ON DISCRETE MODELS OF THE EULER EQUATION

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Abstract. We consider two discrete models for the Euler equation describing incompressible fluid dynamics. These models are infinite coupled systems of ODEs for the functions u_j which can be thought of as wavelet coefficients of the fluid velocity. The first model has been proposed and studied by Katz and Pavlović. The second has been recently discussed by Waleffe and goes back to Obukhov studies of the energy cascade in developed turbulence. These are the only basic models of this type satisfying some natural scaling and conservation conditions. We prove that the Katz-Pavlović model leads to finite time blowup for any initial datum, while the Obukhov model has a global solution for any sufficiently smooth initial datum.

1. Introduction

The regularity of solutions to the incompressible Euler equation in dimension three remains one of the most important open problems of mathematical fluid dynamics. Recently, a number of simpler models have been proposed and studied by several authors as a way to gain insight into the possible behavior of solutions to Euler and Navier-Stokes equations. Different models have been suggested by Katz and Pavlović [9], Friedlander and Pavlović [7], Dinaburg and Sinai [3] and Waleffe [13]. Although these models are fairly drastic simplifications of the original problem, they do keep a few of the most important characteristic features of Euler equations. Moreover, we will argue below that some of these models are quite natural in their own right as they constitute the simplest class satisfying certain scaling and dimensional conditions.

A model proposed by Katz and Pavlović [9] is based, formally, on a wavelet expansion of a scalar function $u(x,t)$, $x \in \mathbb{R}^3$, over a set of dyadic cubes in $\mathbb{R}^3$. The dyadic cubes are cubes with the side lengths 2^j, $j \in \mathbb{Z}$, with vertices at the points of $2^j\mathbb{Z}^3$. If Q is a dyadic cube of size 2^j, then its parent $\tilde{Q}$ is a cube with side length 2^{j+1} containing Q. Define $C^1(Q)$ the set of all 8 children of Q, each having side length 2^{j-1}, and more generally $C^m(Q)$ the set of all 2^{3m} m^{th} generation "descendants" of Q. The Katz-Pavlović model equations describing the evolution of the wavelet coefficient of $u(x,t)$ corresponding to the cube Q are the given by [9]

$$\frac{du_Q}{dt} = 2^{5j/2}u_{\tilde{Q}}^2 - 2^{5(j+1)/2}u_Q \sum_{Q' \in C^1(Q)} u_{Q'}. \tag{1.1}$$

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The model has quadratic nonlinearity and (formally) conserves the energy $\sum_Q u_Q(t)^2$. It has been motivated to some extent by the work [8], where partial regularity of the weak solutions to the Navier-Stokes equations with hyperdissipation was studied. The approach of [8] is based on controlling the "wavelet coefficients" of the solution $u_Q = \|\phi_Q(x)P_j u\|_2$, where P_j are Littlewood-Paley projections restricting the Fourier transform $\hat{u}(\xi)$ to the annulus of size $\sim 2^j$, and ϕ_Q is a certain smooth function supported on a cube Q of size $2^{-j(1-\epsilon)}$, $\epsilon > 0$. The coupled system one gets for the wavelet coefficients from the Navier-Stokes (or, in our case, Euler) equations is complex, and (1.1) can be obtained from it by dropping all but a few terms. Thus, u_Q can be roughly thought of as "wavelet coefficients" describing parts of the solution localized in the cube Q and in the Fourier space at about $|\xi| \sim 2^j$. The choice of the scaling factors in (1.1) is determined by the relation $\|w_Q\|_\infty \sim 2^{3j/2}\|w_Q\|_2$ for a wavelet w_Q supported on a dyadic cube Q of side length 2^j in $\mathbb{R}^3$ and the bound $\|(u\cdot\nabla)u\|_2 \leq \|u\|_\infty\|\nabla u\|_2$ (see [8, 9] for more details).

In [9] Katz and Pavlović showed, in particular, that for any $\varepsilon > 0$, there exist initial data $u_j(0) \in H^{3/2+\epsilon}$ which lead to blowup in a finite time. Friedlander and Pavlović [7] considered a related vector model where they also prove blowup in a finite time. Recently, Waleffe [13] proposed a simplified model which instead of the branching structure of the coupled coefficients constitutes a linear tree of the functions $u_j(t)$ satisfying an infinite system of differential equations

$$u_j' = \lambda^j u_{j-1}^2 - \lambda^{j+1} u_j u_{j+1}, \;\; j > j_0, \;\; u_{j_0}' = -\lambda^{j_0+1} u_{j_0} u_{j_0+1}. \tag{1.2}$$

Here $\lambda > 1$ is a parameter, and j_0 is an index corresponding to the largest relevant space scale (for instance, a period in the periodic setting). Without loss of generality, we will set $j_0 = 0$ for the rest of the paper. The original Katz-Pavlović model reduces to the system (1.2) with $\lambda = 2$ if one assumes that the coefficients of all cubes of the same side length are the same. It is natural to define the Sobolev spaces associated with (1.2) as

$$H^s := \{u_j \mid \|\{u_j\}\|_{H^s}^2 \equiv \sum_{j \geq j_0} \lambda^{2sj}|u_j|^2 < \infty\}.$$

Waleffe proved that there exist initial data for which the blowup in (1.2) happens in any H^s, $s > 0$, and suggested a different model, given by

$$u_j' = \lambda^j u_{j-1} u_j - \lambda^{j+1} u_{j+1}^2, \;\; j > 0, \;\; u_0' = -\lambda u_1^2. \tag{1.3}$$

This model goes back to the work of Obukhov [11] who proposed it in a paper devoted to atmosphere studies as a simple model for studying the cascade mechanism of energy transfer in the developed turbulence. It has been shown in [13] that the model (1.2) may be related to the inviscid Burger's equation, making blowup not surprising. In particular, this model has a built in mechanism of transferring the energy to higher modes. On the other hand, the Obukhov model lacks this mechanism and is thus more subtle and perhaps more realistic. Moreover, in Proposition 2.4 we prove that these models constitute two basic building blocks of all linear tree coupled mode models satisfying four natural conditions: a quadratic nonlinearity, appropriate scaling corresponding to the $(u \cdot \nabla)u$ term, energy conservation, and nearest neighbor coupling. All of these except the last one are the features derived from the Euler equation; the last condition is clearly a simplification designed to make the problem tractable. Our main goal in this note is to prove the following two theorems, which to some extent confirm the above sentiment. For the rest of the paper we call, following Waleffe, model (1.2) the KP model and model (1.3) Obukhov model.

Theorem 1.1. *In the KP model, any non-zero initial datum belonging to H^1 leads to a finite time blowup (in H^1).*

We note that the H^1 condition is needed in general to show local existence of solutions; we discuss this point in section 2. If one accepts a parallel between the KP model and inviscid Burger's equation, the result is not surprising. Indeed, any non-constant initial datum for the Burger's equation with periodic boundary conditions leads to blowup in finite time.

On the other hand, solutions of the Obukhov model are regular.

Theorem 1.2. *In the Obukhov model, the solution corresponding to any initial datum in H^s, $s > 1$, is regular for all times. That is, for any $u_0 \in H^s$ with $s > 1$ and for any $T > 0$ there exists a unique solution $\{u_j\} \in C([0,T], H^s)$ such that $u_j(0) = (u_0)_j$.*

This theorem is probably the most interesting, and certainly the most subtle and difficult to prove result of this paper. It demonstrates an intriguing dichotomy between the properties of two basic dyadic models.

For generic initial data in the Obukhov model, we have a stronger regularity and even dissipation properties, in the following sense.

