}\left|\frac{\partial\psi}{\partial\theta}\right|^2d\theta dr\ge\varepsilon^2\int_{2\delta r_0}^{r_0}\frac{1}{2\pi r}dr=\frac{\varepsilon^2}{2\pi}|\log(2\delta)|.$$

Taking $\delta\to 0$ yields a contradiction, so ψ must be continuous. ☐

Lemma 5.3. *Let u be a 1-periodic incompressible Lipschitz flow on $\mathbb{R}^n$ and let $\mathcal{M}\equiv\prod_{j=1}^n k_j\mathbb{T}$ for some $k_j\in\mathbb{N}$.*

(i) *If ψ is an H^1 eigenfunction of u on $\mathcal{M}$, then $|\psi|$ is an H^1 eigenfunction of u on $\mathcal{M}$ with eigenvalue 0.*

(ii) *The flow u on $\mathcal{M}$ has an H^1 eigenfunction with a non-zero eigenvalue if and only if the same is true for u on $\mathbb{T}^n$.*

(iii) *The flow u on $\mathcal{M}$ has a non-constant H^1 eigenfunction if and only if the same is true for u on $\mathbb{T}^n$.*

Remark. The exclusion of constants in (iii) is natural as these are always eigenfunctions of u. We will us this part in Section 9.

Proof. (i) is an easy computation using

$$\big(\nabla|\psi|\big)(x)=\begin{cases}\frac{\bar\psi(x)\nabla\psi(x)+\psi(x)\nabla\bar\psi(x)}{2|\psi(x)|} & \psi(x)\neq 0,\\ 0 & \psi(x)=0,\end{cases}\tag{5.5}$$

the fact that u is real, and that all eigenvalues of $u\cdot\nabla$ are purely imaginary.

(ii) Let us first consider the case $\mathcal{M}=2\mathbb{T}\times\mathbb{T}^{n-1}$. If ψ is an H^1 eigenfunction of u on $\mathbb{T}^n$, then $\phi(x_1,x')\equiv\psi(\{x_1\},x')$ is obviously an H^1 eigenfunction on $\mathcal{M}$ with the same eigenvalue (here $\{x_1\}$ is the fractional part of x_1 and $x'=(x_2,\dots,x_n)$). This proves one implication.

Let us now assume ψ is an H^1 eigenfunction of u on $\mathcal{M}$ with eigenvalue $i\lambda\in i\mathbb{R}$, and define $\psi_e(x_1,x')\equiv\frac12[\psi(x_1,x')+\psi(x_1+1,x')]$ and $\psi_o(x_1,x')\equiv\frac12[\psi(x_1,x')-\psi(x_1+1,x')]$. Periodicity of u shows that ψ_e,ψ_o are also H^1 eigenfunctions on $\mathcal{M}$ with the same eigenvalue. If $\psi_e\not\equiv 0$ then it is an H^1 eigenfunction on $\mathbb{T}^n$ because it is 1-periodic. If $\psi_e\equiv 0$, then $\psi_o\not\equiv 0$, and we let $\phi\equiv\psi_o^2/|\psi_o|$. Again using (5.5) we find that ϕ is an H^1 eigenfunction of u on $\mathcal{M}$, with eigenvalue $2i\lambda$. But ϕ is 1-periodic (because $\psi_o(x_1+1,x')=-\psi_o(x_1,x')$), and so it is also an H^1 eigenfunction of u on $\mathbb{T}^n$. Since $i\lambda$ and $2i\lambda$ are either both zero or both non-zero, this proves (ii) for $\mathcal{M}=2\mathbb{T}\times\mathbb{T}^{n-1}$.

If now $\mathcal{M}=k\mathbb{T}\times\mathbb{T}^{n-1}$, we use the same argument but with ψ_e,ψ_o replaced by

$$\left\{\psi_j(x_1,x')\equiv\frac{1}{k}\sum_{m=0}^{k-1}e^{2\pi ijm/k}\psi(x_1+m,x')\right\}_{j=0}^{k-1}$$

and $\phi \equiv \psi_j^k/|\psi_j|^{k-1} \in H^1$ associated to the eigenvalue $ki\lambda$ (when $\psi_j \not\equiv 0$). The general case is treated by subsequently repeating this "unfolding" for each coordinate.

(iii) The proof is essentially identical to that of (ii) after noting that $\psi_j^k/|\psi_j|^{k-1}$ cannot be a constant function when $\psi_j \in H^1(\mathcal{M})$ is non-constant. $\square$

The final lemma is based on [4, 5].

Lemma 5.4. *For each $p \in [1, \infty]$ and each integer $d \geq 1$ there exists $C(d) \geq 1$ such that for any $D = \mathbb{R}^n \times \prod_{j=1}^m k_j\mathbb{T}$ with $n + m = d$ and $n, m \geq 0$, any 1-periodic incompressible flow $v \in \operatorname{Lip}(D)$, any $\psi_0 \in L^1(D)$, and any $t \leq 1$ the solution of* (2.3) *on D satisfies*

$$\|\psi(\cdot, t)\|_{L^\infty(D)} \leq C(d)t^{-d/2p}\|\psi_0\|_{L^p(D)}. \tag{5.6}$$

Proof. Interpolation and (2.6) for $p = \infty$ imply that we only need to obtain (5.6) for $p = 1$.

Consider first $d = 2$. When $D = \mathbb{T}^2$ and $\bar{\psi}_0 \equiv |D|^{-1}\int_D \psi_0(x)dx = 0$ (in which case ψ is mean zero at all times because the evolution given by (2.3) preserves its mean), then this is just Lemma 3.3 in [5]. That is,

$$\|\psi(\cdot, t) - \bar{\psi}_0\|_{L^\infty} \leq Ct^{-d/2}\|\psi_0 - \bar{\psi}_0\|_{L^1} \tag{5.7}$$

for any ψ_0. Using

$$\|\bar{\psi}_0\|_{L^\infty} = \|\bar{\psi}_0\|_{L^1} \leq \|\psi_0\|_{L^1}$$

and $t \leq 1$, we obtain (5.6) for $D = \mathbb{T}^2$ and $p = 1$.

Take now any other D with $n + m = d = 2$ and let $\tilde{\psi}$ solve (2.3) on $\mathbb{T}^2$ with $\tilde{\psi}_0(x_1, x_2) \equiv \sup_{j,m} |\psi_0(x_1 + j, x_2 + m)|$. Then by the comparison principle [26], $\tilde{\psi}(x_1, x_2, t) \geq \sup_{j,m} |\psi(x_1 + j, x_2 + m, t)|$, and so

$$\|\psi(\cdot, t)\|_{L^\infty} \leq \|\tilde{\psi}(\cdot, t)\|_{L^\infty} \leq Ct^{-d/2}\|\tilde{\psi}_0\|_{L^1} \leq Ct^{-d/2}\|\psi_0\|_{L^1}$$

with the same C (we then have $C(2) = \max\{C, 1\}$).

If $d \geq 3$, then the proof is identical, using Lemma 5.6 in [4] in place of Lemma 3.3 in [5] to obtain (5.7). Finally, the case $d = 1$ is obvious since the only incompressible flows in one dimension are the constant ones, so (2.3) is just the heat equation in a moving frame. $\square$

Next we show that given our assumptions on u, we have for each fixed $\tau > 0$,

$$\|\mathcal{P}_\tau(Au)\|_{L^1(D) \to L^\infty(D)} \to 0 \qquad \text{as } A \to \infty. \tag{5.8}$$

More precisely, we let $\tilde{\phi}_0 \in L^1(D)$ be such that

$$\|\tilde{\phi}_0\|_{L^1} \leq C^{-1/2}\tau^{1/2} \tag{5.9}$$

with $C \equiv C(2)$, and we will show that for each $\delta \in (0, 1)$ and $A > A_1(\tau, \delta)$ (for some $A_1(\tau, \delta) < \infty$), the solution $\tilde{\phi}^A$ of (1.1) with initial condition $\tilde{\phi}_0$ satisfies

$$\|\tilde{\phi}^A(\cdot, 3\tau)\|_{L^\infty} \leq 3\delta. \tag{5.10}$$

Equality (2.6) with $p = \infty$ shows that it is only necessary to consider $\tau \leq 1$.

We will actually replace the problem on D by the same problem on $\mathcal{M} \equiv k\mathbb{T} \times \mathbb{T}$, with $k > 270/\tau\delta^2$ and with $\tilde{\phi}_0$ replaced by $\sup_{j \in \mathbb{Z}} |\tilde{\phi}_0(x_1 + jk, x_2)|$. This new $\tilde{\phi}_0$ also satisfies (5.9), and by the argument in the proof of Lemma 5.4, it is sufficient to show (5.10) for the new $\tilde{\phi}^A$. Note that if $D = \mathbb{R}^2$, then we consider the problem on $\mathcal{M} \equiv (k\mathbb{T})^2$ and change $\tilde{\phi}_0$ accordingly. In either case Lemma 5.3 shows that u can only have H^1 eigenfunctions with eigenvalue 0 on $\mathcal{M}$.

