ON THE RATE OF MERGING OF VORTICITY LEVEL SETS FOR THE 2D EULER EQUATIONS

ANDREJ ZLATOŠ

ABSTRACT. We show that two distinct level sets of the vorticity of a solution to the 2D Euler equations on a disc can approach each other along a curve at an arbitrarily large exponential rate.

1. INTRODUCTION

In this note we study the question of how fast two distinct level sets of the vorticity of a solution to the Euler equations in two dimensions can approach each other. We are here interested in the approach along a curve rather than just at a single point.

The two-dimensional Euler equations model the motion of an incompressible inviscid fluid on a domain $D \subseteq \mathbb{R}^2$, and we will use here their vorticity formulation

$$\omega_t + u \cdot \nabla \omega = 0 \qquad (1.1)$$

on $D \times (0, \infty)$, with initial data

$$\omega(\cdot, 0) = \omega_0. \qquad (1.2)$$

We will consider the case when the vorticity $\omega = -\nabla \times u$ (which will be more convenient for us than the more standard $\omega = \nabla \times u$) is bounded, that is, $\omega_0 \in L^\infty(D)$. The customary no-flow boundary condition $u \cdot n = 0$ on $\partial D \times (0, \infty)$, with n the unit outer normal vector, then yields the Biot-Savart law

$$u(x, t) = -\int_D \nabla^\perp G_D(x, y) \omega(y, t) dy$$

for computing u from ω. Here $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and G_D is the Dirichlet Green's function for D (i.e., $u = \nabla^\perp(-\Delta_D)^{-1}\omega$, with Δ_D the Dirichlet Laplacian on D).

It has been known since the works of Hölder [4] and Wolibner [11] that solutions to the Euler equations on smooth two-dimensional domains remain globally regular, and that $\|\nabla \omega(\cdot, t)\|_{L^\infty}$ cannot grow faster than double-exponentially as $t \to \infty$ (although this bound seems to have first explicitly appeared in [12]). That is, for each $\omega_0 \in W^{1,\infty}(D)$ there is $C < \infty$ such that

$$\|\nabla \omega(\cdot, t)\|_{L^\infty} \le C e^{e^{Ct}} \qquad \text{for each } t \ge 0.$$

Whether the double-exponential rate of growth is attainable had been a long-standing open problem. The first examples of smooth solutions for which the vorticity gradient grows without bound as $t \to \infty$ were constructed by Yudovich [13, 14]. Later Nadirashvili [10] and Denisov [1] provided examples with at least linear and superlinear growth, respectively. The period of relatively slow progress in this direction was ended by a striking recent result of

Kiselev and Šverák [5]. Motivated by numerical simulations of Luo and Hou [8, 9] which suggest blow-up for axisymmetric 3D Euler equations, they proved that solutions exhibiting double-exponential growth of the vorticity gradient indeed do exist in two dimensions. This result, which was proved on a disc, was extended to general smooth two-dimensional domains with an axis of symmetry by Xu [15].

The double-exponential growth in [5] is proved to occur on the boundary ∂D, whose presence is therefore crucial. We note that the fastest growth currently known to occur on a domain without a boundary (i.e., on $\mathbb{R}^2$ or $\mathbb{T}^2$) is exponential growth of the vorticity gradient for solutions $\omega(\cdot, t) \in C^{1,\alpha}(\mathbb{T}^2)$ ($\alpha < 1$), proved by the author in [16] (see also [7]). Smooth solutions that grow super-linearly have been shown to exist as well, by Denisov [1], who also constructed solutions exhibiting double-exponential growth rate on arbitrarily long but finite time intervals [2] as well as patch solutions subject to a prescribed (regular) stirring for which the two patches approach each other double-exponentially in time [3]. Finally, we note that Kiselev and the author proved that on domains whose boundaries are not everywhere smooth finite time blow-up can occur [6].

We will consider here (1.1) on a disc, as in [5], although our results easily extend to general smooth two-dimensional domains with a symmetry axis via [15]. For convenience we will work with the unit disc $D := B_1(e_2)$ centered at $e_2 = (0, 1)$, and we will denote its right/left halves by $D^{\pm} := D \cap (\mathbb{R}^{\pm} \times \mathbb{R})$. Then we have

$$G_D(x, y) = \frac{1}{2\pi} \ln \frac{|x - y|}{|x - \bar{y}||y - e_2|}$$

for $x \neq y \neq e_2$, with $\bar{y} := e_2 + (y - e_2)|y - e_2|^{-2}$, as well as

$$u(x, t) = -\int_D \left[\frac{(x - y)^{\perp}}{|x - y|^2} - \frac{(x - \bar{y})^{\perp}}{|x - \bar{y}|^2} \right] \omega(y, t) dy, \tag{1.3}$$

where $(a_1, a_2)^{\perp} := (-a_2, a_1)$.