Theorem 1.3. *Let $b_j(\omega)$ be independent uniformly bounded random variables such that the probability of $b_j(\omega)$ being nonpositive is uniformly bounded away from zero: $P[b_j(\omega) \le 0] > \rho > 0$. Assume $a_j > 0$ are such that $\sum_j \lambda^{2sj}|a_j|^2 < \infty$, $s > 1$. Then with probability one a solution $\{u_j(t)\}$ of the Obukhov model corresponding to the initial datum $u_j(0) = a_j b_j(\omega)$ satisfies $\|u\|_{H^r} \le C(r,\omega)$ for all times t and any $r < s$. Moreover, as $t \to \infty$, we have*

$$\lim_{t\to\infty} \|u(t)\|^2_{H^r} = \lim_{t\to\infty} u_0(t)^2 = E_0 \equiv \sum_{j\ge 0} u_j(0)^2, \tag{1.4}$$

that is, the solution u converges in H^r to a constant solution with all energy concentrated in the lowest mode.

We describe some finer properties of the dynamics of the KP and Obukhov models as well.

There are many interesting questions that remain open. In particular, whether Theorem 1.2 holds for $s = 1$. Other natural questions include global existence of solutions in the branched Obukhov model (an analog of (1.1)) and in the Navier-Stokes version of (1.3). It seems reasonable to expect that regularity results for (1.3) should carry over to these cases. Clearly, the analog of the Laplacian term only adds dissipation, and branching is likely to make energy cascade towards high level modes harder to realize. However, on the technical level, the questions are not trivial due to the subtleties of the proof of Theorem 1.2. We did not attempt to address these issues here to keep the present paper from becoming overly technical.

We note that models similar in spirit to (1.2) and (1.3) — shell models — have been studied in the physics literature for a long time (see, e.g., [5, 6, 12, 10], and [1] for a recent review). One version of these models, the "Sabra" shell model, has recently been studied analytically in [2]. What makes questions like existence and (in some sense) regularity of solutions easier to treat in the shell models setting than in (1.2), (1.3) is a weaker scaling factor in the equations (corresponding, generally speaking, to the scaling assumption $\|(u \cdot \nabla)u\|_2 \sim \|u\|_2 \|\nabla u\|_2$). This leads to the shell models being "subcritical", that is the nonlinearity is controlled by the dissipation term. However the models (1.2), (1.3), even when a term representing Laplacian with appropriate scaling is added, are "supercritical". It is only certain monotonicity properties of these models and detailed analysis of their dynamics that make answering the basic regularity/blowup questions possible. Many of the subtler results established for the "Sabra" model in [2] appear harder to establish for the dyadic models of Navier-Stokes equations at this time.

In the next section we collect some preliminary results, postponing the proof of local existence in H^1 of solutions to our models to an appendix. The proofs of our main theorems appear in Sections 3–5.

2. Preliminaries

In this section we collect and prove some simple useful facts about the KP and Obukhov models. Let us start by stating the result on local existence of solutions.

Proposition 2.1. *Assume that the initial datum $u_j(0)$ for either KP or Obukhov model lies in H^s for some $s \geq 1$. Then there exists a unique solution $u \in C([0,T], H^s)$, for some time $T = T(\|u(0)\|_{H^s}) > 0$. The H^s norm of this solution satisfies*

$$\|u(t)\|_{H^s} \leq \|u(0)\|_{H^s} e^{C\int_0^t \sup_j\{\lambda^j u_j(r)\}\, dr}. \tag{2.1}$$

In particular, the solution blows up in finite time τ only if $\int_0^\tau \sup_j\{\lambda^j u_j(r)\}\, dr = \infty$.

Proof. Local existence of solutions has been proved in [7] using fixed point arguments. The argument in [7] is given for the case of KP model with a specific choice of λ (hence our H^1 notation corresponds to $H^{5/2}$ in their setting), but it can be adapted easily to the Obukhov model as well. We sketch this argument in the Appendix. Therefore, here we will only discuss (2.1). Carrying out the differentiation and substituting the expression for the time derivatives from (1.2) (resp. (1.3)) we find

$$\frac{d}{dt}\sum_j \lambda^{2sj} u_j^2(t) \leq C\sup_j\{\lambda^j u_j(t)\} \sum_{j=0}^{\infty} \lambda^{2sj} u_j^2(t),$$

providing the required bound. □

Now we make a few critical observations on the monotonicity properties of our models. From now on, all properties are stated for the solutions described in Proposition 2.1, and hold on the existence interval described in that proposition.

Proposition 2.2. *The following properties hold for KP and Obukhov models.*

- *Both KP and Obukhov models conserve the energy* $E_0 \equiv \sum_{j\geq 0} |u_j(t)|^2$.
- *In the KP model, if* $u_j(t_0) \geq 0$ *for some* t_0, *then* $u_j(t) \geq 0$ *for all times* $t \geq t_0$.
- *In the Obukhov model, if* $u_j(t_0) \leq 0$ *for some* t_0, *then* $u_j(t) \leq 0$ *for all times* $t \geq t_0$.

Proof. The first property is checked directly by differentiating the energy. Clearly each $u_j(t)$ is differentiable, and the fact that solution is H^1 allows us to sum the right hand side, obtaining zero. To prove the last two properties, one just writes explicitly the expression for $u_j(t)$. For example, in the Obukhov model we have

$$u_j(t) = e^{\lambda^j \int_{t_0}^t u_{j-1}(r)\,dr} \left(u_j(t_0) - \lambda^{j+1} \int_{t_0}^t e^{-\lambda^j \int_{t_0}^\rho u_{j-1}(r)\,dr} u_{j+1}^2(\rho)\,d\rho \right).$$

□

Let us define $E_j(t) \equiv \sum_{l\geq j} |u_l(t)|^2$. Note that $E_j'(t) = 2\lambda^j u_{j-1}^2 u_j$ in the KP and $E_j'(t) = 2\lambda^j u_{j-1} u_j^2$ in the Obukhov model. Hence, in both models positive coefficients generate energy transfer to higher modes and negative coefficients transfer energy to lower modes. Since Proposition 2.2 shows that positive coefficients are stable in the KP model and negative ones are stable in the Obukhov model, it is not surprising that the latter is more regular. One more indication of this regularity is the following description of the dynamics corresponding to initial data with only finite number of excited modes.

Proposition 2.3. *In the Obukhov model, if* $u_j(0) = 0$ *for any* $j > j_1$, *then* $u_j(t) = 0$ *for any* t *and* $j > j_1$. *In this case, as time goes to infinity, all energy concentrates in the first mode* u_0. *Moreover, if* u *is any solution that remains in* H^1 *for all time, then* $u_j(t) \to 0$ *as* $t \to \infty$ *for all* $j > 0$.

Proof. The first statement is obvious. Let us prove the third statement (which in turn proves the second in the case of eventually vanishing $u_j(0)$). It is clear from (1.3) that $u_1(t) \to 0$, or else u_0 grows unboundedly large negative, contradicting the energy conservation. This holds since $|u_1'(t)| \leq \lambda^2 E_0$, so the function $u_1(t)$ cannot just have increasingly narrow spikes. Now, if $u_j(t) \to 0$, then $u_{j+1}(t) \to 0$. Otherwise the equation

$$u_j'(t) = \lambda^j u_{j-1} u_j - \lambda^{j+1} u_{j+1}^2$$

and $|u_{j-1}| \leq \sqrt{E_0}$ give us a contradiction as in the case $j = 0$ above. □

Finally, before proving our main results, we state the following observation, which is elementary to verify. It shows that the KP and Obukhov models are basic building blocks of all mode couplings with certain natural properties.