From (5.6) and (5.9) we get $\|\tilde{\phi}^A(\cdot,\tau)\|_{L^\infty} \le C^{1/2}\tau^{-1/2}$ and (2.6) gives $\|\tilde{\phi}^A(\cdot,\tau)\|_{L^1} \le \|\tilde{\phi}_0\|_{L^1} \le C^{-1/2}\tau^{1/2}$, so $\|\tilde{\phi}^A(\cdot,\tau)\|_{L^2} \le 1$. It is important here that C is independent of k and A. Let us now define $\phi_0(x) \equiv \tilde{\phi}^A(x,\tau)$ and $\phi^A(x,t) \equiv \tilde{\phi}^A(x,\tau+t)$ so that $\|\phi_0\|_{L^2} \le 1$ and ϕ^A solves (1.1).

We now use the abstract framework of Theorem 2.4 with $\mathcal{H} \equiv L^2(\mathcal{M})$, $\Gamma \equiv -\Delta$, and $L \equiv iu\cdot\nabla$. It is easy to see [4] that the hypotheses of Theorem 2.4 are satisfied in this setting since $\mathcal{M}$ is a compact manifold and u is Lipschitz. However, instead of directly applying the result we will need to adjust the proof a little, as has been mentioned at the beginning of the present section. Namely, we replace (3.18) by

$$K_1 \equiv K_0 \cap \left\{ \phi \,\middle|\, \|(P_p - P_h)\phi\|_{L^2}^2 \ge \frac{\tilde{\delta}}{2} \text{ or } \left|W_{\phi,\delta}\right| \ge \tilde{\delta} \right\}, \tag{5.11}$$

where $\tilde{\delta} \equiv \delta^2\tau^2/C^2$ and

$$W_{\phi,\gamma} \equiv \left\{ x \,\middle|\, |(P_h\phi)(x)| \ge \gamma \right\}.$$

Note that K_1 is again compact because $\phi_n \to \phi_\infty$ implies $P_h\phi_n \to P_h\phi_\infty$, and so (3.19) will be meaningful provided we show

$$\sum_j \|P_j\phi\|_{H^1}^2 \ge \frac{30}{\tau} \tag{5.12}$$

for all $\phi \in K_1$.

So assume $\phi \in K_1$. Since u can only have H^1 eigenfunctions for eigenvalue zero, $P_h\mathcal{H}$ must be a subspace of the eigenspace of $u\cdot\nabla$ corresponding to the eigenvalue zero. This shows that if $(P_p - P_h)\phi \ne 0$, then at least one of $P_j\phi \notin H^1$ and so $\sum_j \|P_j\phi\|_{H^1}^2 = \infty > \frac{30}{\tau}$. If, on the other hand, $(P_p - P_h)\phi = 0$, then we must have $|W_{\phi,\delta}| \ge \tilde{\delta}$ (and thus also $P_p\phi = P_h\phi \ne 0$). If now $P_h\phi \notin H^1$ (recall that $P_h\mathcal{H}$ need not be contained in H^1), then again $\sum_j \|P_j\phi\|_{H^1}^2 = \|P_h\phi\|_{H^1}^2 = \infty$. If $P_h\phi \in H^1$, then $\psi \equiv P_h\phi$ is an H^1 eigenfunction of u with eigenvalue zero. Lemma 5.2 shows that ψ is continuous, and $|W_{\phi,\delta}| > 0$ together with the density of unbounded streamlines of u (by Lemma 5.1) and the fact that ψ is constant on them imply that there is a streamline of u joining $\{0\} \times \mathbb{T}$ and $\{k\} \times \mathbb{T}$ inside $[0,k] \times \mathbb{T}$ on which $|\psi|$ is greater than $2\delta/3$. This is because any unbounded streamline must wind infinitely many times around $\mathcal{M}$ in the first coordinate. Since obviously $\|\psi\|_{L^2} \le 1$ and $k > 9\delta^{-2}$, the same reasoning shows that there must also be a streamline of u joining $\{0\} \times \mathbb{T}$ and $\{k\} \times \mathbb{T}$ on which $|\psi|$ is smaller than $\delta/3$. Therefore

$$\|\psi\|_{H^1}^2 \ge \int_0^k \int_0^1 \left|\frac{\partial\psi}{\partial x_2}\right|^2 dx_2 dx_1 \ge \int_0^k \left(\int_0^1 \left|\frac{\partial\psi}{\partial x_2}\right| dx_2 \right)^2 dx_1 \ge \int_0^k \left(\frac{\delta}{3}\right)^2 dx_1 \ge \frac{30}{\tau}.$$

In particular, $\sum_j \|P_j\phi\|_{H^1}^2 = \|\psi\|_{H^1}^2 \ge \frac{30}{\tau}$, and hence (5.12) holds for all $\phi \in K_1$.

We note that in the case $D = \mathbb{R}^2$ the last argument has to be changed slightly. Namely, we obtain that there must be a streamline of u joining either $\{0\} \times k\mathbb{T}$ and $\{k\} \times k\mathbb{T}$ inside

$[0,k]\times k\mathbb{T}$, or one joining $k\mathbb{T}\times\{0\}$ and $k\mathbb{T}\times\{k\}$ inside $k\mathbb{T}\times[0,k]$, on which $|\psi|$ is greater than $2\delta/3$. Assume the former. Then for each $x_1\in[0,k]$ there is $x_2(x_1)$ such that $|\psi(x_1,x_2(x_1))|>2\delta/3$. But since $\|\psi\|_{L^2}\le 1$ and $k>18\delta^{-2}$, there must be at least measure $\frac{k}{2}$ set of $x_1\in[0,k]$ such that $|\psi(x_1,x_2(x_1)+z(x_1))|<\delta/3$ for some $|z(x_1)|\le\frac{1}{2}$. As above, $\|\psi\|_{H^1}^2\ge\frac{30}{\tau}$ follows.

We have thus shown that N_1,T_1 are well defined, and so Theorem 3.2(b) must hold whenever $\phi^\varepsilon(\cdot,t)\in K_1$ (with $\varepsilon=A^{-1}$ and ϕ^ε as in Section 3). This allows us to strengthen the condition in Theorem 3.2(c) by adding the requirement $|W_{\phi^\varepsilon(\cdot,t),\delta}|<\tilde\delta$. Ultimately we obtain Theorem 2.4 on $L^2(\mathcal{M})$ with the conclusion that if $\|\phi_0\|_{L^2}\le 1$ (which is our case) and $A>A_1(\tau,\delta)$ (with A_1 only dependent on τ,δ because $k=k(\tau,\delta)$ and C is universal), then the set of all times t for which

$$\|(I-P_h)\phi^A(\cdot,t)\|_{L^2}^2\ge\tilde\delta \qquad\text{or}\qquad |W_{\phi^A(\cdot,t),\delta}|\ge\tilde\delta$$

has measure less than τ. Since $\tilde\phi^A(x,t)=\phi^A(x,t-\tau)$, this says that there must be a time $t_0\in[\tau,2\tau]$ such that

$$\|(I-P_h)\tilde\phi^A(\cdot,t_0)\|_{L^2}^2<\tilde\delta \qquad\text{and}\qquad \left|W_{\tilde\phi^A(\cdot,t_0),\delta}\right|<\tilde\delta. \tag{5.13}$$

We now let χ be the characteristic function of $W_{\tilde\phi^A(\cdot,t_0),\delta}$, define

$$\begin{aligned}\psi_0^1(\cdot)&\equiv(I-P_h)\tilde\phi^A(\cdot,t_0),\\ \psi_0^2(\cdot)&\equiv\chi(\cdot)P_h\tilde\phi^A(\cdot,t_0),\\ \psi_0^3(\cdot)&\equiv(1-\chi(\cdot))P_h\tilde\phi^A(\cdot,t_0),\end{aligned}$$

and let ψ^j solve (1.1) with initial condition ψ_0^j so that $\tilde\phi^A(x,t)=\sum_{j=1}^3\psi^j(x,t-t_0)$. Lemma 5.4, (5.13), $C=C(2)\ge 1$, and $\tau\le 1$ give $\|\psi^1(\cdot,\tau)\|_{L^\infty}\le C\tau^{-1/2}\tilde\delta\le\delta$, and obviously $\|\psi^2(\cdot,\tau)\|_{L^\infty}\le\|\psi_0^2\|_{L^\infty}\le\delta$. Finally, we have

$$\|\psi_0^3\|_{L^1}\le|\mathrm{supp}(\psi_0^3)|^{1/2}\|\psi_0^3\|_{L^2}\le\tilde\delta^{1/2}\|P_h\tilde\phi^A(\cdot,t_0)\|_{L^2}\le\tilde\delta^{1/2}\|\tilde\phi^A(\cdot,t_0)\|_{L^2}\le\tilde\delta^{1/2}\|\tilde\phi^A(\cdot,\tau)\|_{L^2}\le\tilde\delta^{1/2},$$

and Lemma 5.4 again gives $\|\psi^3(\cdot,\tau)\|_{L^\infty}\le C\tau^{-1}\tilde\delta^{1/2}=\delta$. It follows using (2.6) that

$$\|\tilde\phi^A(\cdot,3\tau)\|_{L^\infty}\le\|\tilde\phi^A(\cdot,t_0+\tau)\|_{L^\infty}\le 3\delta,$$

that is, (5.10) holds and (5.8) follows.

