TRANSITION FRONTS IN INHOMOGENEOUS FISHER-KPP REACTION-DIFFUSION EQUATIONS

ANDREJ ZLATOŠ

Abstract. We use a new method in the study of Fisher-KPP reaction-diffusion equations to prove existence of transition fronts for inhomogeneous KPP-type non-linearities in one spatial dimension. We also obtain new estimates on entire solutions of some KPP reaction-diffusion equations in several spatial dimensions. Our method is based on the construction of sub- and super-solutions to the non-linear PDE from solutions of its linearization at zero.

1. Introduction and Main Results

We introduce a new elementary method for the study of certain solutions to reaction-diffusion equations with *Kolmogorov-Petrovskii-Piskunov (KPP) type* non-linearities. We use it to prove existence of transition front solutions for very general spatially inhomogeneous KPP reaction-diffusion equations in one dimension as well as some special ones in several dimensions, and to obtain very good estimates on these solutions. Our method is based on relating the solutions of the original non-linear equation to those of its linearization at $u = 0$.

Let us first consider the reaction-diffusion equation

$$u_t = u_{xx} + f(x, u) \tag{1.1}$$

with $x \in \mathbb{R}$ and f an *inhomogeneous KPP reaction function*. That is, we assume that f is Lipschitz, $a(x) \equiv f_u(x, 0) > 0$ exists,

$$f(x, 0) = f(x, 1) = 0 \qquad \text{and} \qquad a(x)g(u) \le f(x, u) \le a(x)u \quad \text{for } (x, u) \in \mathbb{R} \times [0, 1], \tag{1.2}$$

where $g \in C^1([0, 1])$ is such that

$$g(0) = g(1) = 0, \qquad g'(0) = 1, \qquad \text{and} \qquad 0 < g(u) \le u \quad \text{for } u \in (0, 1). \tag{1.3}$$

We will also assume

$$\int_0^1 \frac{u - g(u)}{u^2} du < \infty \qquad \text{and} \qquad g'(u) \le 1 \quad \text{for } u \in (0, 1). \tag{1.4}$$

We define $a_- \equiv \inf_{x \in \mathbb{R}} a(x) \ge 0$ and also assume existence of $a_+ < \infty$ such that

$$a(x) \le a_+ \qquad \text{for } x \in \mathbb{R}. \tag{1.5}$$

A (right-moving) *transition front* for (1.1) is an *entire* (global-in-time) solution $0 \le u \le 1$ connecting 0 and 1 in the sense of

$$\lim_{x \to -\infty} u(t, x) = 1 \qquad \text{and} \qquad \lim_{x \to +\infty} u(t, x) = 0 \tag{1.6}$$

for each $t \in \mathbb{R}$. It models an *invasion* of the unstable state $u \equiv 0$ by the asymptotically stable state $u \equiv 1$. Moreover, we also require that for any $\varepsilon > 0$ there is $L_\varepsilon < \infty$ such that

$$\sup_{t\in\mathbb{R}} \operatorname{diam} \{x \in \mathbb{R} \,|\, \varepsilon \le u(t,x) \le 1-\varepsilon\} \le L_\varepsilon, \tag{1.7}$$

that is, the width of the transition region between ε and $1-\varepsilon$ is uniformly bounded in time. This definition of transition fronts has first appeared in [3, 11].

It has been well known since the seminal works of Fisher [5] and Kolmogorov-Petrovskii-Piskunov [10] that in the homogeneous case $f(x,u) = f(u)$, there exist transition fronts with constant-in-time speed and profile. More specifically, (1.1) has solutions of the form $u(t,x) = U(x-ct)$ with $U(-\infty) = 1$ and $U(\infty) = 0$ precisely when the front speed $c \ge c_f^*$, with $c_f^* \equiv 2\sqrt{f'(0)}$ the *minimal front speed*. These fronts have a constant-in-time profile U with $U' < 0$, are unique for each c up to a translation, and are usually called *traveling fronts*. There are also other transition fronts in this case [8], which are obtained as a combination of two or more traveling fronts with different speeds (we will discuss this in more detail below). Later, existence of KPP transition fronts with time-periodic profiles (called *pulsating fronts*) was proved for x-periodic reactions f, again for all speeds $c \ge c_f^*$ with some $c_f^* > 0$ [2].