Proposition 2.4. *Assume that real valued functions* $u_j(t)$ *satisfy an infinite system of differential equations such that:*

- *The right hand side is quadratic in* u
- *The coupling is nearest neighbor only, that is only* u_{j-1}, u_j *or* u_{j+1} *may appear in the equation for* u_j'
- *Each term on the right hand side of the equation for* u_j *has a factor of* λ^j *times a constant independent of* j

- *The energy* $\sum_j u_j^2$ *is conserved.*

Then the system must have the form

$$u_j' = \alpha(\lambda^j u_{j-1}^2 - \lambda^{j+1} u_j u_{j+1}) + \beta(\lambda^j u_j u_{j-1} - \lambda^{j+1} u_{j+1}^2), \tag{2.2}$$

that is, the right hand side must be a linear combination of the KP and Obukhov models.

Theorems 1.1 and 1.2 show that if the initial datum is in H^s, $s > 1$, then the solution of (2.2) always blows up resp. stays regular if $\alpha = 1$, $\beta = 0$ resp. $\alpha = 1$, $\beta = 0$. It is an interesting open question how the competition of these two phenomena affects the behavior of solutions of (2.2) when both $\alpha, \beta \neq 0$. Notice that when $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta)$, then we do not have at our disposal a version of the maximum principle, as are the second and third claims of Proposition 2.2. This structural difference in the general case will present an extra difficulty in the analysis of the dynamics of the problem.

3. Blowup in the Katz-Pavlović model

In this section we prove Theorem 1.1. We therefore assume, towards contradiction, that the solution exists in H^1 for all times and $\|u\|_{H^1}$ is locally bounded. Let us define the "positive" and "negative" energies by

$$E_{\pm,j}(t) = \sum_{l \geq j, \pm u_l \geq 0} u_l(t)^2.$$

The following lemma shows that for any non-zero initial datum and any j, $E_{+,j}(t) > 0$ for $t > t_j$.

Lemma 3.1. *For any non-zero initial datum and any* $j > 0$, *we have* $u_j(t) > 0$ *for* $t > t_j$.

Proof. Recall that $j_0 = 0$. Note that

$$u_0(t) = u_0(0) e^{-\lambda \int_0^t u_1(s)\, ds}.$$

Assume that $u_1(0) < 0$, and never turns positive. Then at least we must have $u_1(t) \to 0$ as $t \to \infty$, or else u_0 grows unbounded. But then we get a contradiction with the equation

$$u_1' = \lambda u_0^2 - \lambda^2 u_1 u_2,$$

since $|u_0(t)| \geq |u_0(0)| > 0$ for all times (if $u_0(0) = 0$, it is never in the play, and so we should start from $j = 1$). Thus u_1 must become positive. Now if $u_j(t_j) > 0$, then $u_{j+1}(t)$ must turn positive at some finite time too, by an argument identical to the above. ☐

Next, we show that the positive energy is always increasing.

Lemma 3.2. *For any* j, $E_{+,j}(t)$ *is monotone increasing. The negative energy* $E_{-,j}(t)$ *is monotone decreasing.*

Proof. At any given moment, $E_{+,j}(t)$ can be written as a sum of sums $\sum_{j_1 \leq l \leq j_2} u_l^2$, where $u_l(t) \geq 0$ for $j_1 \leq l \leq j_2$, and $u_{j_1-1}(t), u_{j_2+1}(t) < 0$ (or $j_1 = 0$). Then

$$\frac{d}{dt} \sum_{j_1 \leq l \leq j_2} u_l^2 = 2 \sum_{j_1 \leq l \leq j_2} u_l(\lambda^l u_{l-1}^2 - \lambda^{l+1} u_l u_{l+1}) = 2(\lambda^{j_1} u_{j_1} u_{j_1-1}^2 - \lambda^{j_2+1} u_{j_2}^2 u_{j_2+1}) \geq 0.$$

Moreover, we see from the above argument that

$$E'_{+,j}(t) \geq 2\lambda^j u_j u_{j-1}^2. \tag{3.1}$$

This bound is not relevant if $u_j(t) \leq 0$, but we will need it later in the case when we know that u_j is positive. The proof for $E_{-,j}$ is similar. □

Theorem 1.2 will be a simple consequence of the following key lemma.

Lemma 3.3. *Let $q \in (\lambda^{-1}, 1)$ and $\rho \equiv (\lambda q)^{-1} \in (0,1)$, and assume that j is large enough (depending on λ, q, and E_0). Then for any $C > 0$ there is $A = A(C, \lambda, q) < \infty$ (independent of j) so that if $E_{+,j}(t_0) \geq Cq^j$ for some t_0, then there exists a time $t \in [t_0, t_0 + 2\tau_j]$, with $\tau_j \equiv A\rho^j$, such that either $E_{+,j+1}(t) \geq Cq^{j+1}$ or $E_{+,j}(t) \geq 2Cq^j$.*

Proof. Assume that for all $t \in [t_0, t_0 + 2\tau_j]$ we have $E_{+,j+1}(t) \leq Cq^{j+1}$. Then by $E_{+,j}(t) \geq E_{+,j}(t_0) \geq Cq^j$, we must have $u_j(t) \geq 0$ and $u_j^2(t) \geq Cq^j(1-q)$ for any $t \in [t_0, t_0 + 2\tau_j]$. Let

$$A \equiv \max\left\{1, \frac{1+\lambda q}{\sqrt{C}(1-q)^2}, \frac{4\sqrt{E_0}}{C(1-q)^2}\right\}$$

Consider first the case where $u_{j+1}(t_1) \geq 0$ for some $t_1 \in [t_0, t_0 + \tau_j]$. The amount of energy transfer from j^{th} to $(j+1)^{\text{st}}$ mode is bounded from below by (recall (3.1))

$$\int_{t_1}^{t_1+\tau_j} E'_{+,j+1}(t)\,dt \geq 2\lambda^{j+1} \int_{t_1}^{t_1+\tau_j} u_j(t)^2 u_{j+1}(t)\,dt.$$

It must not exceed Cq^{j+1} to avoid contradiction, so

$$\int_{t_1}^{t_1+\tau_j} u_{j+1}(t)\,dt \leq \frac{q}{(1-q)\lambda^{j+1}}.$$

But

$$u'_{j+1}(t) = \lambda^{j+1}u_j^2 - \lambda^{j+2}u_{j+1}u_{j+2};$$

thus

$$u_{j+1}(t_1+\tau_j) - u_{j+1}(t_1) \geq \lambda^{j+1}Cq^j(1-q)\tau_j - \lambda^{j+2}\frac{q}{(1-q)\lambda^{j+1}}\sqrt{C}q^{(j+1)/2}. \tag{3.2}$$

The above bound follows from the fact that $u_{j+1} \geq 0$ and $u_{j+2} \leq \sqrt{C}q^{(j+1)/2}$ on $[t_1, t_1 + \tau_j]$, the latter by our assumption on $E_{+,j+1}$. The right hand side of (3.2) equals

$$\sqrt{C}q^{(j+1)/2}\left(\sqrt{C}\lambda^{j+1}\tau_j(1-q)q^{(j-1)/2} - \frac{\lambda q}{1-q}\right). \tag{3.3}$$

Since $A \geq (1+\lambda q)/\sqrt{C}(1-q)^2$ and $\tau_j = A\rho^j$, the expression in the brackets in (3.3) is greater than one.