Interpolation and (2.6) then give (2.5) for any $1\le p<q\le\infty$, thus yielding Theorem 2.1(i)–(iii) for $p<q$. The case $p=q\in(1,\infty)$ in part (ii) follows by splitting $\phi_0=\phi_0'+\phi_0''$ with $\phi_0'\in L^1$ and $\|\phi_0''\|_{L^p}$ small. Using (2.5) for ϕ_0' and (2.6) for ϕ_0'' then gives the result.

6. Proof of Theorem 2.1: Part II

In the present section we complete the proof of Theorem 2.1. We now assume that u is a 1-periodic incompressible Lipschitz flow on $D=\mathbb{R}\times\mathbb{T}$ that either leaves a bounded open subset of D invariant or has an $H^1(\mathbb{T}^2)$ eigenfunction with a non-zero eigenvalue. We will then show that Theorem 2.1(i)–(iii) do not hold. Again the cases of $D=\mathbb{R}^2$ and/or of other periods are handled similarly.

The main point here is that flows with the above properties do not "stretch" compactly supported initial data in the way the flows considered in the previous section do, which means the exposure of the solution to the effects of diffusion is limited (at least for a short time), regardless of the flow strength. More precisely, we will show

Lemma 6.1. *Under the above assumptions on u, there is a bounded non-zero compactly supported $\phi_0 \in H^1(D)$ and $b < \infty$ such that the solution of* (3.3) *on D satisfies $\|\phi^0(\cdot,t)\|_{H^1(D)} \le b$ for all $t \ge 0$.*

Assume for the moment that Lemma 6.1 holds. Then (3.9) with $\Gamma \equiv -\Delta$ and $L \equiv iu\cdot\nabla$ on $\mathcal{H} \equiv L^2(D)$, and after setting $A = \varepsilon^{-1}$ and rescaling time appropriately, shows that for each A,

$$\|\phi^A(\cdot,t) - \phi^0(\cdot,At)\|_{L^2}^2 \le \frac{b^2 t}{2}\|\phi_0\|_{L^2}.$$

Note that (3.9) extends to the non-compact setting of D, where Γ does not have a discrete spectrum. Since the measure of the set

$$\big\{x \,\big|\, |\phi^0(x,t)| \ge \gamma\big\}$$

is constant in t for each γ, this means that $\phi^A(\cdot,t)$ cannot be small in any L^p norm for t sufficiently small, regardless of the choice of A. Thus none of Theorem 2.1(i)–(iii) can be valid and we are left with establishing Lemma 6.1.

Proof of Lemma 6.1. Let us first assume that u leaves an open bounded domain $Y \subseteq D$ invariant. If $u \equiv 0$ on some such Y, then we only need to take ϕ_0 to be any bounded H^1 function supported in Y.

If this is not the case, then we know from the proof of Lemma 5.1 that there is such a domain Y with ∂Y a union of one or two non-trivial streamlines of u. If H is a stream function for u, we then have $H(\partial Y) = \{\beta,\gamma\}$ (with possibly $\beta=\gamma$). Since H cannot be constant inside Y (because $u(y) \neq 0$ on ∂Y), there is $y_0 \in Y$ with $H(y_0) \notin \{\beta,\gamma\}$. Then

$$\phi_0(x) \equiv \chi_Y(x)(H(x)-\beta)(H(x)-\gamma) \not\equiv 0.$$

is a compactly supported Lipschitz (and therefore H^1) function that is constant on the streamlines of u and thus $\phi^0(x,t) \equiv \phi_0(x)$ for all t. The claim of the lemma follows.

It remains to consider the case that u on $\mathbb{T}^2$ has an eigenfunction $\psi \in H^1(\mathbb{T}^2)$ with eigenvalue $i\lambda \in i\mathbb{R}\setminus\{0\}$. Notice that the first paragraph of the proof of Lemma 5.2 again shows that ψ has to be continuous at each x for which $u(x) \neq 0$ (the only difference is that now we obtain $(1,0)\cdot\nabla\omega \equiv i\lambda\omega$ and so $\omega(t,s) = e^{i\lambda t}\tilde{\omega}(s)$).

Let $x_0 \in \mathbb{T}^2$ be such that $\psi(x_0) \neq 0 \neq u(x_0)$. Such x_0 exists because $\lambda \neq 0$ implies $u(x_0) \neq 0$ for almost all x_0 with $\psi(x_0) \neq 0$. Without loss of generality we can assume that $x_0 = 0$ and on a neighborhood V of 0 we have $u(x) \equiv (1,0)$; otherwise a Lipschitz change of coordinates as in the proof of Lemma 5.2 will bring us to this situation. Then

$$\psi(x_1,x_2) = e^{i\lambda x_1}\tilde{\psi}(x_2) \tag{6.1}$$

(with $\tilde{\psi}$ continuous) for $|x_1|, |x_2| \le 2\alpha$ and some small $\alpha \in (0,\frac{\pi}{\lambda})$. Also, $\nabla H \equiv (0,-1)$ on V.

Choose a non-negative function $\omega : \mathbb{C} \to \mathbb{R}$ that is smooth as a function from $\mathbb{R}^2$ to $\mathbb{R}$ and is supported on a small ball around $\tilde{\psi}(0)$, so that for some small $\beta, \gamma > 0$ we have $\omega(\psi(x_1, x_2)) = 0$ for $(x_1, x_2) \in ([\alpha, \alpha+\gamma] \cup [-\alpha-\gamma, -\alpha]) \times [-\beta, \beta]$. This is possible because of the continuity of $\tilde{\psi}$ and $\lambda \neq 0$ in (6.1), together with $\alpha < \pi/\lambda$ (this is where we crucially use $\lambda \neq 0$). We also let θ with $\theta(0) \neq 0$ be a smooth non-negative function supported in $[-\alpha-\gamma, \alpha+\gamma] \times [-\beta, \beta]$ which only depends on x_2 in $R \equiv [-\alpha, \alpha] \times [-\beta, \beta] \subseteq V$. Since $H(x) = c - x_2$ on V (for some c), we have $\theta(x) = \tilde{\theta}(H(x))$ for all $x \in R$ and a smooth compactly supported $\tilde{\theta}$.

Now extend ψ periodically and θ by 0 onto D and consider

$$\phi_0(x) \equiv \theta(x)\omega(\psi(x)) = \chi_R(x)\tilde{\theta}(H(x))\omega(\psi(x)) \in H^1(D).$$

Then $\phi^0(x,t) = \phi_0(X(x,-t))$ (with X from (2.1)) is supported in $R_t \equiv X(R,t)$ and

$$\omega(\psi(X(x,-t))) = \omega(e^{-i\lambda t}\psi(x))$$

because $u \cdot \nabla\psi = i\lambda\psi$. Since constancy of H on the streamlines of u gives

$$\theta(X(x,-t)) = \tilde{\theta}(H(X(x,-t))) = \tilde{\theta}(H(x)) \tag{6.2}$$

for $x \in R_t$, we have

$$\phi^0(x,t) = \chi_{R_t}(x)\,\tilde{\theta}(H(x))\,\omega(e^{-i\lambda t}\psi(x)). \tag{6.3}$$

Note that since $R \subseteq \mathbb{T}^2$, the domain $R_t \subseteq D$ is simply connected and the natural map from D onto $\mathbb{T}^2$ is one-to-one when restricted to R_t. Hence

$$\int_{R_t} |\nabla[\omega(e^{-i\lambda t}\psi(x))]|^2\,dx \le \int_{\mathbb{T}^2} |\nabla[\omega(e^{-i\lambda t}\psi(x))]|^2\,dx \le \|\nabla\omega\|^2_{L^\infty(\mathbb{T}^2)}\|\psi\|^2_{H^1(\mathbb{T}^2)}.$$

Since $\tilde{\theta}$ and ω are bounded and ϕ^0 vanishes on ∂R_t, to obtain the claim of the lemma, we only need to show that $\int_{R_t} |\nabla[\tilde{\theta}(H(x))]|^2\,dx$ is uniformly bounded in t. But $|R_t| \le 1$ and

$$|\nabla[\tilde{\theta}(H(x))]| \le \|\nabla\tilde{\theta}\|_{L^\infty(\mathbb{R})}\|\nabla H\|_{L^\infty(R_t)} \le \|\nabla\tilde{\theta}\|_{L^\infty(\mathbb{R})}\|u\|_{L^\infty(\mathbb{T}^2)},$$

for $x \in R_t$, so this is obvious. $\square$

We note that θ is only needed when $|\tilde{\psi}(x_2)|$ is constant on an open interval containing zero. Otherwise $\phi_0(x) \equiv \chi_R(x)\omega(\psi(x))$ does the job.

This finishes the proof of Theorem 2.1.

7. Other Boundary Conditions and Examples

In the case $D = \mathbb{R} \times (0,1)$ we have so far only considered periodic boundary conditions on ∂D. It turns out that there is no change to Theorem 2.1 when we impose Dirichlet or Neumann boundary conditions, provided $u(x) \cdot (0,1) = 0$ for $x \in \partial D$.

