VIRTUAL LINEARITY FOR KPP REACTION-DIFFUSION EQUATIONS

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ABSTRACT. We show that long time solution dynamic for general reaction-advection-diffusion equations with KPP reactions is virtually linear in the following sense. Its leading order depends on the non-linear reaction only through its linearization at $u = 0$, and it can also be recovered for general initial data by instead solving the PDE for restrictions of the initial condition to unit cubes on $\mathbb{R}^d$ (the latter means that non-linear interaction of these restricted solutions has only lower order effects on the overall solution dynamic). The result holds under a uniform bound on the advection coefficient, which we show to be sharp. We also extend it to models with non-local diffusion and KPP reactions.

1. INTRODUCTION AND MAIN RESULTS

Many processes in nature are modeled by the reaction-diffusion equation

$$u_t = \mathcal{L}u + f(t, x, u). \tag{1.1}$$

The unknown function u represents concentration of a substance or density of a species, which is subject to diffusion as well as some possibly space-time dependent reactive process (which may be a combination of birth and death processes), modeled by the two terms on the right-hand side of (1.1). The basic case is $\mathcal{L} = \Delta$, but when diffusion may be inhomogeneous, non-isotropic, and time-dependent, and an underlying advective motion may also be present (such as for processes occurring in fluid media), one instead considers the more general case

$$\mathcal{L}u(t, x) := \sum_{i,j=1}^{d} A_{ij}(t, x)u_{x_i x_j}(t, x) + \sum_{i=1}^{d} b_i(t, x)u_{x_i}(t, x). \tag{1.2}$$

We will study here this model and its non-local version

$$\mathcal{L}u(t, x) := \text{p.v.} \int_{\mathbb{R}^d} K(t, x, \nu) \left[u(t, x + \nu) - u(t, x)\right] d\nu, \tag{1.3}$$

with KPP (a.k.a. Fisher-KPP) reactions f. Named after Kolmogorov, Petrovskii, and Piskunov [6] and Fisher [4], who first studied them in 1937, these reactions are defined as follows.

Definition 1.1. A Lipschitz function $f : \mathbb{R}^+ \times \mathbb{R}^d \times [0, 1] \to \mathbb{R}$ is a *KPP reaction* if $f(\cdot, \cdot, 0) \equiv 0 \equiv f(\cdot, \cdot, 1)$ and $f(t, x, u) \le f_u(t, x, 0)u$ for all $(t, x, u) \in \mathbb{R}^+ \times \mathbb{R}^d \times [0, 1]$ (with $f_u(\cdot, \cdot, 0)$ existing pointwise), plus the following uniform hypotheses are satisfied. We have $\inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d} f(t, x, u) > 0$ for each $u \in (0, 1)$, as well as $\inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d} f_u(t, x, 0) > 0$ and

$$\lim_{u\to 0} \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d} \left(f_u(t, x, 0) - \frac{f(t, x, u)}{u} \right) = 0.$$

Hence we will have $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, and we define $f(t,x,\cdot)$ only on $[0,1]$ because we will consider here solutions $0 \le u \le 1$ (i.e., with normalized concentration or density u). While both $u \equiv 0$ and $u \equiv 1$ are stationary solutions for (1.1), the former is unstable while the latter is asymptotically stable because $f > 0$ on $\mathbb{R}^+ \times \mathbb{R}^d \times (0,1)$, and one is interested in the transition of solutions from values near 0 to those near 1, which models invasion of the low concentration region $u \approx 0$ by the modeled substance or population.

1.1. **Virtual Linearity in the Classical Diffusion Case.** There is a vast literature on (1.1) with KPP reactions, and it would be futile to try to include here a list of even the most relevant works. The reader can consult the reviews [1, 10] and references therein, although many more important papers appeared since their publication. When reviewing any of them, one notices that essentially all the results concerning asymptotic speeds of propagation of solutions (as opposed to, e.g., those relating to precise locations of the transition regions between values $u \approx 0$ and $u \approx 1$, such as in [3, 5, 7]) have one thing in common: *they depend on f only through $f_u(\cdot,\cdot,0)$*. This is because the main KPP hypothesis $f(t,x,u) \le f_u(t,x,0)u$ means that the reaction strength $\frac{f(t,x,u)}{u}$ (i.e., the zeroth order coefficient when $u = u(t,x)$ is any solution to (1.1) and the PDE is viewed as a linear PDE for it) is greatest at values $u \approx 0$, so the leading order of the solution dynamic is determined at these values (and therefore spatially at the leading edge of the invading population). This is also sometimes referred to as *pulled* dynamic, as opposed ot the *pushed* dynamic for some other types of reactions, such as ignition or bistable, whose reaction strength is largest at intermediate values of u (and therefore the solution dynamic is also primarily determined at these values [12, 13]).

Since $f(t,x,u) \approx f_u(t,x,0)u$ at $u \approx 0$, one might then think that the leading order of the solution dynamic for (1.1) with a KPP reaction is the same as for the linear PDE with $f_u(t,x,0)u$ in place of $f(t,x,u)$ (at least when considering the minimum of a solution of the latter and 1). This is not true in general because even if a spatial region has large $f_u(\cdot,\cdot,0)$ for all time (such regions can drive the linear dynamic in all space-time), its effect on the non-linear dynamic vanes once the solution reaches values away from 0 on that region. Nevertheless, dependence of the leading order of the solution dynamic on $f_u(\cdot,\cdot,0)$ only (rather than on all of f) is a version of linearity, and the author is not aware of any prior result that formally establishes this phenomenon for general KPP dynamics.

Our first main result, Theorem 1.2 below, therefore appears to be the first such result. Moreover, it also demonstrates that (1.1) with KPP reactions shares another property with linear equations. Namely, that the leading order of the solution dynamic for a general initial condition $u(0,\cdot)$ can be recovered from solving the PDE with initial conditions that are obtained by restricting $u(0,\cdot)$ to members of a partition of $\mathbb{R}^d$ into compact sets (we use below the unit cubes $C_n := (n_1, n_1+1) \times \cdots \times (n_d, n_d+1)$ with $n \in \mathbb{Z}^d$, but our proofs can be easily adapted to other choices). That is, nonlinear interaction between the resulting initially compactly supported solutions does not affect the leading order of the solution dynamic, so in this sense the solution operator for (1.1) is also close to being linear.

This *virtual linearity* of (1.1) means that to investigate the leading order of the solution dynamic, it suffices to replace a general KPP reaction by some "template" reaction sharing the same $f_u(\cdot,\cdot,0)$ (e.g., f' in Theorem 1.2), as well as to only consider solutions with initial data supported inside small compact sets. We demonstrate it for general KPP reactions f and uniformly elliptic diffusion matrices A, together with advection vectors b that are uniformly bounded above by twice the square root of the product of the ellipticity constant of A and $\inf f_u(\cdot,\cdot,0)$. Moreover, this bound on b is in fact sharp (see below).