It remains to consider the case where $u_{j+1}(t) < 0$ for $t \in [t_0, t_0 + \tau_j]$. Recall that we have $u_j^2 \geq Cq^j(1-q)$, and $-u_{j+1}u_{j+2} \geq u_{j+1}F_0$, where $F_0^2 = E_0$ is the total (conserved) energy of the solution. Then from (1.2) we obtain for any $t_1 \in [t_0, t_0 + \tau_j]$,

$$u_{j+1}(t) \geq u_{j+1}(t_1)e^{\lambda^{j+2}F_0(t-t_1)} + C\int_{t_1}^{t} \lambda^{j+1}q^j(1-q)e^{\lambda^{j+2}F_0(t-s)}\,ds$$

$$\geq e^{\lambda^{j+2}F_0(t-t_1)}\left(u_{j+1}(t_1)+C\lambda^{-1}q^j(1-q)F_0^{-1}(1-e^{-\lambda^{j+2}F_0(t-t_1)})\right). \tag{3.4}$$

Assume without loss of generality that j is large enough, so that $\lambda^j F_0\rho^j >> 1$ (then also $\lambda^{j+2}F_0\tau_j >> 1$ because $A \geq 1$). If for some $t_1 \in [t_0, t_0+\tau_j/2]$ the value of $u_{j+1}(t)$ goes above $-\frac{1}{2}C\lambda^{-1}q^j(1-q)F_0^{-1}$, we see from (3.4) that $u_{j+1}(t)$ will become positive before $t_0+\tau_j$. Thus, we must have

$$u_{j+1}(t) \leq -\tfrac{1}{2}C\lambda^{-1}q^j(1-q)F_0^{-1}$$

for $t \in [t_0, t_0+\tau_j/2]$. But then for these t,

$$\frac{d}{dt}u_j^2 \geq -2\lambda^{j+1}u_j^2u_{j+1} \geq \lambda^jC^2q^{2j}(1-q)^2F_0^{-1}.$$

This implies

$$u_j(t_0+\tau_j/2)^2 \geq u_j(t_0+\tau_j/2)^2 - u_j(t_0)^2 \geq \tfrac{1}{2}\tau_j\lambda^jC^2q^{2j}(1-q)^2F_0^{-1} \geq 2Cq^j$$

since $A \geq 4F_0/C(1-q)^2$. Thus, $E_{+,j}(t_0+\tau_j/2) \geq 2Cq^j$, and the lemma is proved. □

The second alternative in Lemma 3.3 is needed since if u_{j+1} is very large negative, it seems reasonable that it may take some time before it becomes positive and the positive energy starts being transferred up. The proof is based on the observation that in this case, the negative energy from the $(j+1)^{\text{st}}$ mode is quickly transferred into the positive one at the j^{th} mode. The following corollary shows that actually the lemma holds in a simpler form, without the second alternative, if we increase the waiting time slightly.

Corollary 3.4. *In the setting of Lemma 3.3, there exists* $t \in [t_0, t_0+2\log_2(E_0/Cq^j)\tau_j]$, *such that* $E_{+,j+1}(t) \geq Cq^{j+1}$.

Proof. Recall that the total energy of the solution is equal to E_0. Applying Lemma 3.3 repeatedly on the j^{th} level, we see that the second alternative cannot hold more than $\log_2(E_0/Cq^j)$ times. □

Now we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Pick some $q \in (\lambda^{-1}, 1)$ and denote $\tilde{\tau}_j = 2\log(E_0/Cq^j)\tau_j$. It is clear that

$$\tilde{\tau} = \sum_j \tilde{\tau}_j < \infty.$$

Lemma 3.1 shows that each u_j (in particular, those to which Lemma 3.3 applies) will eventually become positive. Using Corollary 3.4 one then shows by induction that for some $t_0 < \infty$, $C > 0$, and for all large j, there exists $t_j \in [t_0, t_0+\tilde{\tau}]$ such that $E_{+,j}(t) \geq Cq^j$. Note that t_j can be chosen to be increasing. But then the H^1 norm satisfies

$$\|u(t_j)\|_{H^1}^2 \geq C\lambda^{2j}q^j \to \infty$$

because $q > \lambda^{-1}$. The proof is finished. □

4. Almost sure estimates in the Obukhov model

In this section we prove Theorem 1.3 as a warmup. This result is rather straightforward, relying only on the fact that negative coefficients are stable in the Obukhov model and that the energy always flows to the lower modes across any negative site.

Proof of Theorem 1.3. Consider a realization of $\{b_j(\omega)\}$, that has infinitely many sites $j_1(\omega) < \cdots < j_n(\omega) < \dots$ at which $b_{j_l}(\omega) \le 0$. Such realizations occur with probability 1, by the hypothesis. We also set $j_0(\omega) = 0$ by convention. Since

$$E'_{j_l(\omega)+1}(t) = 2\lambda^{j_l(\omega)+1} u_{j_l(\omega)}(t) u_{j_l(\omega)+1}(t)^2,$$

we see that by Proposition 2.2, $E'_{j_l(\omega)+1}(t) \le 0$, for all $t > 0$ for which the solution exists. Therefore, for all such times we have the following estimate

$$\|u(t)\|^2_{H^r} \le \sum_{l=0}^{\infty} \lambda^{2j_l(\omega)r} \left(\sum_{m=j_{l-1}(\omega)}^{j_l(\omega)} |u_m(0)|^2 \right) \le C_1 \sum_{l=0}^{\infty} \lambda^{2j_l(\omega)r - 2j_{l-1}(\omega)s}, \tag{4.1}$$

since $|u_m(0)| \le C\lambda^{-ms}$ by assumption. We claim that for any $\alpha > 0$, with probability one we have

$$j_l(\omega) - j_{l-1}(\omega) \le \alpha j_{l-1}(\omega) \tag{4.2}$$

for all but finitely many l. If that were the case, take $\alpha = (s-r)/2r$. Then

$$2j_l(\omega)r - 2j_{l-1}(\omega)s \le -(s-r)j_{l-1}(\omega)$$

almost surely for all but finitely many l. In that case, the sum (4.1) converges almost surely, proving $\|u\|_{H^r} \le C(r,\omega)$. This and local existence in H^1 (note that $1 < s$) now gives the existence of the solution in H^r, $r < s$, for all times.

To prove (4.2), split natural numbers into non-overlapping intervals $L_n \equiv \{j \,|\, 3^{n-1} < j \le 3^n\}$. It is clear that for all α small enough, any interval $I_l = (j_{l-1}, j_l)$ satisfying $j_l - j_{l-1} > \alpha j_{l-1}$ will have an intersection of size at least $\alpha 3^{n-2}$ with some L_n. The probability of having such an interval of negative $b_j(\omega)$'s in L_n is less than $3^n(1-\rho)^{\alpha 3^{n-2}}$. Since the events of having such an interval in L_n for different n are independent, we find that the probability of having an infinite number of such intervals is zero by the Borel-Cantelli lemma.

The fact that $\|u(t)\|^2_{H^r}$ converges to E_0 follows from the above argument and Proposition 2.3. Indeed, with probability one $u_0(t)^2 \le \|u(t)\|^2_{H^r} \le u_0(t)^2 + \sum_{l \ge 1} A_l(t)$, where

$$A_l(t) = \sum_{m=j_{l-1}(\omega)+1}^{j_l(\omega)} |u_m(t)|^2 \lambda^{2mr},$$

and we saw that $A_l(t) \le C(\omega)\lambda^{-(s-r)l}$. But Proposition 2.3 also implies $A_l(t) \to 0$ as $t \to \infty$ for any l. Thus, by the dominated convergence theorem,

$$\lim_{t\to\infty} (\|u(t)\|^2_{H^r} - u_0(t)^2) = 0.$$

In particular, for $r = 0$ we get using energy conservation

$$E_0 = \|u(t)\|_{L^2}^2 = \lim_{t\to\infty} u_0(t)^2$$

which yields (1.4). □

5. Regularity in the Obukhov model

We will now prove Theorem 1.2. Assume, towards contradiction, that for some initial datum $u(0)$ with $\|u(0)\|_{H^s} \le 1$ (this can be assumed without loss of generality, by scaling in u and t), u blows up at time $T < \infty$, that is,

$$\limsup_{t\to T} \|u(t)\|_{H^s} = \infty \tag{5.1}$$

and $\|u(t)\|_{H^s}$ is bounded for $t \in [0, T - \varepsilon]$ and any $\varepsilon > 0$ (using (2.1) and $\|u(t)\|_{H^s} \ge \sup_j\{\lambda^j u_j(t)\}$, one can actually show that the lim sup must be lim). Proposition 2.1 shows that this is only possible if

$$\limsup_{t\to T} \sup_j \{\lambda^j u_j(t)\} = \infty. \tag{5.2}$$

Although a priori it only follows from the proposition that the lim sup is ∞ for some $T^* \le T$, it is immediate from $s > 1$ that in that case (5.1) would hold for T^* and so $T^* = T$. We have

$$u_j(0) \le \lambda^{-sj} \tag{5.3}$$

and by $\|u(t)\|_{L^2} = \|u(0)\|_{L^2} \le \|u(0)\|_{H^s} \le 1$,

$$|u_j(t)| \le 1. \tag{5.4}$$

Finally, we recall that

$$E_j(t) \equiv \sum_{l\ge j} u_l(t)^2$$

satisfies

$$E_j'(t) = 2\lambda^j u_{j-1}(t) u_j(t)^2 \tag{5.5}$$

(with $u_{-1} \equiv 0$).