The following is our main result.

Theorem 1.1. *For $D = B_1(e_2)$ and each $A \geq 1$, there is $\delta > 0$ and $\omega_0 \in C^{\infty}(D)$ with $\|\omega_0\|_{L^{\infty}} = 1$ such that the solution ω to (1.1)–(1.3) satisfies the following. For any $t \geq 0$ we have*

$$\omega(0, \beta, t) = 0 \qquad \text{for each } \beta \in (0, \delta),$$

and there is a function $\alpha_t : (0, \delta) \to (0, e^{-At})$ such that the set of those $\beta \in (0, \delta)$ for which

$$\omega(\alpha_t(\beta), \beta, t) = 1$$

has measure at least $\delta - 2e^{-At}$.

Remark. This result implies that $\|\omega(\cdot, t)\|_{W^{s,p}}$ also grows exponentially as $t \to \infty$ when $sp > 1$. On the other hand, the result in [5] yields double-exponential growth of these norms when $sp > 2$ (as well as exponential growth for $(s, p) = (1, 2)$).

The solutions that we will consider here are the ones from [5], but we will track their dynamics in the neighborhood of the whole segment $\{0\} \times [0, \delta]$ rather than only near the origin.

A crucial extra ingredient in our argument will also be an *explicit* use of incompressibility of the flow u, and the corresponding measure-preserving property of its flow map (see (3.1)).

We also note that if one were able to establish an additional estimate on these solutions, then one would obtain a super-exponential rate of merging of distinct level sets of ω (see Theorem 4.1 and Corollary 4.2 below). In the theoretically best possible case one could even prove a double-exponential rate of merging (see the discussion after Theorem 4.1), although it is not clear whether this case can occur.

In Section 2 we collect some estimates from [5] that we will use. For the convenience of the reader and in order to provide more insight into the arguments that follow, we include the derivation of most of these, with the exception of the key Lemma 2.1. We prove Theorem 1.1 in Section 3, and the discussion of its extension when we are able to obtain additional estimates on ω appears in Section 4.

2. Some Estimates From [5]

To prove Theorem 1.1, we will employ some estimates from [5], including the following key lemma. Just as in that paper, we will consider solutions ω that are odd in x_1 and non-negative on D^+. That is,

$$0 \leq \omega_0(x_1, x_2) = -\omega_0(-x_1, x_2) \tag{2.1}$$

for any $(x_1, x_2) \in D^+$. Of course, then $\omega(\cdot, t)$ has the same properties for any $t \geq 0$ and $u_1(0, x_2, t) = 0$ for any $(x_2, t) \in (0, 2) \times (0, \infty)$. Oddness of ω in x_1 also means that, with $\tilde{x} := (-x_1, x_2)$ the reflection across the vertical axis, we have

$$u(x, t) = -\int_{D^+} \left[\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-\bar{y})^\perp}{|x-\bar{y}|^2} - \frac{(x-\tilde{y})^\perp}{|x-\tilde{y}|^2} + \frac{(x-\bar{\tilde{y}})^\perp}{|x-\bar{\tilde{y}}|^2} \right] \omega(y, t) dy.$$

We denote

$$D_1^\gamma := \{x \in D^+ \; : \; x_1 > \gamma x_2\},$$
$$D_2^\gamma := \{x \in D^+ \; : \; x_2 > \gamma x_1\}$$

for $\gamma > 0$, which are obtained by removing from D^+ sectors close to the x_2 and x_1 axes, respectively. Finally, we let

$$Q(x) := D^+ \cap ([x_1, \infty) \times [x_2, \infty))$$

for $x \in D^+$.

Lemma 2.1 ([5]). *For any $\gamma > 0$ there is $C_\gamma < \infty$ such that for each $\omega_0 \in L^\infty(D)$ that satisfies (2.1) we have*

$$u_j(x, t) = (-1)^j \left(\frac{4}{\pi} \int_{Q(x)} \frac{y_1 y_2}{|y|^4} \omega(y, t) dy + B_j(x, t) \right) x_j \tag{2.2}$$

when $x \in D_j^\gamma$ ($j = 1, 2$), with B_j satisfying

$$|B_j(x, t)| \leq C_\gamma \|\omega_0\|_{L^\infty}. \tag{2.3}$$

Note that when such ω_0 is close to $\|\omega_0\|_{L^\infty}$ on all of D^+ except of a small enough region (this property is then preserved by the evolution because ω is odd in x_1), then the first term in the parenthesis in (2.2) will dominate the second term for all x close enough to the origin, regardless of where the small exceptional region is located. Indeed, if