Theorem 1.2. *Let f be a KPP reaction and $\mathcal{L}$ be from (1.2), where $A=(A_{ij})$ is a bounded symmetric matrix with $A \geq \lambda I$ for some $\lambda>0$, and the vector $b=(b_1,\dots,b_d)$ satisfies $\|b\|_{L^\infty}^2 < 4\lambda \inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d} f_u(t,x,0)$. Let $g(u):=\min\{u,1-u\}$ and*

$$f'(t,x,u) := f_u(t,x,0)g(u) \tag{1.4}$$

for $(t,x,u)\in\mathbb{R}^+\times\mathbb{R}^d\times[0,1]$. Then there is $\phi:\mathbb{R}^+\to\mathbb{R}^+$ with $\lim_{s\to\infty}\phi(s)=0$, and for each $\delta\in(0,\frac12]$ there is $\tau_\delta\geq 1$, such that the following holds (see also Remark 3 below for the dependence of ϕ,τ_δ on A,b,f).

If $u:\mathbb{R}^+\times\mathbb{R}^d\to[0,1]$ solves (1.1), and for each $n\in\mathbb{Z}^d$ we let $u_n':\mathbb{R}^+\times\mathbb{R}^d\to[0,1]$ solve (1.1) with f' in place of f and with $u_n'(0,\cdot):=u(0,\cdot)\chi_{C_n}$, then for each $(t,x)\in[\tau_\delta,\infty)\times\mathbb{R}^d$,

$$\sup_{n\in\mathbb{Z}^d} u_n'(t-\delta t,x)-\phi(\delta t)\leq u(t,x)\leq \sup_{n\in\mathbb{Z}^d} u_n'\left(t+t^\delta,x\right)+\phi(t^\delta). \tag{1.5}$$

Moreover, if there is $\gamma>0$ such that for each $(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$ the function $u\mapsto\frac{f(t,x,u)}{u}$ is non-increasing on $(0,\gamma]$ and $\sup_{u\in[\gamma,\infty)}\frac{f(t,x,u)}{u}=\frac{f(t,x,\gamma)}{\gamma}$, then (1.5) also holds with u_n' instead solving (1.1) with f, and with the first inequality being just $\sup_{n\in\mathbb{Z}^d}u_n'(t,x)\leq u(t,x)$.

Remarks. 1. So up to $o(t)$ time shifts (or even $o(t^{0+})$ time shifts in the second claim) and $o(1)$ errors, we have $u\approx\sup_{n\in\mathbb{Z}^d}u_n'$. This of course also means that if v is a solution to (1.1) with f replaced by another KPP reaction that has the same linearization at $u=0$, then $v\approx u$ in the same sense provided v has the same initial datum (and perturbations of the latter can also be handled easily, via the maximum principle).

2. The proof can be easily adjusted so that both t^δ in (1.5) are replaced by δt.

3. The proof shows that if some non-decreasing function ψ with $\lim_{u\to0}\psi(u)=0$ satisfies

$$\psi(u)\geq \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d}\left(f_u(t,x,0)-\frac{f(t,x,u)}{u}\right),$$

some Lipschitz f_0 with $f_0'(0)>\frac{1}{4\lambda}\|b\|_{L^\infty}^2$ satisifes $\inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d}f(t,x,u)\geq f_0(u)>0$ for each $u\in(0,1)$ (such f_0 exists because ψ does), and we let $0<B\leq 2\sqrt{f_0'(0)\lambda}-\|b\|_{L^\infty}$ and

$$0<\alpha\leq\min\left\{\min_{i,j}\|A_{i,j}\|_{L^\infty}^{-1},\ \|f_u(\cdot,\cdot,0)\|_{L^\infty}^{-1}\right\},$$

then in the first claim of the theorem, ϕ and τ_δ depend only on $\alpha,\lambda,\psi,f_0,B,d$ (and τ_δ also on δ), while in the second claim they also depend on γ.

4. One could in fact replace $\sup_{n\in\mathbb{Z}^d}u_n'$ by $\min\left\{\sum_{n\in\mathbb{Z}^d}u_n',1\right\}$ in (1.5), which has a more "linear" feel but is also less convenient to use — including in applications of Theorem 1.2 to homogenization for KPP reaction-diffusion dynamics in random environments, which we provide in the companion papers [11, 14].

5. Here g could be any other (t,x)-independent KPP reaction with $g(u)\equiv u$ on $[0,\frac12]$. However, we cannot use the linear dynamics, with $g(u)$ replaced by u for all $u\geq0$. Indeed, consider for instance $\mathcal{L}:=\Delta$ and $f_u(t,x,0)=1+C\chi_{B_1(0)}(x)$. Then the asymptotic speed of propagation of solutions to (1.1) with both f and f' is well known to be 2, but this

speed increases when f is replaced by $f_u(t,x,0)u$ and $C \geq 0$ is large enough. It is in fact 2 when $\lambda \leq 2$ and $\frac{\lambda}{\sqrt{\lambda-1}}$ otherwise, where λ ($\to \infty$ as $C \to \infty$) is the principal eigenvalue of $\Delta + 1 + C\chi_{B_1(0)}$ on $\mathbb{R}^d$ (with $\sqrt{\lambda - 1}$ being the asymptotic exponential decay rate of the corresponding radial eigenfunction).

6. One cannot hope for this result to extend to general non-KPP reactions. This is the case even if we let $f' := f$; in this case $\sup_{n\in\mathbb{Z}^d} u'_n \leq u$ is obvious but the second inequality in (1.5) need not hold even with t^δ replaced by δt (see the remark after [14, Theorem 1.4]).

The bound on $\|b\|_{L^\infty}$ in Theorem 1.2 is sharp. Indeed, consider $u_t = u_{xx} + \bar{b}u_x + g(u)$ with $\bar{b} > 0$ a constant and $u(0,\cdot) := \chi_{(0,1)}$, in which case $2\sqrt{f'_0(0)\lambda} = 2$ (taking $f_0 := g$). It is well known (see, e.g., [7]) that if w solves this Cauchy problem without the first-order term, there is a strictly decreasing continuous function $q : (0,1) \to \mathbb{R}$ such that if we denote by $x_\theta(t) > 0$ the unique point with $w(t, x_\theta(t)) = \theta = w(t, 1 - x_\theta(t))$, then for each $\theta_0 \in (0,1)$ we have

$$\lim_{t\to\infty} \sup_{\theta\in[\theta_0,1-\theta_0]} \left| x_\theta(t) - \left(2t - \frac{3}{2}\ln t + q(\theta)\right) \right| = 0.$$

Of course, the original PDE is solved by $u(t,x) := w(t, x+\bar{b}t)$; this also equals u'_0 in the theorem, while all other u'_n are zero. So for $y_t := (2-b)t - \frac{3}{2}\ln t + q(\frac{1}{2})$ we have $\lim_{t\to\infty} u(t, y_t) = \frac{1}{2}$, while for each $\delta \in (0,1)$ we have

$$y_t = y_{t-\delta t} - (\bar{b}-2)\delta t - \frac{3}{2}|\ln(1-\delta)| = y_{t+t^\delta} + (\bar{b}-2)t^\delta + \frac{3}{2}\ln(1 + t^{\delta-1}).$$

Hence whenever $\bar{b} \in (2, 2 + \frac{4}{\delta})$, we obtain

$$\lim_{t\to\infty} \sup_{n\in\mathbb{Z}} u'_n(t - \delta t, y_t) = 1 \qquad \text{and} \qquad \lim_{t\to\infty} \sup_{n\in\mathbb{Z}} u'_n(t + t^\delta, y_t) = 0,$$

which contradicts both inequalities in (1.5) (and for any other KPP reaction f with $f_u(0) = 1$ we get the same result because the above asymptotics still hold, albeit with a different q). Even for $\bar{b} = 2$ we find that

$$\lim_{t\to\infty} \sup_{n\in\mathbb{Z}} u'_n(t - \delta t, y_t) = q^{-1}\left(q\left(\frac{1}{2}\right) - \frac{3}{2}|\ln(1-\delta)|\right) > \frac{1}{2},$$

which still contradicts the first inequality (and if one wants the last claim in the theorem to technically not apply, it suffices to change the reaction on a short time interval $[0, t_0]$ and at all large x so that the hypothesis of that claim is not satisfied but the change does not affect any point $(t, x, u(t,x))$; then the particular solution u considered here is also unchanged).