Our strategy will be to first narrow down the possibility of blowup to a specific scenario (Lemma 5.1) and then exclude blowup under this scenario (Lemma 5.3). Let $t_j < T$ be the first time such that

$$u_j(t_j) = \lambda^{-j} \tag{5.6}$$

(if there is no such time we let $t_j \equiv \infty$). If $t_j < \infty$, then

$$u_j(t) > 0 \text{ for } t \in [0, t_j], \tag{5.7}$$

by Proposition 2.2. Therefore we can use

$$\frac{u_j'}{u_j} = \lambda^j u_{j-1} - \lambda^{j+1} \frac{u_{j+1}^2}{u_j} \tag{5.8}$$

for $j > 0$ to obtain from (5.3) and (5.6)

$$(s-1)j\log\lambda \le \log\frac{u_j(t_j)}{u_j(0)} \le \lambda^j \int_0^{t_j} u_{j-1}(t)\,dt \le \lambda^j T \sup_{t\le t_j} u_{j-1}(t).$$

Hence

$$\sup_{t\le t_j} u_{j-1}(t) \ge \frac{(s-1)\log\lambda}{T\lambda} j\lambda^{-(j-1)}, \tag{5.9}$$

which means that $t_{j-1} \le t_j$ once $j > T\lambda((s-1)\log\lambda)^{-1}$ (this is obviously true also when $t_j = \infty$). Therefore t_j is eventually non-decreasing and has a limit τ. Now (5.2) and (5.4) imply that $\tau \neq \infty$ and so $\tau \le T$. Then (5.9) shows $t_{j-1} < t_j$ for large j as well as $\sup_{t\le t_j} u_{j-1}(t)\lambda^{j-1} \to \infty$ as $j \to \infty$, and so $T \le \tau$ (since blowup cannot happen before T). Hence t_j is eventually increasing and $t_j \to T$. From now on we will consider j large enough so that $T-1 < t_j < t_{j+1} < T$ and set $I_j \equiv [t_j, t_{j+1}]$. Note that

$$u_j(t_{j+1}) \ge 0 \tag{5.10}$$

because t_{j+1} is the first time when u_{j+1} reaches λ^{-j-1}, and u_{j+1} (if it is positive) has to decrease when $u_j < 0$. At various places in the argument below we will further increase the size of j under consideration.

We choose $\varepsilon \in (0, \frac{s-1}{5})$. For $t \le t_j$ and $l \ge 1$ we have by (5.7) and (5.8),

$$\log\frac{u_{j+l}(t)}{u_{j+l}(0)} \le \lambda^{j+l}\int_0^t u_{j+l-1}(\tau)\,d\tau \le \lambda T$$

since $t < t_{j+l}$. Therefore by (5.3),

$$u_{j+l}(t) \le e^{\lambda T}\lambda^{-s(j+l)} \le \lambda^{-(s-\varepsilon)(j+l)} \tag{5.11}$$

for large enough j, $t \le t_j$, and $l \ge 1$. This and (5.8) gives

$$(s-1-\varepsilon)j\log\lambda \le \log\frac{u_{j+1}(t_{j+1})}{u_{j+1}(t_j)} \le \lambda^{j+1}\int_{I_j} u_j(t)\,dt. \tag{5.12}$$

Thus for all large j, u_j has to become large compared to λ^{-j} somewhere on I_j, while u_{j+1} increases to λ^{-j-1} and all the higher modes are tiny. This shows that for blowup at T to occur there must be a "wave" of large $\lambda^j u_j$ moving from low to high modes, reaching infinity in finite time. Next we will show that this wave has to be eventually very thin. Namely, we will show that modes just behind the head of the wave quickly become negative when j is large.

Lemma 5.1. *For all large enough j we have $u_{j-1}(p_j) \le 0$ with $p_j \in I_j$ defined by*

$$\lambda^{j+1}\int_{t_j}^{p_j} u_j(t)\,dt = \frac{3(s-1-\varepsilon)}{4} j\log\lambda. \tag{5.13}$$

Remark. Note that this p_j is unique by (5.12) and Proposition 2.2

Lemma 5.1 will be a consequence of the following weaker formulation of the thin wave property.

Lemma 5.2. *For any j_1 there is $j > j_1$ such that $u_{j-2}(r_j) \le 0$ for $r_j \in I_j$ defined by*

$$\lambda^{j+1} \int_{t_j}^{r_j} u_j(t)\, dt = \frac{s-1-\varepsilon}{2} j \log \lambda. \tag{5.14}$$

Proof. Note that r_j is again unique. Let us assume that the statement is not true and consider large enough j_1 so that $u_{j-2}(r_j) > 0$ for all $j > j_1$. This also means that

$$u_{j-1}(t), u_j(t) > 0 \text{ for } t \in I_j \tag{5.15}$$

because $r_{j+1}, r_{j+2} > t_j$.

We have $u_{j+1}(t) \le \lambda^{-j-1}$ for $t \in I_j$ and so by (1.3) and (5.15)

$$u_j(t) \ge u_j(t_j) - \lambda^{j+1} \int_{I_j} \lambda^{-2j-2}\, dt \ge \lambda^{-j-1}(\lambda - |I_j|) \ge \lambda^{-j-1} \tag{5.16}$$

for $t \in I_j$ when j is large. This, (5.4), (5.6) and (5.8) give

$$\begin{aligned} \lambda^j \int_{I_j} u_{j-1}(t)\, dt &= \log \frac{u_j(t_{j+1})}{u_j(t_j)} + \lambda^{j+1} \int_{I_j} \frac{u_{j+1}(t)^2}{u_j(t)}\, dt \\ &\le j \log \lambda + \lambda^{j+1} |I_j| \lambda^{-j-1} \end{aligned} \tag{5.17}$$

and so

$$\lambda^{j+1} \int_{I_j} u_{j-1}(t)\, dt \le (\lambda + \varepsilon) j \log \lambda \tag{5.18}$$

if j is large. We conclude from (5.14) and (5.18) that there exists $a_j < r_j$, the first time in I_j such that

$$\frac{u_j(a_j)}{u_{j-1}(a_j)} \ge \frac{s-1-\varepsilon}{4(\lambda+\varepsilon)}. \tag{5.19}$$

Of course, sharp inequality can possibly hold only if $a_j = t_j$. Moreover, this choice of a_j and (5.18) ensure that

$$\lambda^{j+1} \int_{t_j}^{a_j} u_j(t)\, dt \le \frac{s-1-\varepsilon}{4} j \log \lambda$$

and hence by (5.14),

$$\lambda^{j+1} \int_{a_j}^{r_j} u_j(t)\, dt \ge \frac{s-1-\varepsilon}{4} j \log \lambda. \tag{5.20}$$