$$\left|\left\{y \in D^+ : \omega_0(y) < \alpha\|\omega_0\|_{L^\infty}\right\}\right| \le \delta^2 \le \frac{1}{16} \tag{2.4}$$

for some $\alpha > 0$ (this then also holds for $\omega(\cdot, t)$ and any $t \ge 0$) and $x \in D^+ \cap B_{2\delta}(0)$, then a simple analysis of the kernel $y_1 y_2 |y|^{-4}$ (which decreases radially and is maximized at $y_1 = y_2$ for any fixed $|y|$) shows that

$$\int_{Q(x)} \frac{y_1 y_2}{|y|^4} \omega(y, t) dy \ge \alpha\|\omega_0\|_{L^\infty} \int_{[D^+ \setminus B_{4\delta}(0)] \cap D_1^{1/2} \cap D_2^{1/2}} \frac{y_1 y_2}{|y|^4} dy \ge \alpha\|\omega_0\|_{L^\infty} \int_{4\delta}^1 \int_{\pi/6}^{\pi/3} \frac{\sin 2\phi}{2r} d\phi dr$$

for each $t \ge 0$. Therefore, by Lemma 2.1,

$$(-1)^j u_j(x, t) \ge \|\omega_0\|_{L^\infty} \left(\frac{\alpha|\ln 4\delta|}{4} - C_\gamma\right) x_j \tag{2.5}$$

for each $\gamma, \alpha, t > 0$ and $j = 1, 2$ when (2.1) and (2.4) hold and $x \in D^+ \cap B_{2\delta}(0) \cap D_j^\gamma$. Of course, this estimate is only useful when $\delta > 0$ is sufficiently small (depending on γ, α).

We next consider the case $\|\omega_0\|_{L^\infty} = 1$, for the sake of simplicity, although it is clear that the argument below works for any fixed $\|\omega_0\|_{L^\infty} > 0$. Let us take any $A \ge 1$, choose $\gamma = \frac{1}{2}$ and $\alpha = 1$, and let

$$\delta := \frac{1}{4} e^{-4(A+C_{1/2})}. \tag{2.6}$$

If now ω_0 satisfies (2.1), $\|\omega_0\|_{L^\infty} = 1$, and (2.4) with $\alpha = 1$, then (2.5) yields for $j = 1, 2$,

$$(-1)^j u_j(x, t) \ge A x_j \qquad \text{for } (x, t) \in (D^+ \cap B_{2\delta}(0) \cap D_j^{1/2}) \times (0, \infty). \tag{2.7}$$

We pick such $\omega_0 \in C^\infty(D)$, which also equals 1 on the set $\{x \in D^+ : x_2 \le x_1 \in [\delta^2, \delta]\}$.

Following [5], for such ω_0 and the corresponding solution ω and velocity u, let us denote

$$\underline{u}_1(x_1, t) := \inf_{(x_1, x_2) \in D^+ \ \& \ x_2 \le x_1} u_1(x_1, x_2, t),$$

$$\overline{u}_1(x_1, t) := \sup_{(x_1, x_2) \in D^+ \ \& \ x_2 \le x_1} u_1(x_1, x_2, t),$$

and let a, b be the solutions of

$$a'(t) = \overline{u}_1(a(t), t), \qquad a(0) = \delta^2, \tag{2.8}$$

$$b'(t) = \underline{u}_1(b(t), t), \qquad b(0) = \delta. \tag{2.9}$$

Note that a, b are decreasing due to (2.7), and they are positive on $(0, \infty)$ because (2.2) yields the bound

$$|u_1(x, t)| \le \frac{4}{\pi} \|\omega_0\|_{L^\infty} (\ln 2 - \ln |x| + C_\gamma) x_1 \tag{2.10}$$

for any $x \in D^+$ with $x_2 \le x_1$. Since (2.7) shows that

$$u_1(x, t) < 0 < u_2(x, t) \qquad \text{when } 0 < x_1 = x_2 \le \delta \text{ and } t > 0,$$

it follows that

$$\omega(\cdot, t) = 1 \qquad \text{on } \{x \in D^+ : x_2 \le x_1 \in [a(t), b(t)]\} \tag{2.11}$$

for any $t \ge 0$ such that $\inf_{s \in [0,t]}(b(s) - a(s)) > 0$. For any such t, comparing (2.2) for points $(a(t), x_2) \in D^+$ with $x_2 \le a(t)$ and for points $(b(t), y_2) \in D^+$ with $y_2 \le b(t)$ (and also using the properties of $\omega(\cdot, t)$) yields

$$\frac{d}{dt}(\ln a(t) - \ln b(t)) \le -\frac{4}{\pi} \int_{y \in D^+ \,\&\, a(t) < y_2 < y_1 \in [a(t), b(t)]} \frac{y_1 y_2}{|y|^4} dy + \frac{4}{\pi} \int_{[b(t), 1] \times [0, b(t)]} \frac{y_1 y_2}{|y|^4} dy + 2C_{1/2}.$$