Nevertheless, it is possible that a version of Theorem 1.2 does hold with a larger uniform upper bound on b, provided $u'_n(t - \delta t, \cdot)$ and $u'_n(t + t^\delta, \cdot)$ are evaluated at some (t, x, δ)-dependent points instead of at x. Theorem 2.1 below with $U \equiv 1 \equiv U'$ would yield such a result if its hypotheses (2.2) and (2.3) can be verified in some relevant setting. A trivial example of this is the setting of Theorem 1.2 with a large constant vector $\bar{b}$ added to b, when we clearly obtain (1.5) with $u'_n(t - \delta t, x + \bar{b}\delta t)$ and $u'_n(t + t^\delta, x - \bar{b}t^\delta)$ on the left and right.

1.2. **Extension to Non-local Diffusions.** Theorem 1.2 extends to (1.1) with non-local diffusion operators from (1.3) under suitable hypotheses. Firstly, it is crucial that solutions do not propagate faster than ballistically, which requires the diffusion kernels K to decay exponentially as $\nu \to \infty$. Secondly, even when K is close in some sense to being even in ν

(here we will only assume it to be even) one can only allow $O(|\nu|^{-d-2+\alpha})$ growth as $\nu \to 0$ (with some $\alpha > 0$) if $\mathcal{L}$ is to be well-defined. This is the same growth as for $\mathcal{L} = -(-\Delta)^{\alpha/2}$, for which well-posedness, comparison principle, and the parabolic Harnack inequality are known to hold [2]. The methods used to establish these should equally apply to various kernels with $O(|\nu|^{-d-2+\alpha})$ asymptotics as $\nu \to 0$ that decay exponentially as $\nu \to \infty$. However, instead of trying to prove them in any level of generality, we will state these properties as hypotheses so that our result applies whenever these can be established. We note that the Harnack inequality referred to here is the forward one (unlike for $\mathcal{L}$ from (1.2), equations with non-local diffusions may also satisfy backward-in-time Harnack inequalities, such as in [2]). Our main result in this setting is now the following analog of Theorem 1.2.

Theorem 1.3. *Let f be a KPP reaction and let f' be from* (1.4)*. Assume that $\mathcal{L}$ is from* (1.3)*, with K from some family F of even-in-ν kernels such that for some $\alpha \in (0,1]$ and any $K \in F$, there is $\mathcal{K} : (0,\infty) \to [0,\infty)$ with $\chi_{(0,\alpha]}(r) \le \mathcal{K}(r) \le \chi_{(0,\alpha]}(r) r^{-d-2+\alpha}$ on $(0,\infty)$ and*

$$\alpha \mathcal{K}(|\nu|) \le K(t,x,\nu) \le \alpha^{-1} \max\left\{\mathcal{K}(|\nu|), e^{-\alpha|\nu|}\right\} \tag{1.6}$$

for each $(t,x,\nu) \in \mathbb{R}^+ \times \mathbb{R}^{2d}$. Assume that (1.1) *with any such K, any KPP reaction f, and locally BV initial data $0 \le u(0,\cdot) \le 1$ is well-posed in some subspace $\mathcal{A} \subseteq L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$, where the comparison principle for sub- and supersolutions to* (1.1) *as well as the parabolic (forward) Harnack inequality also hold (the latter with uniform constants for all $K \in F$ and all f with the same Lipschitz constant), and the solutions for $u(0,\cdot) \equiv 0, 1$ are $u \equiv 0, 1$, respectively. Then the claims in Theorem 1.2 hold for such $\mathcal{L}$.*

Remark. One could also extend this result to mixed diffusion operators, with $\mathcal{L}$ being the sum of the right-hand sides of (1.2) and (1.3), but we will not do so here.

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2. Classical Diffusion Case

The key to Theorem 1.2 will be the following result, which provides a version of (1.5) on general $D \subseteq \mathbb{R}^d$ and without quantitative restrictions on b. The price to pay is that the spatial arguments in the three expressions in (1.5) may be different, and one also needs to guarantee certain exponential growth of solutions from small initial data (which we later show to hold under the hypotheses of Theorem 1.2).

Consider (1.1) on some open $D \subseteq \mathbb{R}^d$, with $\mathcal{L}$ from (1.2). In the following theorem, all solutions are strong (from $W^{(1,2),d+1}_{\text{loc}}(\mathbb{R}^+ \times D) \cap C(\overline{\mathbb{R}^+ \times D})$) and are assumed to satisfy homogeneous Dirichlet boundary conditions on $\mathbb{R}^+ \times \partial D$. We will call such solutions *SD solutions*. We note that the relevant well-posedness theory as well as comparison principle for sub- and supersolutions follow from, e.g., the corresponding linear theory in [8, Chapter 7] (specifically, Theorems 7.1 and 7.32).

Theorem 2.1. *Let $\mathcal{L}$ be given by* (1.2)*, with $A = (A_{ij})$ a bounded symmetric matrix with $A \ge \lambda I$ for some $\lambda > 0$, and $b = (b_1, \dots, b_d)$ a bounded vector. Let $f, f' : \mathbb{R}^+ \times D \times [0,\infty) \to \mathbb{R}$ be Lipschitz with $f(\cdot,\cdot,0) \equiv 0 \equiv f'(\cdot,\cdot,0)$, and let $U, U' : \mathbb{R}^+ \times D \to [0,1]$ be some functions.*

Assume that there are $\gamma \in (0, \frac{1}{2}]$ *and non-decreasing* $\psi : (0,1) \to [0,\infty)$ *with* $\lim_{u\to 0} \psi(u) = 0$ *such that* $0 \le \frac{f'(\cdot,\cdot,u)}{u} - \frac{f(\cdot,\cdot,u)}{u} \le \psi(u)$ *for each* $u \in (0,\gamma]$*, for each* $(t,x) \in \mathbb{R}^+ \times D$ *the function* $u \mapsto \frac{f'(t,x,u)}{u}$ *is non-increasing on* $(0,\gamma]$ *and* $\sup_{u\in[\gamma,\infty)} \frac{\max\{f(t,x,u),f'(t,x,u)\}}{u} \le \frac{f'(t,x,\gamma)}{\gamma}$*, and*

$$\max\left\{\max_{i,j} \|A_{i,j}\|_{L^\infty},\, \max_i \|b_i\|_{L^\infty},\, \|f_u(\cdot,\cdot,0)\|_{L^\infty}\right\} \le \gamma^{-1}. \tag{2.1}$$