Corollary 7.1. *Assume that u is a periodic, incompressible, Lipschitz flow on* $D = \mathbb{R} \times (0,1)$ *with a cell of periodicity* $\mathcal{C} = \alpha\mathbb{T} \times (0,1)$ *such that* $u(x) \cdot (0,1) = 0$ *for* $x \in \partial D$. *Let* ϕ^A *be the solution of* (1.1) *on D with either Dirichlet or Neumann boundary conditions on* ∂D. *Then Theorem 2.1(i)–(iv) are again equivalent.*

Remarks. 1. The operator $u \cdot \nabla$ is again anti-self-adjoint on $L^2(\mathcal{C})$ due to $u_2 \equiv 0$ on $\partial\mathcal{C}$.

2. Notice that there is no distinction between dissipation-enhancing flows in the Dirichlet and Neumann boundary conditions cases. This is because $u_2 \equiv 0$ on ∂D means that boundary conditions do not considerably affect dissipation away from ∂D on short time scales.

Proof. Extend u to $D' \equiv \mathbb{R} \times 2\mathbb{T}$ by letting

$$(u_1(x_1, x_2), u_2(x_1, x_2)) \equiv (u_1(x_1, 2 - x_2), -u_2(x_1, 2 - x_2))$$

for $x_2 \in [1, 2]$. That is, u is periodic and symmetric across $x_2 = 1$. Consider the the Dirichlet boundary conditions case first. It is sufficient to show that each of Theorem 2.1(i)–(iv) holds on D if and only if u is dissipation-enhancing on D'.

The "if" part of this claim is immediate. Indeed, if ϕ^A is a solution on D with Dirichlet boundary conditions, then we can extend it to a solution on D' by letting $\phi^A(x_1, x_2) \equiv -\phi^A(x_1, 2 - x_2)$. Hence any of Theorem 2.1(i)–(iii) on D' implies its counterpart on D. The same is true in the case of part (iv) because if ψ is an eigenfunction of u in $H^1(\mathcal{C})$, then by letting $\psi(x_1, x_2) \equiv \psi(x_1, 2 - x_2)$ one extends ψ to an eigenfunction of u in $H^1(\alpha\mathbb{T} \times 2\mathbb{T})$.

As for the "only if" part, assume u on D' is not dissipation-enhancing. Take some ϕ_0 that satisfies Lemma 6.1 for D' and that is supported inside D. This can be done because the streamlines of u do not cross ∂D. For the same reason ϕ^0 from Lemma 6.1 stays inside D, and so if we extend ϕ^0 to D by letting $\phi^0(x_1, x_2) \equiv -\phi^0(x_1, 2 - x_2)$, then this ϕ^0 satisfies all conditions of that lemma. The corresponding ϕ^A vanishes on ∂D and as in Section 6, it follows that none of Theorem 2.1(i)–(iii) can hold on D. The same is true for part (iv) after realizing that the restriction to D of a bounded open subset of D' invariant under u (or the restriction to $\mathcal{C}$ of an $H^1(a\mathbb{T} \times 2\mathbb{T})$ eigenfunction of u) has the same property on D (on $\mathcal{C}$).

This finishes the case of Dirichlet boundary conditions. Neumann boundary conditions are treated identically, with ϕ^A and ϕ^0 extended evenly (rather than oddly) to D'. ☐

We will now present a simple example of flows on $\mathbb{R}^2$ that demonstrates the independence of the two conditions in Theorem 2.1(iv).

Example 7.2. Let $p : \mathbb{T} \to \mathbb{T}$ and $\tilde{H} : \mathbb{T} \to \mathbb{R}$ be C^1 with $\int_0^1 p'(s)ds = 0$. Define $H(x_1, x_2) \equiv \tilde{H}(y(x_1, x_2))$ with $y(x_1, x_2) \equiv \{p(x_1) - x_2\}$ and consider the flow

$$u(x_1, x_2) \equiv \nabla^\perp H(x_1, x_2) = \big(\tilde{H}'(y), p'(x_1)\tilde{H}'(y)\big) \tag{7.1}$$

on $\mathbb{R} \times \mathbb{T}$ or on $\mathbb{R}^2$. In particular, we have $u(x_1, x_2) = 0$ if and only if $\tilde{H}'(y(x_1, x_2)) = 0$. If $p \equiv 0$ then this is a mean-zero shear flow. For general p (and $\tilde{H}' \not\equiv 0$) these are examples of *percolating flows*.

It is easy to see that the flow preserves y, and its first coordinate $\tilde{H}'(y)$ is therefore constant on the streamlines. The unbounded streamlines are those corresponding to y's for which $\tilde{H}'(y) \neq 0$ (they are then 1-periodic functions of x_1 due to $\int_0^1 p'(s)ds = 0$). This means that there is an open bounded domain invariant under the flow if and only if $\tilde{H}'$ has a plateau (a non-trivial interval where it is constant) with $\tilde{H}' = 0$. Note that (7.1) on $\mathbb{T} \times \mathbb{R}$ always has invariant bounded open domains.

There are always many H^1 eigenfunctions of such u, since each $\tilde{\psi}(y)$ is a first integral. On the other hand, it turns out that u has H^1 eigenfunctions other than the first integrals if and only if $\tilde{H}'$ has plateaus with $\tilde{H}' \neq 0$.

Indeed, if $\tilde{H}'(y) \equiv C \neq 0$ for $y \in [a,b]$ and θ is a smooth function supported on $[a,b]$, then $\psi(x_1,x_2) \equiv e^{2\pi i x_1}\theta(y)$ is an H^1 eigenfunction of u with eigenvalue $2\pi iC$. On the other hand, any H^1 eigenfunction with an eigenvalue $i\lambda \neq 0$ must be continuous a.e. where $u \neq 0$ (i.e., $\tilde{H}' \neq 0$) and zero a.e. where $u = 0$ (i.e., $\tilde{H}' = 0$). This means that it must be of the form $\psi(x_1,x_2) \equiv e^{i\lambda x_1/\tilde{H}'(y)}\theta(y)$ with θ continuous where $\tilde{H}' \neq 0$ and zero where $\tilde{H}' = 0$. But for ψ to be well defined as a function on $\mathbb{T}^2$, $2\pi\lambda/\tilde{H}'(y)$ must be an integer where $\theta(y) \neq 0$. Since $\psi \not\equiv 0$ and so $\theta \not\equiv 0$, this means that there must be a plateau of $\tilde{H}'$ with $\tilde{H}' \neq 0$.

Finally, since the existence of a plateau of $\tilde{H}'$ with $\tilde{H}' = 0$ and the existence of a plateau of $\tilde{H}'$ with $\tilde{H}' \neq 0$ are "independent", we can construct flows u given by (7.1) that demonstrate all four possibilities of the conditions in Theorem 2.1(iv) either being satisfied or not.

Notice that if $\tilde{H}'(y) \equiv C$ for $y \in [a,b]$, then the solutions of (2.1) starting inside the "channel" given by $y(x_1,x_2) \in [a,b]$ move along this channel with the same (horizontal) velocity $H'(y)$. This shows that any initial datum supported inside the channel will not get stretched too much regardless of the amplitude A of the flow, as was mentioned at the beginning of Section 6. On the other hand, $\tilde{H}'(y)$ not locally constant means any compactly supported initial datum will be stretched quickly when A is large because "neighboring" streamlines move at different horizontal speeds and this difference is magnified by A.

We also mention that in the case of shear flows (i.e., $p \equiv 0$) Theorem 2.1 follows from the results of [19] (the earlier paper [3] also considers shear flows and can treat all $\tilde{H}$ except of those that have no plateaus but all their derivatives vanish at some y_0). The above stretching argument was made rigorous there using probabilistic methods (Malliavin calculus in particular), but unlike our functional-analytic method, the approach does not seem to be applicable to general non-shear flows.

Finally, notice that if $\tilde{H}' \neq 0$ only on a dense set of a small measure and $\tilde{H}'$ has no plateaus, then u vanishes on a large set but it is dissipation-enhancing nevertheless.

We end this section by proving the following claim from Section 2.

Theorem 7.3. *Let u be a periodic, incompressible, Lipschitz flow on $D = \mathbb{R}^2$ or $D = \mathbb{R} \times \mathbb{T}$ with a cell of periodicity $\mathcal{C}$. If* (2.1) *on $\mathcal{C}$ has a stable solution and no dense orbits, then u is not dissipation-enhancing.*

Remark. Recall that the claim is false in general if a dense orbit exists.

Proof. We first assume that u on $\mathcal{C}$ is a *Hamiltonian flow* with (multivalued) Hamiltonian $H : \mathcal{C} \to a\mathbb{T}$. That is, there is $a > 0$ such that if $H : \mathbb{R}^2 \to \mathbb{R}$ is the stream function of u on $\mathbb{R}^2$, then $H(1,0) - H(0,0)$ and $H(0,1) - H(0,0)$ are both integer multiples of a. Let $X(x_0,\cdot)$ be a stable solution of (2.1) on $\mathcal{C}$. If $X(x_0,t) \equiv x_0$, then solutions starting near x_0 must either all be constant or there is one whose orbit is a closed curve (homotopic in D to a point because it is contained in a small neighborhood of x_0). In either case there is a bounded open subset of D invariant under u.