The second integrand is no more than y_1^{-2} so the integral is bounded above by 1, and the first integral is bounded below by

$$\int_{2a(t)}^{b(t)} \int_{\pi/6}^{\pi/4} \frac{\sin 2\phi}{2r} d\phi dr \ge \frac{\sqrt{3}\pi}{48}(\ln b(t) - \ln a(t) - \ln 2) \ge \frac{\pi}{32}(\ln b(t) - \ln a(t) - \ln 2),$$

provided $b(t) \ge 2a(t)$. This yields

$$\frac{d}{dt}(\ln a(t) - \ln b(t)) \le \frac{1}{8}(\ln a(t) - \ln b(t)) + C$$

for such t, with $C := \frac{\ln 2}{8} + \frac{4}{\pi} + 2C_{1/2}$. Gronwall's lemma then shows

$$\ln \frac{a(t)}{b(t)} \le \left(\ln \frac{a(0)}{b(0)} + C\right) e^{t/8} \tag{2.12}$$

on any interval $[0, T]$ such that $\sup_{t \in [0,T]} \frac{a(t)}{b(t)} \le \frac{1}{2}$. Since $\delta < e^{-1-C}$ due to (2.6), the parenthesis in (2.12) is less than -1 $(< \ln \frac{1}{2})$, and we thus obtain (2.12) for all $t \ge 0$. We then have

$$a(t) \le \frac{a(t)}{b(t)} \le (\delta e^C)^{e^{t/8}} \le e^{-e^{t/8}} \tag{2.13}$$

for all $t \ge 0$. In particular, $\|\nabla \omega(\cdot, t)\|_{L^\infty}$ grows double-exponentially in time (because (2.11) holds and $\omega(0, t) = 0$ for all $t \ge 0$), which is the main result of [5].

3. Proof of Theorem 1.1

Consider now any $A \ge 1$, and let δ be from (2.6) and ω_0 as in the two sentences following (2.6). Then a, b from the previous section satisfy (2.11) and (2.13). Fix any $T \ge 0$, let

$$V_T := (0, a(T))^2 \cap D^+,$$

and for $s \ge 0$ let

$$U_T^s := \Phi_T^s(V_T),$$

where Φ is the flow map for u, given by

$$\frac{d}{ds}\Phi_T^s(x) = u(\Phi_T^s(x), T + s), \qquad \Phi_T^0(x) = x \in D.$$

That is, U_T^s is the set to which the flow u transports V_T between times T and $T + s$. We also let

$$\tilde{U}_T^s := \{\Phi_T^s(x) : x \in V_T \,\&\, \Phi_T^r(x) \in (0, \delta)^2 \text{ for each } r \in [0, s]\}$$

be the set of all points from U_T^s which never left $(0, \delta)^2 \cap D^+$ as they were transported by u between times T and $T + s$. Since u is divergence free, we have

$$|\tilde{U}_T^s| \leq |U_T^s| = |V_T| < a(T)^2 \tag{3.1}$$

for each $T, s \geq 0$.

The estimate (2.7) applied at $\{x \in \overline{D^+} : 0 \leq x_{3-j} \leq x_j = \delta\}$ shows that any point starting in V_T at time T and leaving $(0, \delta)^2 \cap D^+$ at a later time must do so across the line $[0, \delta] \times \{\delta\}$, as well as that it will also leave $[0, \delta]^2$ at the same time. It therefore can lie in $\partial \tilde{U}_T^s$ at time $T + s$ only if that is the time of its first departure from V_T, in which case it also lies in $[0, \delta] \times \{\delta\}$. It therefore follows that

$$\partial \tilde{U}_T^s \subseteq \Phi_T^s(\partial V_T) \cup ([0, \delta] \times \{\delta\})$$

for each $T, s \geq 0$. (Note that Φ_T^s extends continuously to ∂D because both ω and u extended continuously to $\partial D \times [0, \infty)$, with u remaining log-Lipschitz because ∂D is smooth.)