Now $u_{j-2}(r_j) > 0$ and (5.5) show that E_{j-1} is increasing on $[t_j, r_j]$, so that

$$\lambda^{-2j} = u_j(t_j)^2 \le E_{j-1}(t_j) \le E_{j-1}(a_j).$$

Notice also that for $t \le r_j$

$$\log \frac{u_{j+1}(t)}{u_{j+1}(t_j)} \le \lambda^{j+1} \int_{t_j}^{t} u_j(\tau)\, d\tau \le \frac{s-1-\varepsilon}{2} j \log \lambda,$$

which together with (5.11) gives for $t \le r_j$

$$u_{j+1}(t) \le \lambda^{-\frac{s+1-\varepsilon}{2} j}.$$

Therefore

$$E_{j+1}(a_j) \le \lambda^{-(s+1-\varepsilon)j} + \sum_{l=j+2}^{\infty} \lambda^{-2(s-\varepsilon)l} \le \lambda^{-(s+1-2\varepsilon)j} \le \lambda^{-2j-1} \tag{5.21}$$

if j is large. From this we have $E_{j+1}(a_j) \le \lambda^{-1}E_{j-1}(t_j)$, and we obtain

$$u_{j-1}(a_j)^2 + u_j(a_j)^2 = E_{j-1}(a_j) - E_{j+1}(a_j) \ge \tfrac{\lambda-1}{\lambda}E_{j-1}(t_j) \ge \tfrac{\lambda-1}{\lambda}u_{j-1}(t_j)^2.$$

This and (5.19) imply that with $c_1 \equiv \frac{\lambda-1}{\lambda}[(\frac{4(\lambda+\varepsilon)}{s-1-\varepsilon})^2+1]^{-1}$,

$$u_j(a_j)^2 \ge c_1 u_{j-1}(t_j)^2. \tag{5.22}$$

Similarly as in (5.17), this in turn gives for $C_1 \equiv -\frac{1}{2}\log c_1 + 1$

$$\lambda^j \int_{a_j}^{r_j} u_{j-1}(t)\,dt \le \log\frac{u_j(r_j)}{u_j(a_j)} + |I_j| \le \log\frac{u_j(r_j)}{u_{j-1}(t_j)} + C_1. \tag{5.23}$$

Next, we claim that for large enough j

$$u_j(r_j) \ge (1-|I_j|)(1-(C_2 j)^{-2})u_{j-1}(t_j) \tag{5.24}$$

with C_2 defined in (5.25) below. Assume this is not true. Note that then $u_j(r_j) \le u_{j-1}(t_j)$, and so (5.20) and (5.23) show that there is $b_j \in [a_j, r_j]$ such that

$$\frac{u_j(b_j)}{u_{j-1}(b_j)} \ge \frac{s-1-\varepsilon}{4\lambda C_1}\,j\log\lambda \equiv C_2 j. \tag{5.25}$$

This improves (5.19) by a factor of j. We now run the same energy argument as above, with a_j replaced by b_j (and ignoring the last inequality in (5.21)), to obtain $E_{j+1}(b_j) \le \lambda^{-(s-1-2\varepsilon)j}E_{j-1}(t_j)$ and

$$u_{j-1}(b_j)^2 + u_j(b_j)^2 \ge (1-\lambda^{-(s-1-2\varepsilon)j})u_{j-1}(t_j)^2 \ge (1-(C_2 j)^{-2})u_{j-1}(t_j)^2$$

for large j. (5.25) now gives $u_j(b_j) \ge (1-(C_2 j)^{-2})u_{j-1}(t_j)$, using that $(1+(C_2 j)^{-2})^{-1} \ge (1-(C_2 j)^{-2})$. But then, as in (5.16), we obtain

$$u_j(r_j) \ge u_j(b_j) - \lambda^{j+1}\int_{b_j}^{r_j} \lambda^{-2j-2}\,dt \ge (1-|I_j|)u_j(b_j), \tag{5.26}$$

where the last inequality follows from (5.16) with $t = b_j$. This shows (5.24) for large enough j. Using (5.26) again, with r_j, b_j replaced by t_{j+1}, r_j, we obtain

$$u_j(t_{j+1}) \ge (1-|I_j|)^2(1-(C_2 j)^{-2})u_{j-1}(t_j).$$

Since $\prod_{j>j_1}(1-|I_j|)^2(1-(C_2 j)^{-2}) > 0$ for large enough j_1, this means that there is $c_2 > 0$ such that for all large enough j we have $u_j(t_{j+1}) \ge c_2$. Moreover, (5.22) and an argument as in (5.26) show that we actually have for large j and any $t \in [a_j, t_{j+1}]$,

$$c_2 \le u_j(t) \le 1 \tag{5.27}$$

(with a new $c_2 > 0$). Then (5.23) gives

$$\lambda^{j+1}\int_{a_j}^{r_j} u_{j-1}(t)\,dt \le C_3$$

for $C_3 \equiv C_1 - \log c_2$. This and (5.20) means that there is $d_j \in [a_j, r_j]$ such that

$$\frac{u_j(d_j)}{u_{j-1}(d_j)} \geq \frac{s-1-\varepsilon}{8C_3}\, j \log \lambda \equiv c_3 j \tag{5.28}$$

and

$$\lambda^{j+1} \int_{a_j}^{d_j} u_j(t)\, dt \leq \frac{s-1-\varepsilon}{8} j \log \lambda,$$

and so

$$\lambda^{j+1} \int_{d_j}^{r_j} u_j(t)\, dt \geq \frac{s-1-\varepsilon}{8} j \log \lambda.$$

Thus $r_j - d_j \geq c_4 j \lambda^{-j}$ for $c_4 \equiv \frac{s-1-\varepsilon}{8\lambda} \log \lambda$.

Finally, by (5.27) we have on $[d_j, r_j]$,

$$u'_{j-1} \leq \lambda^{j-1} u_{j-1} - \lambda^j c_2^2$$

with $u_{j-1}(d_j) \leq u_j(d_j)(c_3 j)^{-1} \leq (c_3 j)^{-1}$ by (5.4) and (5.28). But then for large enough j we have $u_{j-1}(d_j) < \frac{1}{2} c_2^2$ and hence $u'_{j-1}(d_j) < 0$. This means that $u'_{j-1} < 0$ and $u_{j-1} < \frac{1}{2} c_2^2$ on $[d_j, r_j]$. Therefore $u'_{j-1} \leq -\frac{1}{2} \lambda^j c_2^2$ on $[d_j, r_j]$, which implies

$$u_{j-1}(r_j) \leq u_{j-1}(d_j) - \tfrac{1}{2} \lambda^j c_2^2 c_4 j \lambda^{-j} \leq \tfrac{1}{2} c_2^2 (1 - c_4 j)$$

which is negative for large enough j. This contradicts (5.15) and the proof is finished. □

Proof of Lemma 5.1. Let j be as in the statement of Lemma 5.2. We will show that if j is large enough, then it also satisfies the statement of Lemma 5.1.