Let now $X(x_0,\cdot)$ be non-constant. Since u is Lipschitz, $H \in C^{1,1}(\mathbb{R}^2)$ and the Lipschitz Morse-Sard theorem of Bates [1] shows that almost all values of H are regular. If h is such a value then any solution $X(x,t)$ of (2.1) on $\mathcal{C}$ with $H(X(x,t)) = h$ must be periodic. Indeed, $\{X(x,n)\}_{n=1}^\infty$ has a subsequence converging to some x_1 with $H(x_1) = h$. But $|\nabla H| \neq 0$ near x_1, so the intersection of the h-level set of H and a small neighborhood of x_1 is a curve which is also a solution of (2.1). Then $X(x,n)$ must lie on this curve for infinitely many n, so $X(x,\cdot)$ is periodic. Since $|\nabla H(x_0)| \neq 0$, this means that there are periodic solutions arbitrarily close to $X(x_0,\cdot)$. Stability and non-constancy of $X(x_0,\cdot)$ then show that all these must have the same period p, along with $X(x_0,\cdot)$. Thus $|\nabla H(X(x_0,\cdot))|$ is bounded away from zero, and stability of $X(x_0,\cdot)$ shows that all solutions of (2.1) starting near x_0 have period p. If now $\alpha \in C^1(\mathcal{C})$ is supported near the orbit of $X(x_0,\cdot)$ and is only a function of $H(x)$, then $u\cdot\nabla\alpha \equiv 0$. For small $|s|$ and all $t\in\mathbb{R}$ we let $\beta(X(x_0+s\nabla H(x_0),t)) = e^{2\pi i t/p}$. Then β is C^1 near the orbit of $X(x_0,\cdot)$ and $u\cdot\nabla(\alpha\beta) = i\frac{2\pi}{p}\alpha\beta$, so $\alpha\beta$ is an H^1 eigenfunction of $u\cdot\nabla$ with a non-zero eigenvalue. This finishes the case of Hamiltonian u.

Let now u be non-Hamiltonian, that is, such that $a \equiv H(1,0)-H(0,0)$ and $b\equiv H(0,1)-H(0,0)$ are rationally independent. We will show that then there is a bounded open subset of D invariant under u. Again almost all values of H are regular, so assume that h is (then so is $h+ma+nb$ for $m,n\in\mathbb{Z}$ by periodicity of u). Pick x_0 so that $H(x_0)=h$ and let $V\subseteq\mathbb{R}^2$ be the closure of $\bigcup_{n,m\in\mathbb{Z},t\in\mathbb{R}}\{X(x_0+(m,n),t)\}$. Here we consider solutions of (2.1) on $\mathbb{R}^2$. Then $V\neq\mathbb{R}^2$ because otherwise $\bigcup_{t\in\mathbb{R}}\{X(x_0,t) \mod 1\}$ were dense in $\mathcal{C}$. Pick $x_1\in W\equiv\mathbb{R}^2\setminus V$ and denote $W_{m,n}$ the connected component of W containing $x_1+(m,n)$. We can assume they are all simply connected because otherwise $X(x_0,\cdot)$ is bounded in $\mathbb{R}^2$ and so periodic, and its interior is then an open subset of D invariant under u.

We have that each connected component V' of $\partial V = \partial W$ is a subset of a level set of H (if not, then there is $x\in V'$ with $H(x)$ a regular value of H and it easily follows from the definition of V and regularity of $H(x)$ that V' is the orbit of $X(x,\cdot)$, i.e., a subset of a level set of H). This means that all the open sets $W_{m,n}$ are different (and thus disjoint) because the values of H on their boundaries are different (since $W_{m,n}=W_{0,0}+(m,n)$). It follows that each of them has a finite area, and so $\lim_{n\to\infty}|W_{0,0}\cap B_n'|=0$, with B_n' the complement of the ball of radius n centered at the origin. So uniform continuity of H and simple connectedness of $W_{0,0}$ show that $\partial W_{0,0}$ is a subset of a single level set h of H and for each ε there is n such that $H(W_{0,0}\cap B_n')\subseteq(h-\varepsilon,h+\varepsilon)$. If H is constant on $W_{0,0}$, then any subset of $W_{0,0}$ is invariant under u. If not, then $\{x\in W_{0,0}\,\big|\,|H(x)-h|>\varepsilon\}$ is a bounded open set (non-empty for small enough $\varepsilon>0$) invariant under u. In either case we are done. $\square$

8. APPLICATIONS TO REACTION-DIFFUSION EQUATIONS

We now turn to applications of Theorem 2.1 to quenching in *reaction-advection-diffusion equations*. We consider the equation

$$T_t^A(x,t) + Au\cdot\nabla T^A(x,t) = \Delta T^A(x,t) + f(T^A(x,t)), \qquad T^A(x,0)=T_0(x) \tag{8.1}$$

for $x\in\mathbb{R}\times\mathbb{T}$ or $x\in\mathbb{R}^2$. Here $T^A(x,t)\in[0,1]$ is the (normalized) temperature of a premixed combustible gas that is advected by the periodic incompressible flow $Au(x)$. The nonlinear *reaction term* $f(T^A)$ accounts for temperature increase due to burning and will be considered to be of the *ignition type*, that is,

$$\begin{aligned} &\text{(i) } f(0) = f(1) = 0 \text{ and } f(T) \text{ is Lipschitz continuous on } [0,1],\\ &\text{(ii) } \exists \eta_0 \in (0,1) \text{ such that } f(T) = 0 \text{ for } T \in [0,\eta_0] \text{ and } f(T) > 0 \text{ for } T \in (\eta_0, 1). \end{aligned} \tag{8.2}$$

The value η_0 is called the (normalized) *ignition temperature*. We also take $T_0(x)$ to be compactly supported with values in $[0,1]$, so that $T^A(x,t) \in [0,1]$ for all x, t by the maximum principle.

Definition 8.1. We say that the initial "flame" T_0 is *quenched* by the flow Au if

$$\|T^A(\cdot,t)\|_{L^\infty} \to 0 \qquad \text{as } t \to \infty. \tag{8.3}$$

A flow u is said to be *strongly quenching* if for each compactly supported T_0 and each ignition-type reaction f there exists A_0 such that Au quenches T_0 for each $A > A_0$.

That is, strongly quenching flows are those that have the ability to extinguish any initially localized reaction, provided their amplitude is large enough. Notice also that due to the compactness of $\operatorname{supp}(T_0)$ and $\eta_0 > 0$, (8.3) is equivalent to $\|T^A(\cdot,t_0)\|_{L^\infty} \le \eta_0$ for some $t_0 < \infty$.

We can now state

Theorem 8.2. *Assume that u is a periodic, incompressible, Lipschitz flow on $D = \mathbb{R}^2$ or $D = \mathbb{R} \times \mathbb{T}$ with a cell of periodicity $\mathcal{C}$.*

(i) *If u is dissipation-enhancing, then u is strongly quenching.*

(ii) *If either u leaves an open bounded subset of D invariant or u has an eigenfunction $\psi \in C^{1,1}(\mathcal{C})$ that is not a first integral of u, then u is not strongly quenching.*

Remarks. 1. $C^{1,1}(\mathcal{C})$ is the set of all $\psi \in C^1(\mathcal{C})$ with $\nabla\psi \in \operatorname{Lip}(\mathcal{C})$.

2. This of course leaves open the case when no open bounded sets are invariant under u, the flow does have $H^1(\mathcal{C})$ eigenfunctions with non-zero eigenvalues, but none of them belongs to $C^{1,1}(\mathcal{C})$. Such flows can again be constructed using Example 2 in Section 6 of [4], this time with a smooth $Q : \mathbb{T} \to \mathbb{T}$ and a Liouvillean α such that (2.7) has a solution $R \in H^1(\mathbb{T}) \setminus H^2(\mathbb{T})$. We conjecture that u is not strongly quenching in such cases, and hence that the strongly quenching periodic flows in two dimensions are precisely the dissipation-enhancing ones.

Proof. (i) Let c be the Lipschitz constant for f so that $f(T) \le cT$. If ϕ^A solves (1.1) with initial condition $\phi_0 \equiv T_0 \in L^1(D)$, then $T^A(x,t) \le e^{ct}\phi^A(x,t)$. The result follows by choosing A large enough so that $\|\phi^A(\cdot,1)\|_{L^\infty} \le e^{-c}\eta_0$.

(ii) Assume first there is an open bounded domain $Y \subseteq D$ invariant under u. From the proof of Lemma 6.1 we know that then there is such a Y so that either $u \equiv 0$ on Y, or ∂Y consists of one or two closed streamlines of u (one if Y is simply connected, two otherwise). In either case we will construct a stationary subsolution T_0 of (8.1) for some f and any A. From this the claim will follow, because then $T^A(x,t) \ge T_0(x)$ for all A, x, t and so u cannot be quenching.

Assume the first case (i.e., $u \equiv 0$ on Y) and choose a smooth function T_0 supported in Y and bounded above by $\frac{2}{3}$ such that $\Delta T_0(x) \ge 0$ when $T_0(x) < \frac{1}{3}$. We then have

$$\Delta T_0 + f(T_0) \ge 0$$

whenever f is larger than $\|\Delta T_0\|_{L^\infty}$ on $[\frac{1}{3}, \frac{2}{3}]$. Hence $T_0(x)$ is a subsolution of (8.1) for such f and any A.