Let us now denote by R_k ($k = 1, 2, 3, 4$) the four closed segments of ∂V_T with endpoints at $(0, 0)$, $(a(T), 1 - \sqrt{1 - a(T)^2})$, $(a(T), a(T))$, and $(0, a(T))$, with R_1 being the segment between the first two of these points, and the other three segments labeled in counter-clockwise order. Let us also fix

$$s_T := \frac{1}{A} |\ln a(T)|. \tag{3.2}$$

Then (2.7) and $\delta \leq \frac{1}{4}$ show that for each $x \in R_3$ there is $r < s_T$ such that $(\Phi_T^r(x))_2 = \frac{4}{3}\delta$. In particular,

$$\Phi_T^{s_T}(R_3) \cap \partial \tilde{U}_T^{s_T} = \emptyset.$$

Next, (2.7) together with $\Phi_T^{s_T}$ being continuous and satisfying $\Phi_T^{s_T}(\overline{D^+} \cap \overline{D^-}) = \overline{D^+} \cap \overline{D^-}$ and $\Phi_T^{s_T}(0) = 0$ show that $\{0\} \times [0, \delta] \subseteq \Phi_T^{s_T}(R_4)$ and in fact

$$\{0\} \times [0, \delta] \subseteq \partial \tilde{U}_T^{s_T} \tag{3.3}$$

From (2.7), $\Phi_T^{s_T}(\partial D) = \partial D$ and $\Phi_T^{s_T}(0) = 0$ we also have

$$\Phi_T^s(R_1) \subseteq R_1$$

for any $T, s \geq 0$. From all this it follows that

$$B := \partial \tilde{U}_T^{s_T} \setminus [R_1 \cup (\{0\} \times [0, \delta]) \cup ([0, \delta] \times \{\delta\})] \subseteq \Phi_T^{s_T}(R_2). \tag{3.4}$$

Moreover, (3.3) shows that for each $\beta \in (0, \delta)$ there is $\alpha_T^\beta > 0$ such that

$$(\alpha_T^\beta, \beta) \in \partial \tilde{U}_T^{s_T} \qquad \text{and} \qquad (0, \alpha_T^\beta) \times \{\beta\} \subseteq \tilde{U}_T^{s_T}. \tag{3.5}$$

Then from (3.1) we conclude that

$$\left| \left\{ \beta \in (0, \delta) : \alpha_T^\beta < a(T) \right\} \right| \geq \delta - a(T) \qquad \text{for each } T \geq 0. \tag{3.6}$$

Finally, the definition of B and (3.5) show that

$$(\alpha_T^\beta, \beta) \in B \qquad \text{for each } \beta \in (1 - \sqrt{1 - a(T)^2}, \delta), \tag{3.7}$$

and then (3.4) with $1 - \sqrt{1 - a(T)^2} \leq a(T)$ yield

$$\omega(\alpha_T^\beta, \beta, T + s_T) = 1 \qquad \text{for each } \beta \in (a(T), \delta) \text{ and } T \geq 0. \tag{3.8}$$

Indeed, this holds because (1.1) is a transport equation and $\omega(x, T) = 1$ for any $x \in R_2$ due to (2.11). Since $\omega(0, x_2, T + s_T) = 0$ for any $x_2 \in (0, 2)$, the result follows once we notice that (2.13) and (3.2) yield

$$T + s_T \leq 8 \ln |\ln a(T)| + \frac{1}{A} |\ln a(T)|. \tag{3.9}$$

Indeed, since $a(\cdot)$ is continuous, for each

$$t \geq -\frac{1}{A} \ln a(0) = \frac{2}{A} |\ln \delta|$$

there is T_t such that $T_t + s_{T_t} = t$. If we also require $t \geq \frac{2}{A} e^{8A}$, then either

$$|\ln a(T_t)| = As_{T_t} \geq e^{8A} \tag{3.10}$$

or $T_t \geq \frac{1}{A} e^{8A}$. In the latter case we use $e^{8A} \geq (8A)^2$ (due to $A \geq 1$) to conclude $T_t \geq 64A$, and then (3.10) again follows from (2.13). Since $\ln x \leq \frac{x}{8A}$ for $x \geq e^{8A}$, we now obtain from (3.9) that $t \leq \frac{2}{A} |\ln a(T_t)|$. This yields $a(T_t) \leq e^{-At/2}$, which together with (3.6) implies

$$\left| \left\{ \beta \in (0, \delta) : \alpha_{T_t}^\beta \leq e^{-At/2} \right\} \right| \geq \delta - e^{-At/2} \qquad \text{for each } t \geq \max \left\{ 2e^{8A}, 12(1 + C_{1/2}) \right\}, \tag{3.11}$$

where we also used $A \geq 1$ and

$$|\ln \delta| = 4(A + C_{1/2}) + \ln 4 \leq 6(A + C_{1/2}).$$

On the other hand, (3.8) and $a(T_t) \leq e^{-At/2}$ now yield

$$\omega(\alpha_{T_t}^\beta, \beta, t) = 1 \qquad \text{for each } \beta \in \left(e^{-At/2}, \delta \right) \text{ and } t \geq \max \left\{ 2e^{8A}, 12(1 + C_{1/2}) \right\}. \tag{3.12}$$

Replacing A by $2A$ now yields the result for all $t \geq \max \left\{ 2e^{16A}, 12(1 + C_{1/2}) \right\}$. To obtain it also for all $t \in [0, \max \left\{ 2e^{16A}, 12(1 + C_{1/2}) \right\}]$, we only need to pick ω_0 as above which is also equal to 1 on $D^+ \cap ([c, 1] \times \mathbb{R})$ for a small enough $c > 0$ (because u is log-Lipschitz [4, 11], so $|u_1(x, t)| \leq C(|\log x_1|)x_1$ holds for some C and all $(x, t) \in D^+ \times (0, \infty)$).