Also assume that there are $\kappa > 0$ *and* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\lim_{s\to\infty} \phi(s) = 0$*, and for each* $t_0 \ge 1$ *and* $(s,x) \in \mathbb{R}^+ \times D$ *there are* $y^s_{t_0,x}, y^{-s}_{t_0+s,x} \in D$*, such that* $y^{-s}_{t_0+s,y^s_{t_0,x}} = x$ *and for any SD solutions* $u, u' : (t_0-1,\infty) \times D \to [0,1]$ *to* (1.1) *and to* (1.1) *with* f' *in place of* f*, respectively, we have*

$$u(t_0+s,x) \ge \min\left\{e^{\kappa s} u(t_0, y^{-s}_{t_0+s,x}),\, U(t_0+s,x) - \phi(s)\right\}, \tag{2.2}$$

$$u'(t_0+s,x) \ge \min\left\{e^{\kappa s} u'(t_0, y^{-s}_{t_0+s,x}),\, U'(t_0+s,x) - \phi(s)\right\}. \tag{2.3}$$

Let $u : \mathbb{R}^+ \times D \to [0,1]$ *be an SD solution to* (1.1)*, and for each* $n \in \mathbb{Z}^d$ *let* $u'_n : \mathbb{R}^+ \times D \to [0,1]$ *be the SD solution to* (1.1) *with* f' *in place of* f *and with* $u'_n(0,\cdot) := u(0,\cdot)\chi_{cC_n\cap D}$ *for some* $c > 0$*. Then for each* $\delta \in (0,\frac{1}{2}]$ *there is* $\tau_\delta \ge 1$ *(depending also on* $\gamma, \kappa, \psi, c, d$*) such that*

$$u(t,x) \ge \min\left\{\sup_{n\in\mathbb{Z}^d} u'_n\left(t-\delta t, y^{-\delta t}_{t,x}\right),\, U(t,x) - \phi(\delta t)\right\}, \tag{2.4}$$

$$\sup_{n\in\mathbb{Z}^d} u'_n\left(t+t^\delta, y^{t^\delta}_{t,x}\right) \ge \min\left\{u(t,x) - t^{-1/\delta},\, U'\left(t+t^\delta, y^{t^\delta}_{t,x}\right) - \phi(t^\delta)\right\} \tag{2.5}$$

for each $(t,x) \in [\tau_\delta,\infty) \times D$*. If* $f' = f$*, then clearly also*

$$u(t,x) \ge \sup_{n\in\mathbb{Z}^d} u'_n(t,x). \tag{2.6}$$

Remarks. 1. Note that the hypotheses on f, f' guarantee existence of $f_u(t,x,0) = f'_u(t,x,0)$ as well as $\max\{f(t,x,u), f'(t,x,u)\} \le f_u(t,x,0)u$ for all $(t,x,u) \in \mathbb{R}^+ \times D \times \mathbb{R}^+$.

2. A natural choice of U, U' are some SD solutions to (1.1) and to (1.1) with f' in place of f, respectively. For instance, when $D = \mathbb{R}^d$ and $f(\cdot,\cdot,1) \equiv 0 \equiv f(\cdot,\cdot,1)$, one might consider $U \equiv 1 \equiv U'$.

3. The restriction to $t_0 \ge 1$ is needed so that parabolic Harnack inequality guarantees that $u(t_0,\cdot)$ does not vary too much near $y^{-s}_{t_0+s,x}$. Otherwise hypothesis (2.2) could not hold for all u.

4. The point $y^{-s}_{t,x}$ is such that the values of u near $(y^{-s}_{t,x}, t-s)$ provide a good lower bound for the value at (t,x), and $y^s_{t,x}$ is such that the values of u near (t,x) provide a good lower bound for the value at $(y^s_{t,x}, t+s)$. Of course, if $x \mapsto y^{-s}_{t,x}$ is a bijection on D for some t,s, then $z \mapsto y^s_{t-s,z}$ must be its inverse. One natural example of this is $y^s_{t,x} = x$, another is given by the ODE $\frac{d}{ds} y^s_{t,x} = b(t+s, y^s_{t,x})$ with $y^0_{t,x} = x$.

5. One might also consider other boundary conditions, provided the construction of exponentially-decaying ballistically-moving supersolutions from the proof below can be adjusted to that setting.

Proof. Comparison principle gives for each $s \in (0, \gamma]$ and all $(t, x) \in \mathbb{R}^+ \times D$ that

$$se^{-\psi(s)t} \sup_{n \in \mathbb{Z}^d} u'_n(t,x) \le u(t,x) \le \gamma^{-1} \sum_{n \in \mathbb{Z}^d} u'_n(t,x). \tag{2.7}$$

Indeed, the first inequality follows from $v(t,x) := se^{-\psi(s)t} u'_n(t,x)$ ($\le s$) being a subsolution to (1.1) for each $n \in \mathbb{Z}^d$, which holds because the hypotheses on f, f' yield

$$se^{-\psi(s)t} f'(t,x,u'_n(t,x)) \le f'(t,x,se^{-\psi(s)t}u'_n(t,x)) \le f(t,x,se^{-\psi(s)t}u'_n(t,x)) + \psi(s)se^{-\psi(s)t}u'_n(t,x).$$

The second one follows from $v(t,x) := \gamma^{-1} \sum_{n \in \mathbb{Z}^d} u'_n(t,x)$ being a supersolution to (1.1) in the region where $v \le 1$. This holds because if $0 < v(t,x) \le 1$, then $u'_n(t,x) \le \gamma$ for each $n \in \mathbb{Z}^d$, so we can take $v_n := \gamma^{-1} u'_n(t,x) \le 1$ and apply the estimate

$$f\left(\cdot,\cdot,\sum_{n=1}^\infty v_n\right) = \gamma^{-1} \sum_{n=1}^\infty f\left(\cdot,\cdot,\sum_{m=1}^\infty v_m\right)\left(\sum_{m=1}^\infty v_m\right)^{-1} \gamma v_n \le \gamma^{-1} \sum_{n=1}^\infty f'(\cdot,\cdot,\gamma v_n).$$

This estimate holds for all $v_1, v_2, \dots \ge 0$ with $0 < \sum_{n=1}^\infty v_n \le 1$, by the hypotheses on f, f'.

Next, without loss assume that $\kappa \le 1$. Let us take any $\delta \in (0, \frac12]$, pick $s \in (0, \gamma]$ such that $\psi(s) \le \frac{\kappa\delta}{2}$, let $\tau_\delta := \frac{2}{\kappa\delta} |\ln s| \ge 2$, and assume that (2.4) fails for some $(t,x) \in [\tau_\delta, \infty) \times D$. Then $u(t,x) < U(t,x) - \phi(\delta t)$, so (2.2) and (2.7) yield

$$u(t,x) \ge e^{\kappa\delta t} u\left(t - \delta t, y^{-\delta t}_{t,x}\right) \ge e^{\kappa\delta t} se^{-\kappa\delta t/2} \sup_{n \in \mathbb{Z}^d} u'_n\left(t-\delta t, y^{-\delta t}_{t,x}\right) \ge \sup_{n \in \mathbb{Z}^d} u'_n\left(t-\delta t, y^{-\delta t}_{t,x}\right).$$

But this means that (2.4) does hold for (t,x), a contradiction. Hence (2.4) must hold.