Next assume the second case above and assume Y is bounded and simply connected (the other alternative can be handled by a simple modification of the following argument). Notice that we have that $u \neq 0$ on ∂Y by construction (see Section 6) and so $|\nabla H| \ge c$ for some $c > 0$ on some open neighborhood $\tilde{V}$ of ∂Y. This, the fact that we are in two dimensions, and u Lipschitz ensure that all streamlines that are close enough to ∂Y must also be closed. It follows that there is a domain $V \subseteq \tilde{V} \cap Y$ with ∂V consisting of two streamlines of u, one of which is ∂Y. Since $|\nabla H|$ is strictly positive on V and continuous, V can be chosen so that $H(\partial V) = \partial H(V)$.

Let $\tilde{\phi}_0$ be a smooth function on the interval $H(V)$ with $\tilde{\phi}_0(H(\partial Y)) = 0$ and $\tilde{\phi}_0(H(\partial V \setminus \partial Y)) = \frac{2}{3}$, with the first and second derivatives of $\tilde{\phi}_0$ vanishing on $\partial H(V)$, and with

$$\tilde{\phi}_0''(s) \ge c^{-2} \|\Delta H\|_{L^\infty} |\tilde{\phi}_0'(s)| \tag{8.4}$$

when $\tilde{\phi}_0(s) < \frac{1}{3}$. This is possible because $H \in C^{1,1}(D)$ and so $\|\Delta H\|_{L^\infty} < \infty$. We then let

$$M \equiv \|\tilde{\phi}_0''\|_{L^\infty} \|\nabla H\|_{L^\infty}^2 + \|\tilde{\phi}_0'\|_{L^\infty} \|\Delta H\|_{L^\infty} \tag{8.5}$$

and pick f that is larger than M on $[\frac{1}{3}, \frac{2}{3}]$. We define

$$T_0(x) \equiv \begin{cases} \tilde{\phi}_0(H(x)) & x \in V, \\ \frac{2}{3} & x \in Y \setminus V, \\ 0 & x \notin Y, \end{cases}$$

so that $\Delta T_0 + f(T_0) = f(T_0) \ge 0$ outside V and

$$\Delta T_0(x) + f(T_0(x)) = \tilde{\phi}_0''(H(x)) |\nabla H(x)|^2 + \tilde{\phi}_0'(H(x)) \Delta H(x) + f(T_0(x)) \ge 0$$

in V (using (8.4) when $T_0(x) < \frac{1}{3}$ and (8.5) otherwise). This and the fact that T_0 is constant on the streamlines of u means that T_0 is a subsolution of (8.1) for any A.

Let us now assume that u has an eigenfunction $\psi \in C^{1,1}(\mathcal{C})$ with eigenvalue $i\lambda \in i\mathbb{R} \setminus \{0\}$. We will show that if we choose f and the functions ω and θ from the corresponding part of Section 6 appropriately, then the (time-dependent) solution of the fast free linear dynamics $\phi^0(x, At)$ from (6.3) will be a subsolution of (8.1) for each A.

Take x_0 such that $\psi(x_0) \neq 0 \neq u(x_0)$. Without loss of generality we can assume that $x_0 = 0$, $\psi(0) = 1$, and $H(0) = 0$, as this can be achieved by a translation of the problem, multiplication of ψ by a constant, and additon of a constant to H. In what follows we will call C^2 functions smooth.

Assume first that the flow $u(x) \equiv (1, 0)$ in a neighborhood of 0. Repeat the construction from Section 6 to obtain smooth non-negative ω, θ, and a small rectangle $R \equiv [-\alpha, \alpha] \times [-\beta, \beta]$ such the following hold with ψ extended periodically onto D. The product $\theta(x)\omega(\psi(x))$ is supported in R (by slightly enlarging R we can actually assume that $\theta(x)\omega(\psi(x))$ is supported on a compact subset of the interior of R) and on R we have $\theta(x) = \tilde\theta(H(x))$ for some smooth non-negative compactly supported $\tilde\theta$. Moreover, we will also pick ω so that $\omega(z) = \tilde\omega(\Im z)$ on $\psi(R)$ for some compactly supported smooth $\tilde\omega$ and $\Im z$ the imaginary part of z. This can be achieved thanks to $\psi(0) = 1$, the continuity of ψ on R, and $\lambda \neq 0$ in (6.1), provided R is small (recall that so far ω was only required to be supported on a small ball around $\psi(0) = 1$). The picture we are establishing here is that $H(x)$ and $\Im\psi(x)$ determine a coordinate system on R, while inside R each of the functions θ and $\omega\circ\psi$ depends on one of these coordinates only (and their product is supported in the interior of R). The main point is that, as we shall see, this setup will be preserved by the free evolution and hold on $R_t \equiv X(R,t)$.

This time, however, we need to impose additional conditions on R, $\tilde\omega$, and $\tilde\theta$. This will be necessary because we will deal with second derivatives here, and possible because these will not clash with the conditions we imposed so far — that R be small and $\tilde\omega$, $\tilde\theta$ be non-negative, nonzero, smooth, and have small supports containing zero (since $\Im\psi(0) = H(0) = 0$).

We first ask that R is small enough so that

$$|\psi(x) - 1| \le \frac{1}{2} \tag{8.6}$$

for $x \in R$. Since the flow preserves $|\psi|$, we have $|\psi(x)| \ge \frac12$ for $x \in R_t$. This and $u\cdot\nabla\psi = i\lambda\psi$ mean that if

$$\begin{aligned} C &\equiv \max\{\|u\|_{L^\infty}, \|\nabla\psi\|_{L^\infty}, \|\Delta H\|_{L^\infty}, \|\Delta\psi\|_{L^\infty}, \sqrt{\lambda}, 1\} < \infty, \\ c &\equiv \min\left\{\frac{\lambda}{2C}, \frac{1}{2}\left(1 - \sqrt{1 - \frac{\lambda^2}{4C^4}}\right)\right\} > 0, \end{aligned}$$

then

$$|\nabla H(x)| = |u(x)| \ge \frac{\lambda}{2C} \ge c \tag{8.7}$$

for $x \in R_t$. We let $\kappa_t(x) \equiv \Im(e^{-i\lambda t}\psi(x))$ so that

$$u \cdot \nabla\kappa_t(x) = \lambda\Re(e^{-i\lambda t}\psi(x))$$

together with

$$e^{-i\lambda t}\psi(x) = \psi(X(x,-t)) \in \psi(R) \tag{8.8}$$

for $x \in R_t$ and with (8.6) gives

$$|\nabla\kappa_t(x)| \ge \frac{\lambda}{2C} \ge c \tag{8.9}$$

for $x \in R_t$. Finally, we note that $\nabla H \perp u$ and $|\nabla H| = |u|$ give for $x \in R_t$,

$$\begin{aligned} |\nabla H(x)\cdot\nabla\kappa_t(x)| &= \left(|\nabla H(x)|^2|\nabla\kappa_t(x)|^2 - |u(x)\cdot\nabla\kappa_t(x)|^2\right)^{1/2} \\ &= |\nabla H(x)|\,|\nabla\kappa_t(x)|\sqrt{1 - \frac{\left|\lambda\Re(e^{-i\lambda t}\psi(x))\right|^2}{|\nabla H(x)|^2|\nabla\kappa_t(x)|^2}} \\ &\le |\nabla H(x)|\,|\nabla\kappa_t(x)|\sqrt{1 - \frac{\lambda^2}{4C^4}} \\ &\le (1-2c)|\nabla H(x)|\,|\nabla\kappa_t(x)|, \end{aligned} \tag{8.10}$$

where we again used (8.8) and (8.6) in the third step.

As for $\tilde\theta$ and $\tilde\omega$, we ask that they be smooth, bounded above by $\frac{2}{3}$, and satisfy

$$\begin{aligned} |\tilde\omega'(s)| &= kK\tilde\omega(s)^{1-1/k} \qquad \text{and} \qquad \tilde\omega''(s) = k(k-1)K^2\tilde\omega(s)^{1-2/k} \qquad \text{when} \qquad \tilde\omega(s) \le \frac{1}{2}, \\ |\tilde\theta'(s)| &= kK\tilde\theta(s)^{1-1/k} \qquad \text{and} \qquad \tilde\theta''(s) = k(k-1)K^2\tilde\theta(s)^{1-2/k} \qquad \text{when} \qquad \tilde\theta(s) \le \frac{1}{2}, \end{aligned} \tag{8.11}$$

for some $K > 1$ and

$$k \equiv 1 + Cc^{-3}.$$

This can be achieved by making $\tilde\omega, \tilde\theta$ equal to translations of $(K|s|)^k$ close to the edges of their respective supports (with K large to ensure the supports are as small as needed) and taking values from $[\frac{1}{2}, \frac{2}{3}]$ on the remainders of their supports. We then let

$$L \equiv \max\left\{ \max\left\{ \frac{|\tilde\omega'(s)|}{\tilde\omega(s)}, \frac{|\tilde\omega''(s)|}{\tilde\omega(s)} \,\middle|\, \tilde\omega(s) \ge \frac{1}{2} \right\}, \max\left\{ \frac{|\tilde\theta'(s)|}{\tilde\theta(s)}, \frac{|\tilde\theta''(s)|}{\tilde\theta(s)} \,\middle|\, \tilde\theta(s) \ge \frac{1}{2} \right\}, 1 \right\}.$$

From now on $\tilde\omega, \tilde\theta$ will be fixed.