4. Towards a Faster Rate of Merging

Notice that the double-exponential upper bound (2.13) on $a(t)$ is in fact not crucial for proving an exponential rate of approach of level sets of ω along a segment. Indeed, if we only knew $a(T) \leq e^{-cT}$ for some $c > 0$ and all $t \geq 0$, we would obtain

$$t = T_t + s_{T_t} \leq \frac{A + c}{Ac} |\ln a(T_t)|.$$

Then $a(T_t) \leq e^{-Act/(A+c)}$, which would yield (3.11) and (3.12) with $e^{-At/2}$ replaced by $e^{-Act/(A+c)}$ and a different lower bound on t.

The limitation on obtaining a stronger result using (2.13) comes from our use of the bound (2.7) for u_2 on the time interval $[T, T + s_T]$, which dictates our choice of s_T from (3.2).

However, if we could gain more mileage from (2.2) by proving a faster growth of the second coordinate of $\Phi_T^s(x)$ for all $x \in R_3$, we might be able to improve Theorem 1.1 further.

This would be the case, for instance, if we could obtain a better *lower bound* on $b(t)$ than (2.13), such as $b(t) \geq e^{-3e^{t/8}}$ for large t. More generally, let us assume that we have

$$a(t) \leq e^{-ce^{t/C}} \qquad \text{and} \qquad b(t) \geq e^{-\tilde{c}e^{t/C}} \qquad \text{for all large enough } t, \tag{4.1}$$

with some $C > 0$ and $c > 2\tilde{c} > 0$. Let us then, for the sake of simplicity, define V_T in Section 3 with $e^{-ce^{T/C}}$ in place of $a(T)$, and change everything else up to the definition of R_k accordingly (also, below only consider T large enough so that (4.1) holds for all $t \geq T$).

Now for any $t \geq 0$ and any $a(t) \leq x_1 \leq x_2 \leq b(t)$, we obtain

$$u_2(x_1, x_2, t) \geq \left(\frac{4}{\pi} \int_{x_2 \leq y_2 \leq y_1 \leq b(t)} \frac{y_1 y_2}{|y|^4} dy - C_{1/2} \right) x_2$$

from (2.2) and (2.11). The integral can be estimated below by

$$\int_{2x_2}^{b(t)} \int_{\pi/6}^{\pi/4} \frac{\sin 2\phi}{2r} d\phi dr \geq \frac{\sqrt{3}\pi}{48} (\ln b(t) - \ln 2x_2)$$

when $x_2 \leq \frac{b(t)}{2}$, so then

$$u_2(x_1, x_2, t) \geq \frac{-\ln x_2 + \ln b(t) - \ln 2 - 8C_{1/2}}{8} x_2. \tag{4.2}$$

This is a better estimate than (2.7) when $x_2 \leq \frac{b(t)}{2} e^{-8(A+C_{1/2})}$, so it would be useful for $\Phi_T^s(x)$ when $x \in R_3$ as long as

$$(\Phi_T^s(x))_2 \leq \frac{b(T+s)}{2e^{8(A+C_{1/2})}}.$$

Let us therefore take $x \in R_3$ and consider $s \geq 0$ such that $\Phi_T^s(x)$ did not yet exit $(0, \delta)^2$ — thus, in particular, $(\Phi_T^s(x))_1 \leq (\Phi_T^s(x))_2$ — and also such that with $\gamma := \frac{1}{2} e^{-8C_{1/2}}$ we have

$$(\Phi_T^s(x))_2 \leq \gamma^2 b(T+s)^2 \quad \left(\leq \frac{b(T+s)}{2e^{8(A+C_{1/2})}} \text{ when } T \geq 4 \text{ due to (2.6) and (2.7)} \right). \tag{4.3}$$

Then from (4.2) we obtain

$$u_2(\Phi_T^s(x), T+s) \geq \frac{|\ln(\Phi_T^s(x))_2|}{16} (\Phi_T^s(x))_2.$$

Since also $(\Phi_T^0(x))_2 = x_2 = e^{-ce^{T/C}}$, we obtain

$$(\Phi_T^s(x))_2 \geq e^{-ce^{(16T-Cs)/16C}} \qquad \text{until } (\Phi_T^s(x))_2 = \gamma^2 b(T+s)^2. \tag{4.4}$$