To prove (2.5), consider any $\delta \in (0, \frac12]$, let $a := \gamma^{-1}(1 + d + d^2)$ and fix any $(t', x') \in \mathbb{R}^+ \times D$. For each $n \in \mathbb{Z}^d$ let x_n be the point from $c\overline{C}_n$ closest to x' and let $e_n := \frac{x' - x_n}{|x' - x_n|}$ when $x_n \ne x'$ (otherwise pick any $e_n \in \mathbb{S}^{d-1}$). Since $v_n(t,x) := e^{at - (x - x_n)\cdot e_n}$ is a supersolution to (1.1) by (2.1), and $v_n(0,\cdot) \ge \chi_{cC_n} \ge u'_n(0,\cdot)$, we have

$$\gamma^{-1} u'_n(t', x') \le \gamma^{-1} v_n(t', x') \le \gamma^{-1} e^{at' - |x' - x_n|}.$$

Sum of the right-hand sides over all n with $|x' - x_n| \ge (a+1)t'$ is less than $\frac12 (t')^{-1/\delta}$ whenever $t' \ge \tau$ for some (δ, γ, c, d)-dependent $\tau \ge 1$.

So if (2.5) fails for some $(t', x') \in [\tau, \infty) \times D$, then $u(t', x') \ge (t')^{-1/\delta}$ and (2.7) show that there is n with $|x' - x_n| < (a+1)t'$ such that

$$\gamma^{-1} u'_n(t', x') \ge \frac12 (t')^{-1/\delta} \left((a+1)t' + c\sqrt{d}\right)^{-d} c^d V_d^{-1}, \tag{2.8}$$

where V_d is the volume of the unit ball in $\mathbb{R}^d$. We also have

$$u'_n\left(t' + (t')^\delta, y^{(t')^\delta}_{t',x'}\right) < U'\left(t' + (t')^\delta, y^{(t')^\delta}_{t',x'}\right) - \phi((t')^\delta)$$

because (2.5) fails for (t', x'), so (2.3) yields

$$u'_n\left(t' + (t')^\delta, y^{(t')^\delta}_{t',x'}\right) \ge e^{\kappa (t')^\delta} u'_n(t', x')$$

(recall also that $y^{-s}_{t+s, y^s_{t,x}} = x$). But (2.8) shows that the right-hand side is greater than 1 whenever $t' \ge \tau_\delta$, with some $(\delta, \gamma, \kappa, c, d)$-dependent $\tau_\delta \ge \tau$. So (2.5) must hold for all $(t', x') \in [\tau_\delta, \infty) \times D$, otherwise we would have a contradiction.

Finally, comparison principle immediately yields (2.6). $\square$

We note that when $u \mapsto \frac{f(t,x,u)}{u}$ is non-increasing on $(0,1]$, then one can easily show that with $f'=f$ we have $\sup_{n\in\mathbb{Z}^d} u_n' \le u \le \sum_{n\in\mathbb{Z}^d} u_n'$. For $d=1$, $\mathcal{L}=\partial_{xx}$, $f(t,x,u)=f(u)$, and a sum of two initial data (i.e., $u(0,\cdot)=u_1(0,\cdot)+u_2(0,\cdot)$), this has already appeared in [9, Lemma 8.4] and [3, Lemma 3.5].

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let f_0,ψ,B,α be from Remark 3 after Theorem 1.2. We will denote $\beta := f_0'(0)>0$ because it will be used extensively, and let

$$\gamma := \min\left\{\alpha,\, \frac{1}{2\sqrt{\beta\lambda}},\, \frac{1}{2}\right\} > 0$$

(when proving the last claim, add γ from it into the min). Note that we assume

$$\max_i \|b_i\|_{L^\infty} \le 2\sqrt{\beta\lambda} \le \gamma^{-1}.$$

We will show that Theorem 2.1 applies with $y^s_{t_0,x} := x =: y^{-s}_{t_0+s,x}$ for all $(t_0,s,x)\in[1,\infty)\times\mathbb{R}^+\times\mathbb{R}^d$, with $U\equiv 1\equiv U'$, and some κ,ϕ (when proving the last claim, we use instead $f' := f$ in Theorem 2.1). Then (2.4) and (2.5) will immediately yield (1.5), provided we replace $\phi(t)$ by $\phi(t)+t^{-1/2}$.

The only hypotheses of Theorem 2.1 that do not obviously hold are (2.2) and (2.3). The proofs are identical, so let us prove (2.2). The key will be the following construction of a sequence of appropriate subsolutions $\{u_{v,j}\}_{j=0}^\infty$ to (1.1), which are essentially ballistically spreading radially symmetric plateaus (with a common positive spreading speed) at levels converging to 1 as $j\to\infty$, with the first of them starting at level $v>0$ (which can be chosen to be arbitrarily small). We will then show that any solution $0\le u\le 1$ to (1.1) that starts above the first subsolution will not only remain above it forever, but it will also be eventually pushed above any other of the subsolutions — and (2.2) will follow. In the whole proof, all constants and functions may depend on $\gamma,\beta,\lambda,\psi,B,f_0,d$, with only the constant v below (and constants/functions with v in their subscripts) possibly depending on the solution u.

Let $\Lambda := \frac{d}{\alpha}$, so $A\le\Lambda I$. If $u(t,x):=w(t,|x|)$ for $x\in\mathbb{R}^d$ and some $w:\mathbb{R}^+\times\mathbb{R}\to[0,\infty)$, then we have (with arguments (t,x) and $(t,y):=(t,|x|)$, as appropriate)

$$\sum_{i,j=1}^d A_{ij}u_{x_ix_j} = \frac{w_{yy}|x|-w_y}{|x|^3}\sum_{i,j=1}^d A_{ij}x_ix_j + \frac{d}{|x|}w_y \ge \min\{\lambda,\Lambda\operatorname{sgn} w_{yy}\}|w_{yy}| - \frac{d+\Lambda}{|x|}|w_y|.$$

This and $\|b\|_\infty \le 2\sqrt{\beta\lambda}-B$ yield for $C:=\frac{B}{4}\sqrt{\beta/\lambda}$ ($\le\frac{\beta}{2}$ by $B\le 2\sqrt{\beta\lambda}$) and all $|x|\ge\frac{3(d+\Lambda)}{B}$,

$$\mathcal{L}u+(\beta-C)u-\frac{B}{3}|\nabla u| \ge \min\{\lambda,\Lambda\operatorname{sgn} w_{yy}\}|w_{yy}| - \left(2\sqrt{\beta\lambda}-\frac{B}{3}\right)|w_y| + (\beta-C)\,w. \tag{2.9}$$

Let now z be the root of $\lambda z^2 + (2\sqrt{\beta\lambda}-\frac{B}{3})z+\beta-C=0$ that lies in the second quadrant of the complex plane; note that the discriminant of this quadratic equation is

$$4\lambda C - \frac{B}{9}\left(12\sqrt{\beta\lambda}-B\right) \le B\sqrt{\beta\lambda} - \frac{B}{9}10\sqrt{\beta\lambda} = -\frac{B}{9}\sqrt{\beta\lambda} < 0.$$