We have (recall (5.13) and (5.14))

$$\lambda^{j+1} \int_{r_j}^{p_j} u_j(t)\, dt = \frac{s-1-\varepsilon}{4} j \log \lambda. \tag{5.29}$$

We proceed by contradiction, so assume that $u_{j-1}(p_j) > 0$. Notice that then $u_{j-1}, u_j > 0$ on $[t_j, p_j]$ (the latter by (5.10)). Hence (5.16) holds on this interval, and as in (5.17) and (5.18),

$$\lambda^{j+1} \int_{r_j}^{p_j} u_{j-1}(t)\, dt \leq (\lambda + \varepsilon) j \log \lambda. \tag{5.30}$$

Again, (5.29) and (5.30) show that there must be $e_j \in [r_j, p_j]$ such that

$$\frac{u_j(e_j)}{u_{j-1}(e_j)} \geq \frac{s-1-\varepsilon}{8(\lambda+\varepsilon)}. \tag{5.31}$$

and

$$\lambda^{j+1} \int_{e_j}^{p_j} u_j(t)\, dt \geq \frac{s-1-\varepsilon}{8} j \log \lambda. \tag{5.32}$$

Now $u_{j-2}(r_j) \leq 0$ gives $u'_{j-1} \leq -\lambda^j u_j^2$ on $[r_j, p_j]$, that is,

$$u_{j-1}(e_j) - u_{j-1}(p_j) \geq \lambda^j \int_{e_j}^{p_j} u_j(t)^2 dt$$

$$\ge \frac{\lambda^j}{p_j - e_j}\left(\int_{e_j}^{p_j} u_j(t)dt\right)^2$$
$$\ge \frac{s-1-\varepsilon}{8\lambda(p_j-e_j)} j\log\lambda \int_{e_j}^{p_j} u_j(t)dt \tag{5.33}$$

by (5.32). Since $u_{j-1}, u_j > 0$ on $[e_j, p_j]$, a similar computation as in (5.26) shows that for t in this interval $u_j(t) \ge \frac{1}{2}u_j(e_j)$ if j is large. But that, (5.31), and (5.33) yield

$$u_{j-1}(e_j) - u_{j-1}(p_j) \ge c_5 j u_{j-1}(e_j)$$

for $c_5 = \frac{(s-1-\varepsilon)^2}{128\lambda(\lambda+\varepsilon)}\log\lambda$ and all large j. Once $j > c_5^{-1}$, this contradicts $u_{j-1}(p_j) > 0$.

Thus we have showed that if $u_{j-2}(r_j) \le 0$ and j is large enough, then $u_{j-1}(p_j) \le 0$. But then also $u_{j-1}(r_{j+1}) \le 0$ because $p_j < t_{j+1} < r_{j+1}$. Lemma 5.1 and induction finish the proof. □

Hence we have narrowed the possibility of a blowup to a scenario where for all large j there is (a single) $q_j \in [t_j, p_j]$ such that

$$u_{j-1}(q_j) = 0. \tag{5.34}$$

That is, u_{j-1} vanishes while u_{j+1} is still relatively small. Indeed, (5.13) shows that

$$\log\frac{u_{j+1}(t)}{u_{j+1}(t_j)} \le \frac{3(s-1-\varepsilon)}{4} j\log\lambda,$$

for $t \le q_j$ which together with (5.11) gives for $t \le q_j$

$$u_{j+1}(t) \le \lambda^{-(1+\frac{s-1-\varepsilon}{4})j} \le \lambda^{-(1+\varepsilon)j}. \tag{5.35}$$

Of course, blowup can now come only from large $u_j(q_j)$ because all the other modes are controlled by λ^{-j}. Yet since u_{j-1} becomes negative on I_j, we can expect that a portion of u_j energy will be passed to the lower modes, rather than transferred to u_{j+1}, making blowup unlikely. This intuition will be confirmed if we prove the following lemma.

Lemma 5.3. *For all large enough j we have $u_j(q_j) \le \lambda^{-2}u_{j-2}(q_{j-2})$ or $u_{j+1}(q_{j+1}) \le \lambda^{-3}u_{j-2}(q_{j-2})$.*

Let us first complete the proof of Theorem 1.3 given this lemma.

Proof of Theorem 1.3. Choose large enough j_1 and set $C \equiv \lambda^{j_1}u_{j_1}(q_{j_1})$. Then the lemma and induction show that there is a sequence $j_l \to \infty$ such that $\lambda^{j_l}u_{j_l}(q_{j_l}) \le C$. Since on $[q_{j_l}, t_{j_l+1}]$ we have $u'_{j_l} \le 0$, we also have there $\lambda^{j_l}u_{j_l} \le C$ which gives $u'_{j_l+1} \le C\lambda u_{j_l+1}$. But then (5.6) and (5.35) show

$$(\varepsilon j_l - 1)\log\lambda \le \log\frac{u_{j_l+1}(t_{j_l+1})}{u_{j_l+1}(q_{j_l})} \le C\lambda(t_{j_l+1} - q_{j_l}) \le C\lambda.$$

This is a contradiction when l is large. □

Thus, we are left with proving the lemma.

Proof of Lemma 5.3. Notice that (5.16) holds for $t \le q_j$ and so

$$u_j(q_j) \ge \lambda^{-j-1}. \tag{5.36}$$

Also u_j obviously decreases on $[q_j, q_{j+1}]$. Let now j be large enough so that Lemma 5.1 holds for any $j' \ge j-2$ in place of j. In particular, $u_{j'-1}(q_{j'}) = 0$ with $q_{j'}$ defined above. Let us denote

$$B \equiv u_{j-2}(q_{j-2}) \ge \lambda^{-j+1}. \tag{5.37}$$

By (5.11), (5.35), and (5.37) we have $E_{j-2}(q_{j-2}) \le \frac{25}{16}B^2$ if j is large enough. Since $u_{j-3} \le 0$ on $[q_{j-2}, T)$, (5.5) gives $E_l(t) \le \frac{25}{16}B^2$ for $l \ge j-2$ and $t \ge q_{j-2}$, in particular,

$$u_l(t) \le \tfrac{5}{4}\, B \text{ for } l \ge j-2 \text{ and } t \ge q_{j-2}. \tag{5.38}$$

Let us again proceed by contradiction and assume that

$$u_j(q_j) > \tfrac{B}{\lambda^2} \text{ and } u_{j+1}(q_{j+1}) > \tfrac{B}{\lambda^3}. \tag{5.39}$$

We define $f_j \in [q_j, q_{j+1}]$ to be the first time such that

$$u_j(f_j) = u_{j+1}(f_j) \tag{5.40}$$

(recall that $u_j(q_j) \ge \lambda^{-j-1} \ge u_{j+1}(q_j)$ and $u_j(q_{j+1}) = 0 \le u_{j+1}(q_{j+1})$). Then we must have for any $t \in [q_j, f_j]$,

$$u_j(t) \ge \tfrac{B}{2\lambda^3} \quad \big(\ge \tfrac{1}{2}\lambda^{-j-2} \big) \tag{5.41}$$

because otherwise (5.5), (5.11), (5.35), the definition of f_j, and $u_{j-1}(t) \le 0$ show that

$$E_{j+1}(q_{j+1}) = E_j(q_{j+1}) \le E_j(t) \le 3u_j(t)^2 \le \tfrac{B^2}{\lambda^6},$$

which would mean $u_{j+1}(q_{j+1}) \le B\lambda^{-3}$, contradicting the assumption. We assume here again that j is large enough, so that $E_{j+2}(t) \le u_j(t)^2$ for $t \in [q_j, f_j]$. Therefore (5.41) holds and there is a first time $g_j \in [q_j, f_j]$ such that $u_{j+1}(g_j) = 2\lambda^{-(1+\varepsilon)j}$. From (1.3), (5.35), and $u_j(t) \ge u_j(f_j) \ge B(2\lambda^3)^{-1}$ and $u_{j+1}(t) \le u_{j+1}(g_j) = 2\lambda^{-(1+\varepsilon)j}$ for $t \in [q_j, g_j]$ we get

$$\begin{aligned} \lambda^{j+1} \int_{q_j}^{g_j} u_j(t)^2\, dt &\ge \frac{B}{4\lambda^{3-(1+\varepsilon)j}} \lambda^{j+1} \int_{q_j}^{g_j} u_j(t) u_{j+1}(t)\, dt \\ &\ge \frac{B}{4\lambda^{3-(1+\varepsilon)j}} (u_{j+1}(g_j) - u_{j+1}(q_j)) \\ &\ge \frac{B}{4\lambda^3}. \end{aligned} \tag{5.42}$$

Next, we will show that there is $h_j \in [q_j, g_j]$ such that

$$u_{j-1}(h_j) \le -\tfrac{B}{10\lambda^5}, \quad u_j(h_j) \ge \tfrac{B}{2\lambda^3}, \quad u_{j+1}(h_j) \le 2\lambda^{-(1+\varepsilon)j}. \tag{5.43}$$

The second and third inequality are automatic when $h_j \le g_j$ (by (5.41) and the definition of g_j), so let us assume that for all $t \in [q_j, g_j]$ we have $-B(10\lambda^5)^{-1} < u_{j-1}(t) (\le 0)$. Then on $[q_j, g_j]$ we have by $u_j \ge B(2\lambda^3)^{-1}$, (1.3) and (5.38),

$$u'_{j-1} \le \lambda^{j-1} \tfrac{B}{10\lambda^5} \tfrac{5B}{4} - \lambda^j u_j^2 \le -\tfrac{1}{2}\lambda^j u_j^2.$$

Hence from (5.42),

$$u_{j-1}(g_j) \le u_{j-1}(q_j) - \tfrac{B}{8\lambda^4} = -\tfrac{B}{8\lambda^4} \le -\tfrac{B}{10\lambda^5},$$

a contradiction with the assumption.