Very recently, existence of transition fronts was first time proved for some non-periodic inhomogeneous KPP reactions [15] (see [12,13,16,19] for results on ignition reactions, and [19] for results on some non-KPP non-negative reactions). Specifically, if $a_- > 0$ and $a(x) - a_-$ is *compactly supported*, then transition fronts exist when $\lambda_0 \equiv \sup \sigma[\partial_{xx}^2 + a(x)]$, the supremum of the spectrum of the operator $\partial_{xx}^2 + a(x)$, satisfies $\lambda_0 < 2a_-$ (note that always $\lambda_0 \ge a_-$). These fronts do not have a constant profile but for each $c \in (2\sqrt{a_-}, \lambda_0(\lambda_0 - a_-)^{-1/2})$ there is a front which has a *mean speed*

$$\lim_{|t-s|\to\infty} \frac{X(t) - X(s)}{t-s} \tag{1.8}$$

equal to c, where $X(t)$ is the rightmost point such that $u(t, X(t)) = \frac{1}{2}$. Moreover, no transition fronts exist when, in addition, $a(x) \ge a_-$ and $\lambda_0 > 2a_-$ [15]; this is the first *non-existence-of-fronts* result.

We consider here the question of existence of transition fronts in general inhomogeneous media without the assumption of compact support of $a(x) - a_-$ (in which case no constant or mean speed fronts exist in general) and answer it in the affirmative again when $\lambda_0 < 2a_-$. We achieve this by using a new and elementary method which exploits the close connection between the equation (1.1) and its linearization

$$v_t = v_{xx} + a(x)v \tag{1.9}$$

at $u = 0$.

Such a connection is well known, in particular, when $f(x,u) = f(u)$ and so $a(x) \equiv a = f'(0)$ is constant. Then (1.9) has traveling-front-like solutions $e^{-\gamma(x - c_{a,\gamma}t)}$ with $\gamma > 0$ and speed $c_{a,\gamma} \equiv \gamma + a\gamma^{-1} \ge 2\sqrt{a} = c_f^*$. It turns out [18] that if $c > 2\sqrt{a}$ and $\gamma < \sqrt{a}$ is such that $c = c_{a,\gamma}$, then the traveling front for (1.1) with speed c also has asymptotic decay $e^{-\gamma(x-c_{a,\gamma}t)}$ as $x \to \infty$, while for $c = 2\sqrt{a}$, the asymptotic decay is $(x - 2\sqrt{a}\,t)e^{-\sqrt{a}(x-2\sqrt{a}\,t)}$ as $x \to \infty$ (fronts for (1.9) with $\gamma > \sqrt{a}$ do not give rise to fronts for (1.1)). This means that if $U_{f,\gamma}$ is a traveling front profile for (1.1) corresponding to speed $c_{a,\gamma} \geq c_f^*$ with $\gamma \leq \sqrt{a}$, and the function $h : [0,\infty) \to [0,1)$ is given by $U_{f,\gamma}(x) = h(e^{-\gamma x})$ (so that $h(0) = 0$ and $\lim_{v\to\infty} h(v) = 1$), then $h'(0) = 1$ when $\gamma < \sqrt{a}$ and $\lim_{v\to 0} h(v)(-v \ln v)^{-1} = 1$ when $\gamma = \sqrt{a}$, after an appropriate translation of $U_{f,\gamma}$ in x.

The above shows that for $f(x,u) = f(u)$ and for faster-than-minimal speed $c > c_f^*$, the "tails" of the corresponding traveling fronts for (1.1) and (1.9) are asymptotically the same. We will show that this still holds for some transition fronts in general inhomogeneous media when $\lambda_0 < 2a_-$. We will in fact show that *the study of these fronts for* (1.1) *is essentially equivalent to the study of the corresponding front-like solutions for the simpler equation* (1.9).

Similarly to the compactly supported $a(x) - a_-$ setting in [15], examples of the latter can be found in the form $v_\lambda(t,x) \equiv e^{\lambda t}\phi_\lambda(x)$, where $\phi_\lambda(x) > 0$ is a solution of the Schrödinger generalized eigenfunction equation

$$\phi_\lambda'' + a(x)\phi_\lambda = \lambda\phi_\lambda,$$

with $\lim_{x\to\infty} \phi_\lambda(x) = 0$ and $\phi_\lambda(0) = 1$. Notice that if a is constant, then $v_\lambda(t,x) = e^{\lambda t - \sqrt{\lambda - a}\, x} = e^{-\gamma(x - c_{a,\gamma}t)}$ with $\gamma \equiv \sqrt{\lambda - a}$.