Finally, we note that if $u \not\equiv (1,0)$ around 0, we can map u onto $(1,0)$ via a bilipschitz mapping J, construct $R, \omega, \theta, \tilde\omega, \tilde\theta$ as above (using $\psi \circ J$), and then map R, θ back via J^{-1}, keeping $\omega, \tilde\omega, \tilde\theta$ unchanged. This gives us R that is not necessarily a rectangle but has the properties we are interested in. Namely, $\phi_0(x) \equiv \theta(x)\omega(\psi(x))$ is supported in the interior of R, and $\theta(x) = \tilde\theta(H(x))$ and $\omega(\psi(x)) = \tilde\omega(\Im\psi(x))$ for $x \in R$. Therefore $\phi_0(x) = \tilde\theta(H(x))\tilde\omega(\Im\psi(x))$ on its support, and so $\phi_0 \in C^{1,1}$ because $\psi, H \in C^{1,1}$ and $\tilde\theta, \tilde\omega$ are smooth.

Once again the solution $\phi^0(x,t) = \phi_0(X(x,-t))$ of the free linear dynamics (3.3) is supported in the interior of R_t. The introduction of $\tilde\omega$ turns (6.3) into

$$\phi^0(x,t) = \theta(X(x,-t))\,\omega(\psi(X(x,-t))) = \chi_{R_t}(x)\,\tilde\theta(H(x))\,\tilde\omega(\kappa_t(x)).$$

This is because the flow preserves H, and for $x \in R_t$ we have $X(x,-t) \in R$ so that

$$\omega(\psi(X(x,-t))) = \tilde\omega(\Im[\psi(X(x,-t))]) = \tilde\omega(\Im[e^{-i\lambda t}\psi(x)]) = \tilde\omega(\kappa_t(x)).$$

We also have

$$\frac{d}{dt}\phi^0(x,At) + Au\cdot\nabla\phi^0(x,At) = 0, \tag{8.12}$$

and we will show that $\phi^0(x,At)$ is a subsolution of (8.1) with an appropriate f.

Obviously $\Delta\phi^0(x,t) = 0$ for $x \notin R_t$, and for $x \in R_t$,

$$\begin{aligned} \Delta\phi^0(x,t) =& \tilde\theta''(H(x))\,\tilde\omega(\kappa_t(x))\,|\nabla H(x)|^2 + \tilde\theta(H(x))\,\tilde\omega''(\kappa_t(x))\,|\nabla\kappa_t(x)|^2 \\ &+ 2\tilde\theta'(H(x))\,\tilde\omega'(\kappa_t(x))\,\nabla H(x)\cdot\nabla\kappa_t(x) \\ &+ \tilde\theta'(H(x))\,\tilde\omega(\kappa_t(x))\,\Delta H(x) + \tilde\theta(H(x))\,\tilde\omega'(\kappa_t(x))\,\Delta\kappa_t(x). \end{aligned} \tag{8.13}$$

Note that from $\psi, H \in C^{1,1}$ and $\tilde\theta, \tilde\omega$ smooth it follows that

$$\Delta\phi^0(x,t) \ge -M$$

for some large M independent of x, t. Let us now assume that $x \in R_t$ is such that $\tilde{\omega}(\kappa_t(x)) \le \frac{1}{2}$ and $\tilde{\theta}(H(x)) \le \frac{1}{2}$. Then we have (after dropping the arguments)

$$\begin{aligned} \tilde{\omega}''\tilde{\omega} &= \frac{k-1}{k}(\tilde{\omega}')^2, \\ \tilde{\theta}''\tilde{\theta} &= \frac{k-1}{k}(\tilde{\theta}')^2, \end{aligned}$$

and so $a^2 + b^2 \ge 2ab$, (8.10), and $k > c^{-1}$ give

$$\begin{aligned} (1-c)\big(\tilde{\theta}''\tilde{\omega}|\nabla H|^2 + \tilde{\theta}\tilde{\omega}''|\nabla \kappa_t|^2\big) &\ge 2(1-c)\frac{k-1}{k}|\tilde{\theta}'\tilde{\omega}'|\,|\nabla H|\,|\nabla \kappa_t| \\ &\ge 2\frac{1-c}{1-2c}\frac{k-1}{k}|\tilde{\theta}'\tilde{\omega}'|\,|\nabla H \cdot \nabla \kappa_t| \\ &\ge |2\tilde{\theta}'\tilde{\omega}'\,\nabla H \cdot \nabla \kappa_t|. \end{aligned}$$

On the other hand, (8.7) and (8.11) show that

$$c\tilde{\theta}''\tilde{\omega}|\nabla H|^2 \ge c^3(k-1)|\tilde{\theta}'|\tilde{\omega} \ge |c^3 C^{-1}(k-1)\tilde{\theta}'\tilde{\omega}\Delta H| = |\tilde{\theta}'\tilde{\omega}\Delta H|,$$

and the same is true for $\tilde{\theta}$ and $\tilde{\omega}$ exchanged and κ_t in place of H. Therefore $\Delta\phi^0(x,t) \ge 0$.

Next let $x \in R_t$ be such that $\tilde{\omega}(\kappa_t(x)) \ge \frac{1}{2}$ and $\tilde{\theta}(H(x)) \le \frac{1}{2}$. Then

$$(8.13) \ge \tilde{\theta}''\tilde{\omega}c^2 - \tilde{\theta}\tilde{\omega}LC^2 - |\tilde{\theta}'|\tilde{\omega}2LC^2 - |\tilde{\theta}'|\tilde{\omega}C - \tilde{\theta}\tilde{\omega}LC \ge \tilde{\theta}''\tilde{\omega}c^2 - (|\tilde{\theta}'| + \tilde{\theta})\tilde{\omega}3LC^2$$

by the definition of L. But then (8.11) gives

$$\Delta\phi^0(x,t) \ge \tilde{\theta}^{1-2/k}\tilde{\omega}c^2k(k-1)K^2 - \tilde{\theta}^{1-1/k}\tilde{\omega}6kKLC^2 = \tilde{\theta}^{1-1/k}\tilde{\omega}kK(\tilde{\theta}^{-1/k}c^2(k-1)K - 6LC^2).$$

This is greater than zero provided $\tilde{\theta} \le \varepsilon \equiv \min\{(6LC^2c^{-2}(k-1)^{-1}K^{-1})^{-k}, \frac{1}{2}\}$. We get the same conlusion if $\tilde{\omega}(\kappa_t(x)) \le \frac{1}{2}$ and $\tilde{\theta}(H(x)) \ge \frac{1}{2}$.

This all means that $\Delta\phi^0(x,t) \ge 0$ when $x \in R_t$ and either $\tilde{\omega}(\kappa_t(x)) \le \varepsilon$ or $\tilde{\theta}(H(x)) \le \varepsilon$. But then

$$\Delta\phi^0(x,t) + f(\phi^0(x,t)) \ge 0$$

for all $x \in R_t$ (and so for all $x \in D$), provided f is such that $f(T) \ge M$ for $T \in [\varepsilon^2, \frac{4}{9}]$ (recall that $\tilde{\omega}, \tilde{\theta} \le \frac{2}{3}$). Combining this with (8.12), we find that $\phi^0(x, At)$ is indeed a subsolution of (8.1), so that u is not strongly quenching. ☐

We note that the above method of construction of a subsolution to (8.1) does not extend to the case when u only has $H^1 \setminus C^{1,1}$ eigenfunctions with non-zero eigenvalues.

It turns out that dissipation-enhancing flows quench some reactions without an ignition temperature cutoff, in particular, *Arrhenius-type* reactions with $f(T) \equiv e^{-c/T}(1-T)$ and $c > 0$.

Theorem 8.3. *Assume that* u *is a periodic incompressible Lipschitz flow on* $D = \mathbb{R}^2$ *or* $D = \mathbb{R} \times \mathbb{T}$*, and that the reaction function* f *satisfies* (8.2)*(i) and* $f(T) \le cT^\alpha$ *for some* $c > 0$ *and* $\alpha > 2$ *(if* $D = \mathbb{R}^2$*) resp.* $\alpha > 3$ *(if* $D = \mathbb{R} \times \mathbb{T}$*). If* u *is dissipation-enhancing, then* *for each M there is $A_0(M)$ such that when $\|T_0\|_{L^1(D)} \le M$, $T_0 \in [0,1]$, and $A > A_0(M)$, the solution of* (8.1) *satisfies $\|T^A(\cdot,t)\|_{L^\infty(D)} \to 0$ as $t \to \infty$.*

Remarks. 1. It follows from [22] (see also [27]) that if $f(T) \ge cT^\alpha$ for some $c > 0$, $\alpha < 2$ (if $D = \mathbb{R}^2$) resp. $\alpha < 3$ (if $D = \mathbb{R} \times \mathbb{T}$), and all small T, then the conclusion of the theorem does not hold for any A and u.