Then the function $\tilde{\xi}(y) := e^{y \operatorname{Re} z} \sin(y \operatorname{Im} z)$ solves $\lambda\tilde{\xi}'' + (2\sqrt{\beta\lambda} - \frac{B}{3})\tilde{\xi}' + (\beta - C)\tilde{\xi} = 0$. Let $y_3 := \frac{\pi}{\operatorname{Im} z}$ and let $y_2 \in (0, y_3)$ be its greatest inflection point that is smaller than y_3, and let

$$\xi(y) := \begin{cases} 0 & y > y_3, \\ \tilde{\xi}(y) & y \in [y_2, y_3], \\ \tilde{\xi}(y_2) + \tilde{\xi}'(y_2)(y - y_2) & y < y_2. \end{cases}$$

Since $\tilde{\xi}$ is convex and positive on $[y_2, y_3)$, so is ξ on $(-\infty, y_3)$. Hence $\xi'' \geq 0 \geq \xi'$ (in the sense of distributions due to the discontinuity of $\tilde{\xi}$ at y_3; elsewhere it is C^2) and so

$$\lambda|\xi''| - \left(2\sqrt{\beta\lambda} - \frac{B}{3}\right)|\xi'| + (\beta - C)\,\xi \geq 0. \tag{2.10}$$

Indeed, this clearly holds on $[y_2, \infty)$, while on $(-\infty, y_2]$ it follows from

$$\left(2\sqrt{\beta\lambda} - \frac{B}{3}\right)\xi'(y_2) + (\beta - C)\xi(y_2) = 0 \tag{2.11}$$

because on this interval $\xi'' \equiv 0$, $\xi' \equiv \xi'(y_2)$, and ξ is decreasing. Then of course the left-hand side of (2.11) equals $(\beta - C)\xi'(y_2)(y - y_2) \geq 0$ on $(-\infty, y_2]$, and it is easy to show that there are $y_0 < y_1 < y_2$ and $\zeta : \mathbb{R} \to [0, \infty)$ that coincides with ξ on $[y_1, \infty)$, is concave and C^2 on $(-\infty, y_2]$, constant on $(-\infty, y_0]$ (and hence also maximal there), and satisfies

$$\min\{\lambda, \Lambda \operatorname{sgn} \zeta''\}\,|\zeta''| - \left(2\sqrt{\beta\lambda} - \frac{B}{3}\right)|\zeta'| + (\beta - C)\,\zeta \geq 0. \tag{2.12}$$

Let now $v_0 > 0$ be such that $\psi(v_0) \leq \frac{C}{2}$ and for any $v \in (0, v_0]$ let

$$w_v(t, y) := \frac{\min\{e^{Ct/2}v, v_0\}}{\zeta(y_0)}\,\zeta\left(y + y_0 - \frac{3(d + \Lambda)}{B} - \frac{B}{3}t\right).$$

Then $0 \leq w_v \leq v_0$ because $0 \leq \zeta \leq \zeta(y_0)$, and $w_v(t, \cdot)$ is constant on $[-\infty, \frac{3(d+\Lambda)}{B}]$ for each $t \geq 0$. This, (2.9), and (2.12) show that $u_{v,0}(t, x) := w_v(t, |x|)$ satisfies on $\mathbb{R}^+ \times \mathbb{R}^d$,

$$\mathcal{L}u_{v,0} + f(t, x, u_{v,0}) - (u_{v,0})_t \geq \mathcal{L}u_{v,0} + (\beta - C)u_{v,0} - \frac{B}{3}|\nabla u_{v,0}| \geq 0.$$

Hence $u_{v,0}$ is a subsolution to (1.1) such that at each $t \geq 0$ we have

$$\min\{e^{Ct/2}v, v_0\}\chi_{B_{p+qt}(0)} \leq u_{v,0}(t, \cdot) \leq \min\{e^{Ct/2}v, v_0\}\chi_{B_{y_3 - y_0 + p + qt}(0)}, \tag{2.13}$$

where $p := \frac{3(d+\Lambda)}{B}$ and $q := \frac{B}{3}$.

Let now $t_{v,0} := \max\{\frac{2}{C}\ln\frac{v_0}{v}, \frac{y_3 - y_0}{q}\}$, so that

$$v_0\chi_{B_{y_3 - y_0 + p + q(t - t_{v,0})}(0)} \leq u_{v,0}(t, \cdot) \leq v_0 \tag{2.14}$$

for all $t \geq t_{v,0}$. Let $L \geq 1$ be a Lipschitz constant for f_0 and let $r := \frac{f_0(v_0)}{2\beta - C + L}$ $(< \frac{f_0(v_0)}{L} \leq v_0)$. It follows that $f_0(v') \geq (\beta - \frac{C}{2})(v' - (v_0 - r))$ for all $v' \in [v_0 - r, v_0 + r]$, and so as in the above argument, we obtain that

$$u'_v(t, x) := v_0 - r + \frac{\min\{e^{Ct/2}r, 2r\}}{\zeta(y_0)}\,\zeta\left(|x| + y_0 - p - q(t - t_{v,0})\right)$$

is a subsolution to (1.1) on time interval $[t_{v,0},\infty)$. Since $u'_v \le v_0 \le u_{v,0}$ on $\{t_{v,0}\}\times B_{y_3-y_0+p}(0)$ and $u'_v(t,\cdot) = v_0 - r < v_0 = u_{v,0}(t,\cdot)$ on $\partial B_{y_3-y_0+p+q(t-t_{v,0})}(0)$ for each $t \ge t_{v,0}$, it follows that

$$u_{v,1} := \max\left\{u_{v,0}, u'_v\chi_{\{|x|<y_3-y_0+p+q(t-t_{v,0})\}}\right\}$$

is a subsolution to (1.1) on time interval $[t_{v,0},\infty)$, with $u_{v,1}(t_{v,0},\cdot) \le u_{v,0}(t_{v,0},\cdot)$. Also, if we let $v_1 := v_0 + r$ and $t_{v,1} := t_{v,0} + \sigma$, where $\sigma := \max\{\frac{2}{C}\ln 2, \frac{y_3-y_0}{q}\}$, then for $t \ge t_{v,1}$ we have

$$v_1\chi_{B_{y_3-y_0+p+q(t-t_{v,1})}(0)} \le u_{v,1}(t,\cdot) \le v_1.$$

We can use this in place of (2.14) to similarly obtain a subsolution $u_{v,2}$ to (1.1) on time interval $[t_{v,1},\infty)$, such that $u_{v,2}(t_{v,1},\cdot) \le u_{v,1}(t_{v,1},\cdot)$ and for all $t \ge t_{v,2}$ we have

$$v_2\chi_{B_{y_3-y_0+p+q(t-t_{v,2})}(0)} \le u_{v,2}(t,\cdot) \le v_2,$$

where $v_2 := v_1 + \frac{f_0(v_1)}{2\beta-C+L}$ and $t_{v,2} := t_{v,1}+\sigma$. Repeating this argument, we obtain a v-independent sequence $0 < v_1 < v_2 < \dots$ converging to 1 (because Lipschitz $f_0 > 0$ on $(0,1)$) and subsolutions $\{u_{v,k}\}_{k\ge 0}$ to (1.1) on intervals $[t_{v,k-1},\infty)$, with $t_{v,k} := t_{v,0} + k\sigma$ and $t_{v,-1} := 0$, such that for each $k \ge 1$ we have $u_{v,k}(t_{v,k-1},\cdot) \le u_{v,k-1}(t_{v,k-1},\cdot)$ and

$$v_k\chi_{B_{y_3-y_0+p+q(t-t_{v,k})}(0)} \le u_{v,k}(t,\cdot) \le v_k \tag{2.15}$$

for all $t \ge t_{v,k}$. These subsolutions will now allow us to obtain (2.2).