Sturm oscillation theory shows that such $\phi_\lambda > 0$ exists and is unique precisely when $\lambda > \lambda_0$. Moreover, ϕ_λ grows exponentially as $x \to -\infty$ (see (2.11)). Then v_λ is a super-solution of (1.1) and we will show that for any $\lambda \in (\lambda_0, 2a_-)$ there is $h : [0,\infty) \to [0,1)$ such that $w_\lambda(t,x) \equiv h(v_\lambda(t,x))$ is a sub-solution (rather than an outright solution, as in the homogeneous case). Moreover, $\lambda < 2a_-$ will ensure $h(v) \leq v$ so it will follow that there exists a transition front $u \in [w_\lambda, v_\lambda]$ for (1.1). We note that this construction cannot be expected to work for $\lambda \geq 2a_-$ in general because in the homogeneous case this translates to $\gamma \geq \sqrt{a}$, which either gives rise to no front for (1.1) when $\gamma > \sqrt{a}$ or violates $h(v) \leq v$ when $\gamma = \sqrt{a}$.

There is, in fact, a larger class of positive entire solutions of (1.9), of which the v_λ are the extremal points. Indeed, if μ is a finite non-negative non-zero Borel measure on (λ_0, ∞) with a bounded support, then Harnack inequality shows that

$$v_\mu(t,x) \equiv \int_{\mathbb{R}} v_\lambda(t,x)d\mu(\lambda) = \int_{\mathbb{R}} e^{\lambda t}\phi_\lambda(x)d\mu(\lambda) \tag{1.10}$$

is well-defined, and it is obiously an entire solution of (1.9). We will show that v_μ also gives rise to an entire solution of (1.1) provided $\sup\operatorname{supp}(\mu) < 2a_-$.

Finally, our result extends to and will be stated for the more general PDEs

$$u_t = (B(x)u_x)_x + q(x)u_x + f(x,u) \tag{1.11}$$

and

$$v_t = (B(x)v_x)_x + q(x)v_x + a(x)v \tag{1.12}$$

with B, q Lipschitz and satisfying

$$0 < B_- \leq B(x) \leq B_+ < \infty \qquad \text{and} \qquad |q(x)| \leq q_+ < \infty \quad \text{ for } x \in \mathbb{R}. \tag{1.13}$$

Let us define

$$\lambda_0 \equiv \sup_{\psi \in H^1(\mathbb{R})} \frac{\int_{\mathbb{R}} [-B(x)\psi'(x)^2 + q(x)\psi'(x)\psi(x) + a(x)\psi(x)^2]dx}{\int_{\mathbb{R}} \psi(x)^2 dx} \quad (\geq a_-). \tag{1.14}$$

Note that when $q \equiv 0$, then the Rayleigh quotient formula for self-adjoint operators gives

$$\lambda_0 = \sup \sigma \left[\partial_x (B(x)\partial_x) + a(x)\right].$$

As we show below, for $\lambda > \lambda_0$ there is again a unique $\phi_\lambda > 0$ such that

$$(B(x)\phi_\lambda')' + q(x)\phi_\lambda' + a(x)\phi_\lambda = \lambda\phi_\lambda, \tag{1.15}$$

$\lim_{x\to\infty} \phi_\lambda(x) = 0$ and $\phi_\lambda(0) = 1$.

Theorem 1.1. *Assume* (1.2)–(1.5) *and* (1.13)*, let* λ_0 *be as in* (1.14) *and for* $\lambda > \lambda_0$ *let* ϕ_λ *be as in* (1.15)*. Let* $(aB)_- \equiv \inf_{x\in\mathbb{R}}[a(x)B(x)]$*, and assume also that* $q_+ \le 2\sqrt{(aB)_-}$ *and*

$$\lambda_0 < \lambda_1 \equiv \inf_{x\in\mathbb{R}} \left\{ a(x) + \sqrt{(aB)_-} \left[\sqrt{(aB)_-} - |q(x)|\right] B(x)^{-1} \right\}. \tag{1.16}$$

Let μ *be a finite non-negative non-zero Borel measure on* (λ_0, λ_1) *with* $\mu_0 \equiv \inf \operatorname{supp}(\mu)$ *and* $\mu_1 \equiv \sup \operatorname{supp}(\mu)$*, and define* v_μ *as in* (1.10)*.*

(i) If $\mu_1 < \lambda_1$*, then there is an increasing function* $h : [0,\infty) \to [0,1)$ *with* $h(0) = 0$*,* $h'(0) = 1$*,* $\lim_{v\to\infty} h(v) = 1$*, and an entire solution* u_μ *of* (1.11) *satisfying* (1.6)*,* $(u_\mu)_t > 0$*,*

$$h(v_\mu) \le u_\mu \le \min\{v_\mu, 1\}. \tag{1.17}$$

In fact, we can choose $h = h_{g,\alpha}$ *from* (2.1) *below, with any* $\alpha \in (1 - (\lambda_1 - \mu_1)a_+^{-1}, 1)$*.*

(ii) If $\lambda_0 < \mu_0 \le \mu_1 < \lambda_1$*, then* u_μ *from (i) is a transition front (i.e., satisfying also* (1.7)*), with* L_ε *depending only on* $g, a_+, B_\pm, \varepsilon$ *and* ζ*, provided* $\min\{\mu_0 - \lambda_0, \lambda_1 - \mu_1\} \ge \zeta > 0$*.*

Remarks. 1. Condition (1.16) is sharp in this generality, as exhibited by the previously mentioned non-existence of transition fronts in the case of $B \equiv 1$, $q \equiv 0$, and compactly supported $a(x) - a_-$ with $a(x) \ge a_- > 0$ and $\lambda_0 > 2a_-$ [15].