2. Theorem 8.2(ii) trivially extends to this setting since by the comparison principle, solution of (8.1) with $\tilde{f} \ge f$ dominates that of (8.1) with f.

Proof. Let $D = \mathbb{R} \times \mathbb{T}$ and define $I_A \equiv \int_0^\infty \|\phi^A(\cdot,t)\|_\infty^{\alpha-1}\,dt$ where ϕ^A is the solution of (1.1) with $\phi_0 \equiv T_0$. It follows from [22] (see also [27, Lemma 2.1]) that the conclusion of the theorem is valid whenever $c(\alpha-1)I_A < 1$.

Lemma 3.1 in [5] shows that there exists $C < \infty$ such that for each incompressible Lipschitz flow v on D and $t \ge 1$ we have

$$\|\psi(\cdot,t)\|_{L^\infty(D)} \le Ct^{-1/2}\|\psi_0\|_{L^1(D)}, \tag{8.14}$$

with ψ the solution of (2.3). We pick $\tau_0 > 1$ so that

$$c(\alpha-1)(CM)^{\alpha-1}\frac{2}{\alpha-3}\tau_0^{-(\alpha-3)/2} < \frac{1}{3}, \tag{8.15}$$

$\delta > 0$ so that $c(\alpha-1)\tau_0\delta^{\alpha-1} < \frac{1}{3}$, and $\tau \in (0,\tau_0)$ so that $c(\alpha-1)\tau < \frac{1}{3}$. If now $A_0(M)$ is such that

$$\|\mathcal{P}_\tau(Au)\|_{L^1(D)\to L^\infty(D)} \le \delta M^{-1}$$

for all $A > A_0(M)$, then $c(\alpha-1)I_A < 1$ for such A. This is obtained by estimating $\|\phi^A(\cdot,t)\|_{L^\infty}$ by 1 for $t \in [0,\tau)$, by δ for $t \in [\tau,\tau_0)$, and by (8.14) for $t \ge \tau_0$.

The case $D = \mathbb{R}^2$ is identical (with $\tau_0^{-(\alpha-2)}$ in (8.15)) provided we show

$$\|\psi(\cdot,t)\|_{L^\infty(D)} \le Ct^{-1}\|\psi_0\|_{L^1(D)} \tag{8.16}$$

for some C, any $t \ge 1$, any incompressible Lipschitz flow v, and any solution ψ of (2.3) on D. We provide the proof of this claim below, essentially following [5].

Solutions of (2.3) satisfy

$$\tfrac{d}{dt}\|\psi\|_2^2 = -2\|\nabla\psi\|_2^2 \le -C\|\psi\|_2^4\|\psi\|_1^{-2} \le -C\|\psi\|_2^4\|\psi_0\|_1^{-2},$$

where we used the Nash inequality $\|\psi\|_2^2 \le C\|\nabla\psi\|_2\|\psi\|_1$ [23] and (2.6) with $p = 1$. Dividing by $\|\psi\|_2^4$ and integrating in time gives

$$\|\psi(\cdot,t)\|_{L^2} \le Ct^{-1/2}\|\psi_0\|_{L^1}.$$

This shows that $\|\mathcal{P}_t(v)\|_{L^1\to L^2} \le Ct^{-1/2}$. But $\mathcal{P}_t(v)$ is the adjoint of $\mathcal{P}_t(-v)$ which satisfies the same bound, so we obtain

$$\|\mathcal{P}_{2t}(v)\|_{L^1\to L^\infty} \le \|\mathcal{P}_t(v)\|_{L^1\to L^2}\|\mathcal{P}_t(v)\|_{L^2\to L^\infty} = \|\mathcal{P}_t(v)\|_{L^1\to L^2}\|\mathcal{P}_t(-v)\|_{L^1\to L^2} \le C^2t^{-1},$$

which gives (8.16). $\square$

Note that the same proof with the inequality $\|\psi\|_2^{1+2/n} \le C\|\nabla\psi\|_2\|\psi\|_1^{2/n}$ in $\mathbb{R}^n$ [23] gives (8.16) with $t^{-n/2}$ when $D = \mathbb{R}^n$. The claim of the theorem can be extended to this case with $\alpha > 1 + \frac{2}{n}$.

9. Dissipation-Enhancing Flows in More Dimensions

Most of Sections 5 and 6 does not extend to higher dimensions or time-periodic flows. An exception are Lemmas 5.3 and 5.4 which have both been stated in any dimension. They also extend to time-periodic flows. In that case Lemma 5.3 deals with H^1 eigenfunctions of the unitary evolution operator U_{τ_0} generated by the flow (with τ_0 the time-period) rather than those of u, and the proof stays the same. Notice that the two sets of eigenfunctions coincide when u is time-independent. The statement of Lemma 5.4 is unchanged in this case, and the proof uses [18] to obtain (5.7).

We call a time-dependent flow u on $D = \mathbb{R}^n \times \mathbb{T}^m$ dissipation-enhancing if for any $1 \le p < q \le \infty$ and $\tau > 0$,

$$\|\mathcal{P}_\tau(Au^A)\|_{L^p(D)\to L^q(D)} \to 0 \qquad \text{as } A \to \infty, \tag{9.1}$$

where $u^A(x,t) \equiv u(x,At)$. This is the right choice for u^A as it ensures that the solutions of $X'(t) = u^A(X(t),t)$ with $X(0) = x_0$ have the same orbits for different A. The definition of strongly quenching time-dependent flows is changed analogously.

Theorem 9.1. *Assume that u is a space- and time-periodic incompressible Lipschitz flow on $D = \mathbb{R}^n \times \mathbb{T}^m$ with $n \ge 1$, $m \ge 0$, a cell of spatial periodicity $\mathcal{C} \subseteq D$, and time-period τ_0. If the unitary evolution operator U_{τ_0} on $\mathcal{C}$ has no non-constant eigenfunctions in $H^1(\mathcal{C})$, then u is dissipation-enhancing and strongly quenching.*

Proof. The proof essentially follows Section 5, but is simpler due to the absence of non-constant first integrals. Choose any $\tau, \delta > 0$ and let $k \in \mathbb{Z}$ be larger than $\delta^{-2/n}$. Let $\|\tilde\phi_0\|_{L_1(D)} \le 1$ and periodize the problem and $\tilde\phi_0$ onto $\mathcal{M} \equiv (k\mathbb{T})^n \times \mathbb{T}^m$ as we did in Section 5. We define $\phi_0(x) \equiv \tilde\phi^A(x,\tau)$ so that by Lemma 5.4 in $d = n+m$ dimensions,

$$\|\phi_0\|_{L^1(\mathcal{M})} \le 1 \qquad \text{and} \qquad \|\phi_0\|_{L^\infty(\mathcal{M})} \le C\tau^{-d/2}$$

with $C = C(d)$. This then gives

$$\|\phi_0\|_{L^2(\mathcal{M})} \le C^{1/2}\tau^{-d/4} \qquad \text{and} \qquad |\bar\phi_0| \le k^{-n} \le \delta k^{-n/2}$$

where $\bar\phi_0$ is the average of ϕ_0 over $\mathcal{M}$. Consider the operators $\Gamma \equiv -\Delta$ and $L_t \equiv iu(\cdot,t)\cdot\nabla$ on the space $\mathcal{H} \equiv L^2(\mathcal{M})$. From Lemma 5.3(iii) for time-periodic flows we know that U_{τ_0}, now as an operator on $\mathcal{H}$, has no non-constant eigenfunctions in $H^1(\Gamma)$. It follows from Theorem 4.1 that for each $A > A_1(\tau,\delta)$ (with A_1 independent of ϕ_0), there is $t \le \tau$ such that the solution ϕ^A of (4.4) satisfies

$$\|\phi^A(\cdot,t)\|_{L^2(\mathcal{M})} \le \|\phi^A(\cdot,t) - \bar\phi_0\|_{L^2(\mathcal{M})} + \|\bar\phi_0\|_{L^2(\mathcal{M})} \le \delta + (k^n(\delta k^{-n/2})^2)^{1/2} = 2\delta.$$

This is because the average of ϕ^A stays constant and so $P_h\phi^A(\cdot,t) = |\mathcal{M}|^{-1}\int_{\mathcal{M}}\phi^A(x,t)\,dx = \bar\phi_0$. Another application of Lemma 5.4 gives

$$\|\tilde\phi^A(\cdot,3\tau)\|_{L^\infty(\mathcal{M})} = \|\phi^A(\cdot,2\tau)\|_{L^\infty(\mathcal{M})} \le \|\phi^A(\cdot,t+\tau)\|_{L^\infty(\mathcal{M})} \le 2C\tau^{-d/2}\delta$$

and so the same is true for the original problem on D. Since δ was arbitrary and C only depends on d, (9.1) follows with $p = 1$ and $q = \infty$ for each $\tau > 0$. As in Section 5, interpolation provides the other cases, so u is dissipation-enhancing. Strong quenching is then immediate as in Theorem 8.2(i). $\square$

The complete characterization of periodic incompressible dissipation-enhancing flows in more than two dimensions, even in the time-independent case, remains an open problem.

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