The equality in (4.4) will be achieved at some $s' \leq \frac{16}{C} T$. Then considering $s \geq s'$ and applying (2.7) shows that $\Phi_T^s(x)$ exits $(0, \delta)^2$ at some

$$r < s' + \frac{2}{A} |\ln \gamma b(T+s')| \leq \frac{16}{C} T + \frac{2}{A} \left| \ln \gamma b \left(\frac{16+C}{C} T \right) \right| =: s_T.$$

The argument from the proof of Theorem 1.1 then applies with this s_T, and we obtain (3.4)–(3.8) for all large T, with $e^{-ce^{T/C}}$ in place of $a(T)$. Then for any large enough t, the equality $T_t + s_{T_t} = t$ from that proof becomes

$$\frac{16+C}{C}T_t + \frac{2}{A}\left|\ln \gamma b\left(\frac{16+C}{C}T_t\right)\right| = t. \tag{4.5}$$

Since $b(t) \leq e^{-At}$ by (2.7), we have $\frac{16+C}{C}T_t \leq \frac{1}{A}|\ln \gamma b(\frac{16+C}{C}T_t)|$, so we know that T_t satisfies

$$b\left(\frac{16+C}{C}T_t\right) \leq \gamma^{-1}e^{-At/3} = 2e^{8C_{1/2}}e^{-At/3}. \tag{4.6}$$

This finally yields (3.11) and (3.12) with $e^{-At/2}$ replaced by $e^{-ce^{T_t/C}}$ and a different lower bound on t. Note that this argument also requires

$$a(t) \leq \frac{1}{4}e^{-16C_{1/2}}b(t)^2 \qquad \text{for all large enough } t, \tag{4.7}$$

so that (4.3) holds at least for $s = 0$ when $T = T_t$ (since obviously $\lim_{t\to\infty} T_t = \infty$), but this is guaranteed by $c > 2\tilde{c}$ in (4.1). (The power 2 in (4.3) and (4.7) could be replaced by any power $p > 1$, at the expense of some constants above being different, so $c > \tilde{c}$ in fact suffices to obtain a result here.)

We have therefore proved the following result (notice that b is decreasing, so replacing $\leq$ by $=$ in (4.6) can only decrease T_t).

Theorem 4.1. *Let $D = B_1(e_2)$, $D^+ := D \cap (\mathbb{R}^+ \times \mathbb{R})$, $C_{1/2}$ be from Lemma 2.1, $A \geq 1$, and δ be from (2.6). Consider any $\omega_0 \in C^\infty(D)$ satisfying (2.1), $\|\omega_0\|_{L^\infty} = 1$,*

$$\left|\{x \in D^+ : \omega_0(x) < 1\}\right| \leq \delta^2,$$

and equal to 1 on the set $\{x \in D^+ : x_2 \leq x_1 \in [\delta^2, \delta]\}$, and let ω solve (1.1)–(1.3). If a, b from (2.8), (2.9) satisfy (4.1) with $C > 0$ and $c > 2\tilde{c} > 0$, then the claims in Theorem 1.1 hold for all large enough t, with e^{-At} replaced by $e^{-ce^{T_t/C}}$ and T_t solving

$$b\left(\frac{16+C}{C}T_t\right) = 2e^{8C_{1/2}}e^{-At/3}.$$

Let us now consider the theoretically best possible scenario. For ω as above, the best lower bound on b we could hope for is

$$b(t) \geq c'e^{-C't} \qquad \text{for all large enough } t, \tag{4.8}$$

with some $c' \in (0, 1)$ and $C' \geq A$ (due to (2.7)). If this is the case, then (4.1) certainly holds and (4.6) yields

$$T_t \geq \frac{CA}{(50+3C)C'}t \qquad \text{for all large enough } t.$$

That is, (3.11) and (3.12) would hold with $e^{-At/2}$ replaced by $e^{-ce^{At/(50+3C)C'}}$ and a different lower bound on t. Hence, the exponential lower bound (4.8) on $b(t)$ would yield a double-exponential rate of approach of two distinct level sets of ω along a segment.

Of course, it is not at all clear whether (4.8) can hold for some ω satisfying our hypotheses. Nevertheless, even a much weaker lower bound may provide a stronger result than Theorem 1.1. For instance, if we can prove that

$$a(t) \leq e^{-ce^{t/C}} \qquad \text{and} \qquad b(t) \geq e^{-c'e^{t/C'}} \qquad \text{for all large enough } t, \tag{4.9}$$

with $c, c' > 0$ and $C < C'$, then Theorem 4.1 yields a super-exponential rate of approach whenever $C' > 16 + C$ (we obtain (3.11) and (3.12) with $e^{-At/2}$ replaced by $e^{-c(At/4C')^{C'/(16+C)}}$).