Let $u : (t_0-1,\infty)\times\mathbb{R}^d \to [0,1]$ solve (1.1) and pick any $x \in \mathbb{R}^d$. Shift A, b, f, u by $(-t_0-1,-x)$ in space-time so that we have $u : (-2,\infty)\times\mathbb{R}^d \to [0,1]$, and we therefore need to prove (2.2) with $(t_0,x) = (-1,0)$ (note that the above subsolutions are also subsolutions for all space-time translations of A, b, f). This is then the estimate

$$u(s-1,0) \ge \min\{e^{\kappa s}u(-1,0),\, 1-\phi(s)\} \tag{2.16}$$

for some u-independent κ, ϕ as required and all $s \ge 0$. Assume also that $u \not\equiv 0$ because otherwise this holds trivially.

By the parabolic Harnack inequality [8, Corollary 7.42], $u \ge 0$, and $\left|\frac{f(\cdot,\cdot,u)}{u}\right| \le \frac{1}{\gamma}$ for all $u \in (0,1]$ (since f is KPP), there is u-independent $\mu \in (0, v_0 e^{-C(y_3-y_0)/2q}]$ such that $u(0,\cdot) \ge \mu u(-1,0)\chi_{B_{y_3-y_0+p}(0)}$ (note that dependence on λ in Remark 2 after Theorem 1.2 enters through this μ). If we let $v := \mu u(-1,0) \le v_0$, then $u(0,\cdot) \ge u_{v,0}(0,\cdot)$, so $u \ge u_{v,0}$ on $[0,\infty)\times\mathbb{R}^d$. Thus $u(t_{v,0},\cdot) \ge u_{v,0}(t_{v,0},\cdot) \ge u_{v,1}(t_{v,0},\cdot)$, so $u \ge u_{v,1}$ on $[t_{v,0},\infty)\times\mathbb{R}^d$. Continuing this way, we find that $u \ge u_{v,k}$ on $[t_{v,k-1},\infty)\times\mathbb{R}^d$ for each k, hence

$$u(t,0) \ge \max\left\{\min\{e^{Ct/2}v, v_0\}, v_1H(t-t_{v,1}), v_2H(t-t_{v,2}), \dots\right\} \tag{2.17}$$

for all $t \ge 0$, where $H := \chi_{[0,\infty)}$. Now let $\kappa := \frac{C}{16}$ and for each $s \ge 0$ define

$$\phi(s) := \begin{cases} 1 & s < s_0 := \max\left\{2, \frac{16}{C}\ln\frac{1}{\mu}, 2(t_{v,1}-t_{v,0})+1\right\}, \\ 1 - v_{\max\{k\,|\,s\ge 2(t_{v,k}-t_{v,0})+1\}} & s \ge s_0, \end{cases}$$

which converges to 0 as $s\to\infty$ because $v_k \to 1$. Since $t_{v,k}-t_{v,0}$ $(= k\sigma)$ is independent of v (and hence of u) for each k, so is ϕ. Also note that $t_{v,0} = \frac{2}{C}\ln\frac{v_0}{v}$ (and hence $e^{Ct_{v,0}/2}v = v_0$) because $\mu \le v_0 e^{-C(y_3-y_0)/2q}$.

Then (2.16) obviously holds for all $s < s_0$, so let us assume that $s \ge s_0$ and let $k := \max\{j \,|\, s \ge 2(t_{v,j} - t_{v,0}) + 1\}$ (so $\phi(s) = 1 - v_k$). If (2.16) fails, then (2.17) implies that $s < t_{v,k} + 1$, and hence $t_{v,k} < 2t_{v,0}$ by the definition of k. But then $s \in [s_0, 2t_{v,0} + 1]$, so

$$e^{\kappa s} u(-1, 0) = e^{Cs/16} v\mu^{-1} \le e^{Cs/8} v \le e^{C(s-1)/4} v \le \min\{e^{C(s-1)/2} v, v_0\} \le u(s-1, 0)$$

by (2.17). Hence (2.16) also holds for all $s \ge s_0$, and the proof is finished. $\square$

Note that the proof shows that ϕ decays exponentially as $s \to \infty$ if $\liminf_{v\to 1} \frac{f_0(v)}{1-v} > 0$.

3. Non-local Diffusion Case

Let us now consider (1.1) on $\mathbb{R}^d$, with $\mathcal{L}$ from (1.3). Any solutions considered here will be from the space $\mathcal{A}$ from Theorem 1.3.

Theorem 3.1. *Theorem 2.1 holds when f, f' are as in that theorem, $\mathcal{L}$ is from* (1.3) *instead of* (1.2)*, hypotheses on $\mathcal{L}$ and* (2.1) *are replaced by K being even in ν and*

$$\max\left\{ \sup_{(t,x,\nu)\in\mathbb{R}^+\times\mathbb{R}^{2d}} K(t,x,\nu) \max\left\{|\nu|^{d+2-\gamma}, e^{\gamma|\nu|}\right\}, \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d} f_u(t,x,0) \right\} \le \gamma^{-1},$$

and well-posedness and comparison principle hold for (1.1) *on the space $\mathcal{A}$ from Theorem 1.3.*

Proof. The proof is identical to that of Theorem 2.1, with the only change being the choice of the exponential supersolutions v_n, which will now instead be $v_n(t,x) := e^{at-\gamma(x-x_n)\cdot e_n/2}$ with some γ-dependent $a > 0$. This is because any $u \in C^3(\mathbb{R}^d)$ satisfies

$$u(x+\nu) + u(x-\nu) - 2u(x) = 2\nu \cdot D^2 u(x)\nu + O(|\nu|^3),$$

so K being even in ν and local integrability of $|\nu|^{-d+\gamma}$ imply that $(\mathcal{L} + \gamma)v_n \le a v_n$ holds with some $a > 0$ depending only on γ. $\square$

Proof of Theorem 1.3. This proof tracks that of Theorem 1.2, first picking β, f_0, ψ, γ in the same way and then using Theorem 3.1 instead of Theorem 2.1. One then again only needs to prove (2.2) with $y^s_{t_0,x} := x =: y^{-s}_{t_0+s,x}$, and $U \equiv 1$, for some κ, ϕ. This is done via a similar construction of an appropriate sequence of subsolutions to (1.1).