Therefore (5.43) holds for some $h_j \in [q_j, g_j]$. Moreover, $u_{j-1} \le -B(10\lambda^5)^{-1}$ on $[h_j, f_j]$ because whenever equality holds, then by (5.38) and (5.41),

$$u'_{j-1} \le \lambda^{j-1} \tfrac{B}{10\lambda^5} \tfrac{5B}{4} - \lambda^j \left(\tfrac{B}{2\lambda^3}\right)^2 < 0.$$

Therefore by (5.38), on $[h_j, f_j]$

$$\frac{u'_j}{u_j} \le \lambda^j u_{j-1} \le -\lambda^j \frac{B}{10\lambda^5} \le -\lambda^j \frac{\frac{4}{5}u_j}{10\lambda^5} = -\frac{2}{25\lambda^6}\lambda^{j+1} u_j.$$

From this,

$$\log \frac{u_j(f_j)}{u_j(h_j)} \le -\frac{2}{25\lambda^6}\lambda^{j+1} \int_{h_j}^{f_j} u_j(t)\, dt,$$

which together with (5.38) and (5.41) shows that

$$\lambda^{j+1} \int_{h_j}^{f_j} u_j(t)\, dt \le \frac{25\lambda^6}{2} \log \frac{5\lambda^3}{2} \equiv C_6.$$

But then (5.37), (5.40), (5.41), (5.43), and (1.3) yield

$$\log \frac{\lambda^{\varepsilon j}}{4\lambda^2} \le \log \frac{B}{4\lambda^3 \lambda^{-(1+\varepsilon)j}} \le \log \frac{u_{j+1}(f_j)}{u_{j+1}(h_j)} \le \lambda^{j+1} \int_{h_j}^{f_j} u_j(t)\, dt \le C_6,$$

a contradiction when j is large. The lemma is proved. □

6. Appendix

Here, for the sake of completeness, we sketch the argument giving the local existence of solutions in the Obukhov and KP models. These two cases (as well as any combination) are handled identically; for simplicity we will consider the Obukhov model. We will also look at a more general branching case, since the result extends naturally and without extra effort. Thus, we look at a largest dyadic cube Q^0 of generation zero, and assume it has d children Q^1_l belonging to the first generation. Each cube Q of generation j has in its turn d children of generation $j+1$. Given a cube Q of generation j, we denote $\tilde{Q}$ its unique parent and $C^1(Q)$ the set of its d children. Likewise, we denote $C^k(Q)$ the set of all descendants of Q of generation $j+k$. Let us denote $j(Q)$ the generation of any given cube Q in our branching tree. The branched Obukhov model is given by the following system of differential equations

$$\frac{d}{dt} u_Q = \lambda^{j(Q)} u_{\tilde{Q}} u_Q - \lambda^{j(Q)+1} \sum_{Q' \in C^1(Q)} u^2_{Q'} \tag{6.1}$$

for each Q, with $u_{\tilde{Q}} \equiv 0$ when $Q = Q^0$. We say that $U \equiv \{u_Q\}$ belongs to the Sobolev space H^s if

$$\|U\|_{H^s}^2 \equiv \sum_Q \lambda^{2sj(Q)} |u_Q|^2 < \infty.$$

Consider an equivalent integral equation reformulation of (6.1), given by

$$u_Q(t) = u_Q(0) + \int_0^t \left(\lambda^j u_{\tilde{Q}}(\tau) u_Q(\tau) - \lambda^{j+1} \sum_{Q' \in C^1(Q)} u_{Q'}(\tau)^2 \right) d\tau. \tag{6.2}$$

Recall one version of the well-known Picard's fixed point theorem.

Theorem 6.1 (Picard)**.** *Let X be a Banach space and Γ a bilinear operator $\Gamma : X \times X \mapsto X$ such that for any $U, V \in X$ we have*

$$\|\Gamma(U,V)\|_X \le \eta \|U\|_X \|V\|_X. \tag{6.3}$$

Then for any $U_0 \in X$ satisfying $4\eta\|U_0\|_X < 1$ the equation $U = U_0 + \Gamma(U,U)$ has a unique solution $U \in X$ such that $\|U\|_X \le 1/2\eta$.

Using this theorem we are going to prove

Theorem 6.2. *Given any $\{u_Q(0)\} \in H^s$, $s \ge 1$, there exists $T = T(\|u_Q\|_{H^s}) > 0$ such that there is a unique solution $u_Q(t)$ of the branching Obukhov system* (6.2) *which belongs to $C([0,T], H^s)$.*

Proof. Let us define $U_0(t) \equiv \{u_Q(0)\}$ for all t and

$$\gamma(U,V)_Q(t) = \lambda^{j(Q)} u_{\tilde{Q}}(t) v_Q(t) - \lambda^{j(Q)+1} \sum_{Q' \in C^1(Q)} u_{Q'}(t) v_{Q'}(t),$$

and

$$\Gamma(U,V)_Q(T) = \int_0^T \gamma(U,V)_Q(t)\, dt.$$

The result will follow from Picard's theorem if we verify the bound (6.3) for Γ. If $s \ge 1$, we have

$$\begin{aligned}
\|\gamma(U,V)(t)\|_{H^s}^2 &= \sum_Q \lambda^{2sj(Q)} \left(\lambda^{j(Q)} u_{\tilde{Q}}(t) v_Q(t) - \lambda^{j(Q)+1} \sum_{Q' \in C^1(Q)} u_{Q'}(t) v_{Q'}(t) \right)^2 \\
&\le \sum_Q \lambda^{2sj(Q)} (d+1) \left(\lambda^{2j(Q)} u_{\tilde{Q}}(t)^2 v_Q(t)^2 + \lambda^{2j(Q)+2} \sum_{Q' \in C^1(Q)} u_{Q'}(t)^2 v_{Q'}(t)^2 \right) \\
&\le (d+1) \lambda^{2s} \|U(t)\|_{H^s}^2 \left(\sum_Q \lambda^{2j(Q)} v_Q(t)^2 + \sum_Q \sum_{Q' \in C^1(Q)} \lambda^{2j(Q)+2} v_{Q'}(t)^2 \right) \\
&\le 2(d+1) \lambda^{2s} \|U(t)\|_{H^s}^2 \|V(t)\|_{H^s}^2.
\end{aligned}$$

Then

$$\|\Gamma(U,V)\|_{C([0,T],H^s)} \leq C(d,\lambda)\int_0^T \|U(t)\|_{H^s}\|V(t)\|_{H^s}\,dt \leq C(d,\lambda)T\|U\|_{C([0,T],H^s)}\|V\|_{C([0,T],H^s)}.$$

Choosing a small enough $T > 0$ completes the proof. □

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