However, we can do even better in this case because if we only have a double-exponential lower bound on $b(t)$, then the above argument may be further optimized by this time choosing s' so that $(\Phi_T^{s'}(x))_2 = \gamma^2 e^{-2c'e^{(T+s')/C'}}$. We then obtain $s' \leq \frac{16(C'-C)}{(16+C')C}T + o(1)$ (with $o(1) = o(T^0)$ as $T \to \infty$). Therefore

$$T + s' \leq \frac{(16+C)C'}{(16+C')C}T + o(1),$$

so we can replace (4.5) by

$$\frac{(16+C)C'}{(16+C')C}T_t + o(1) + \frac{2}{A}\left|\ln \gamma e^{-c'e^{[(16+C)C'T_t/(16+C')C+o(1)]/C'}}\right| = t.$$

This yields

$$T_t = \frac{(16+C')C}{16+C}\ln\frac{At}{2c'} + o(1),$$

and hence leads to (3.11) and (3.12) with $e^{-At/2}$ replaced by $e^{-\kappa t^{(16+C')/(16+C)}}$ for some $\kappa > 0$, and with a different lower bound on t. So we again have a super-exponential rate of approach of distinct level sets of ω along a segment, but this time whenever $C < C'$:

Corollary 4.2. *Let D, A, δ, ω be as in Theorem 4.1. If a, b from (2.8), (2.9) satisfy (4.9) with $c, c' > 0$ and $C < C'$, then the claims in Theorem 1.1 hold for all large enough t, with e^{-At} replaced by $e^{-\kappa t^{(16+C')/(16+C)}}$ for some $\kappa > 0$.*

References

[1] S. Denisov, *Infinite superlinear growth of the gradient for the two-dimensional Euler equation.* Discrete Contin. Dyn. Syst. A **23** (2009), 755–764.

[2] S. Denisov, *Double-exponential growth of the vorticity gradient for the two-dimensional Euler equation* Proceedings of the AMS **143** (2015), 1199–1210.

[3] S. Denisov, *The sharp corner formation in 2D Euler dynamics of patches: infinite double exponential rate of merging,* Arch. Rational Mech. Anal. **215** (2015), 675–705.

[4] E. Hölder, *Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrentzten inkompressiblen Flüssigkeit (German),* Math. Z. **37** (1933), 727–738.

[5] A. Kiselev and V. Šverák, *Small scale creation for solutions of the incompressible two dimensional Euler equation,* Annals of Math. **180** (2014), 1205–1220 .

[6] A. Kiselev and A. Zlatoš, *Blow up for the 2D Euler equation on some bounded domains,* J. Differential Equations **259** (2015), 3490–3494.

[7] Z. Lei and J. Shi, *Infinite-time exponential growth of the Euler equation on two-dimensional torus,* preprint.

[8] G. Luo and T.Y. Hou, *Potentially singular solutions of the 3D axisymmetric Euler equations,* Proc. Natl. Acad. Sci. U.S.A. **111** (2014), 12968–12973.

[9] G. Luo and T.Y. Hou, *Toward the finite-time blowup of the 3D axisymmetric Euler equations: a numerical investigation,* Multiscale Model. Simul. **12** (2014) 1722–1776.

[10] N. S. Nadirashvili, *Wandering solutions of the two-dimensional Euler equation (Russian),* Funktsional. Anal. i Prilozhen. **25** (1991), 70–71; translation in Funct. Anal. Appl. **25** (1991), 220–221.

[11] W. Wolibner, *Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long (French),* Mat. Z. **37** (1933), 698–726.

[12] V.I. Yudovich, *The flow of a perfect, incompressible liquid through a given region,* Dokl. Akad. Nauk SSSR **146** (1962) 561–564 (Russian); translated as Soviet Physics Dokl. **7** (1962) 789–791.

[13] V. I. Judovič, *The loss of smoothness of the solutions of Euler equations with time (Russian),* Dinamika Splošn. Sredy **16** (1974), 71–78.

[14] V. I. Yudovich, *On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid,* Chaos **10** (2000), 705–719.

[15] X. Xu, *Fast growth of the vorticity gradient in symmetric smooth domains for 2D incompressible ideal flow,* J. Math. Anal. Appl. 439 (2016) 594–607.

[16] A. Zlatoš, *Exponential growth of the vorticity gradient for the Euler equation on the torus,* Adv. Math. **268** (2015), 396–403.

Department of Mathematics, University of California San Diego, La Jolla, CA 92093
Email: zlatos@ucsd.edu