Let $\zeta : \mathbb{R} \to [-\frac{1}{2}, \frac{1}{2}]$ be such that $\|\zeta^{(j)}\|_{L^\infty} \le 1$ for $j = 1, 2, 3$ and for some $0 = y_0 < y_1 < y_2 < y_3 < y_4$ we have that $\zeta \equiv \frac{1}{2}$ on $(-\infty, y_0]$ and $\zeta \equiv -\frac{1}{2}$ on $[y_4, \infty)$, also ζ is concave on $(-\infty, y_1]$ and convex on $[y_1, \infty)$, as well as $\zeta(y_2) = \frac{1}{4}$ and $\zeta(y_3) = 0$ and $\zeta'' \equiv \frac{1}{100}$ on $[y_2, y_3]$. Taylor's theorem then shows that for any $\eta, p > 0$, the function $u(x) := \zeta(\eta|x| - p)$ satisfies

$$\left| u(x+\nu) + u(x-\nu) - 2u(x) - 2\nu \cdot D^2 u(x)\nu \right| \le 2c_d \eta^3 |\nu|^3$$

for some $c_d > 0$ only depending on d, and for all $x, \nu \in \mathbb{R}^d$. A simple computation shows that for any $x, \nu \in \mathbb{R}^d$ we also have

$$\nu \cdot D^2 u(x)\nu = \left(\eta^2 \zeta''(\eta|x| - p) - \frac{\eta\zeta'(\eta|x| - p)}{|x|} \right) \left(\frac{x}{|x|} \cdot \nu \right)^2.$$

If we take $p \ge 200$, then the first parenthesis above is zero when $|x| \le \frac{200}{\eta}$, while for $|x| \ge \frac{200}{\eta}$ it is bounded below by $-\frac{3\eta^2}{200}$, and for $|x| \in [\frac{y_2+p}{\eta}, \frac{y_3+p}{\eta}]$ it is bounded below by $\frac{\eta^2}{200}$. This, evenness of K in ν, and (1.6) (recall also that we have chosen γ to be $\le \alpha$) show that at any $t>0$ we have for any $R\ge 0$,

$$\mathcal{L}u(x) \ge -\int_{B_R(0)} \frac{\max\{|\nu|^{-d-2+\gamma},1\}}{\gamma}\left(\frac{3\eta^2|\nu|^2}{200}+c_d\eta^3|\nu|^3\right)d\nu - \int_{\mathbb{R}^d\setminus B_R(0)} \frac{e^{-\gamma|\nu|}}{\gamma}d\nu \tag{3.1}$$

for all $x\in\mathbb{R}^d$ (recall also that $u(x+\nu)-u(x)\ge -1$), but also

$$\mathcal{L}u(x) \ge \int_{B_\gamma(0)} \mathcal{K}(|\nu|)\left[\gamma\frac{\eta^2}{200}\left(\frac{x}{|x|}\cdot\nu\right)^2 - \frac{c_d\eta^3|\nu|^3}{\gamma}\right]d\nu - \int_{\mathbb{R}^d}\frac{e^{-\gamma|\nu|}}{\gamma}\min\{c_d\eta^3|\nu|^3,1\}d\nu \tag{3.2}$$

when $|x|\in[\frac{y_2+p}{\eta},\frac{y_3+p}{\eta}]$ (note that both these lower bounds are independent of x).

Let now $C' := \frac12\inf_{u\in(0,1/2]}\frac{f_0(u)}{u}>0$, pick R so that the second integral in (3.1) is $\le\frac{C'}{8}$, and then $\eta\in(0,1]$ so that the first integral is $\le\frac{C'}{8}$. Then (3.1) shows that

$$\mathcal{L}u(x)+2C'u(x)\ge C'u(x) \tag{3.3}$$

whenever $u(x)\ge\frac14$. Next decrease η further (this will not compromise (3.3)) so that the right-hand side of (3.2) becomes some $C''>0$ (this is possible because $\mathcal{K}(r)\ge 1$ for $r\le\gamma$). Then (3.2) shows that

$$\mathcal{L}u(x)\ge C'' \tag{3.4}$$

whenever $u(x)\in[0,\frac14]$. But this and $\eta\le\frac14$ mean that if we let $C:=\frac12\min\{C',C''\}>0$, then

$$\mathcal{L}u(x)+2C'u(x)\ge C\left(|\nabla u(x)|+u(x)\right) \tag{3.5}$$

holds whenever $u(x)\ge 0$. This and the definition of C' show that if $v_0:=\frac12$, $v\in(0,\frac12]$, and

$$u_{v,0}(t,x) := \left(\min\{e^{Ct/2}v, v_0\}\, 2\zeta(\eta|x|-200-Ct)\right)_+,$$

then $u_{v,0}$ is a subsolution to (1.1) on $\mathbb{R}^+\times\mathbb{R}^d$. The factor $\frac12$ in the exponent is not necessary but we added it to make $u_{v,0}$ similar to $u_{v,0}$ in the proof of Theorem 1.2.

In particular, we now have

$$v_0\chi_{B_{(200+Ct)/\eta}(0)}\le u_{v,0}(t,\cdot)\le v_0$$

for all $t\ge\frac{2}{C}\ln\frac{v_0}{v}$. As in the proof of Theorem 1.2, we can now find $r,t_{v,0}>0$ (then we let $v_1:=v_0+r$) and a subsolution

$$u'_v(t,x) := v_0-r+\left(\min\{e^{Ct/2}r,2r\}\, 2\zeta\left(\eta|x|-200-C(t-t_{v,0})\right)\right)_+$$

to (1.1) on time interval $[t_{v,0},\infty)$ such that

$$u_{v,1} := \max\left\{u_{v,0}, u'_v\chi_{\{|x|<(200+Ct)/\eta\}}\right\}$$

is also a subsolution to (1.1) on time interval $[t_{v,0},\infty)$. We note that non-locality of $\mathcal{L}$ causes a minor issue here but this can be easily overcome by halving C above so that we can add $Cu(x)$ to the right-hand side of (3.5). This gives us an extra term $C(u'_v(t,x)-(v_0-r))$ in the same estimate for u'_v, which is no less than Cr at all points where $u_{v,1}>u_{v,0}$ (all these have $|x|<(y_2+200+C(t-t_{v,0}))/\eta$). This will dominate the decrease of $\mathcal{L}u_{v'}$ at these points caused by replacing $u_{v'}$ by $u_{v'}\chi_{\{(200+Ct)/\eta\}}$, provided $t_{v,0}$ is chosen large enough (recall that K has a uniform exponential decay as $\nu\to\infty$).

The construction of subsolutions $u_{v,2}, u_{v,3}, \dots$ is analogous (without the need to change C further), with the corresponding times $t_{v,1}, t_{v,2}, \dots$ not anymore forming an arithmetic sequence, but with $t_{v,k} - t_{v,0}$ again independent of v. We again have $u_{v,k}(t_{v,k-1}, \cdot) \le u_{v,k-1}(t_{v,k-1}, \cdot)$ for all $k \ge 1$, and (2.15) instead becomes

$$v_k \chi_{B_{(200+C(t-t_{v,k-1}))/\eta}(0)} \le u_{v,k}(t,\cdot) \le v_k$$

for all $t \ge t_{v,k-1} + \frac{2}{C} \ln 2$. The rest of the proof is identical to that of Theorem 1.2 (in particular, the Harnack inequality for $\mathcal{L}$ is used in this part). $\square$

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