2. The properties of h give $\lim_{x\to\infty} u_\mu(t,x)v_\mu(t,x)^{-1} = 1$ for each $t \in \mathbb{R}$.

3. Note that $a_- + \sqrt{(aB)_-}[\sqrt{(aB)_-} - q_+]B_+^{-1} \le \lambda_1 \le 2a_-$, so (1.16) is satisfied when $\lambda_0 < a_- + \sqrt{(aB)_-}[\sqrt{(aB)_-} - q_+]B_+^{-1}$. In the case $B \equiv 1$ and $q \equiv 0$ we have $\lambda_1 = 2a_-$, so (1.16) simplifies to $\lambda_0 < 2a_-$, the condition mentioned above.

4. Of course, an identical result holds for solutions moving to the left, with ψ_λ defined as ϕ_λ but satisfying instead $\lim_{x\to-\infty} \psi_\lambda(x) = 0$. In addition, a combination of two solutions of (1.12) from (i), moving in opposite directions, gives an entire solution of (1.11) whose spatial infimum converges to 1 as $t \to \infty$.

5. The borderline case $\mu = \delta_{\lambda_1}$, which corresponds to the traveling front with the minimal speed c_f^* and maximal decay $\sim e^{-\sqrt{f'(0)}x}$ as $x \to \infty$ when $f(x,u) = f(u)$, is not covered by our result (because then $\alpha = 1$ in Lemma 2.1 below). It is an open question whether a transition front with a maximal decay as $x \to \infty$ exists in the inhomogeneous setting.

6. The nonlinearity f can in addition depend on time, as long as $f_u(t,x,0)$ is time independent. This is also the case for the other results in this paper.

7. Finally, we note that all our results continue to hold if in (1.2) one does not necessarily require $f(x,1) = 0$. In that case we drop the lower bound on f in (1.2) for $u > 1$, consider

solutions $u \geq 0$ (rather than $0 \leq u \leq 1$) not necessarily converging to 1 as $x \to -\infty$, and the upper bound in (1.17) becomes just $u_\mu(t,x) \leq v_\mu(t,x)$.

Although the "extremal" fronts u_{δ_λ} (corresponding to the extremal measures δ_λ) have a constant speed in homogeneous media, one cannot expect them to have a constant or even a mean speed in general. However, if the medium is random and stationary ergodic, they do have (almost surely) a deterministic *aymptotic speed*

$$c \equiv \lim_{|t|\to\infty} \frac{X(t)}{t} > 0. \tag{1.18}$$

with $X(t)$ as in (1.8).

Theorem 1.2. *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that a measurable function $p \equiv (a, B, q) : \Omega \to L^\infty_{\rm loc}(\mathbb{R})^3$ is Lipschitz in x and satisfies* (1.5) *and* (1.13)*, uniformly in $\omega \in \Omega$. In addition, assume that p is stationary ergodic. That is, there is a group $\{\pi_y\}_{y\in\mathbb{R}}$ of measure preserving transformations acting ergodically on Ω such that $p(\pi_y\omega; x) = p(\omega; x+y)$. Then λ_0, λ_1 from Theorem 1.1 are constant in ω, except on a measure zero set. If $\lambda_0 < \lambda_1$ and a reaction $f(\omega; \cdot, \cdot)$ satisfies* (1.2)–(1.4) *for almost all $\omega \in \Omega$, then for each $\lambda \in (\lambda_0, \lambda_1)$ there is $c_\lambda > 0$ such that the transition front $u_{\delta_\lambda}(\omega; \cdot, \cdot)$ from Theorem 1.1(ii) has asymptotic speed c_λ in the sense of* (1.18) *for almost all $\omega \in \Omega$. The same is true for $u_\mu(\omega; \cdot, \cdot)$ if μ is supported in (λ_0, λ_1), but possibly with different limits $c^-_\mu \leq c^+_\mu$ as $t \to \mp\infty$ in* (1.18)*.*

Remarks. 1. Notice that f itself need not be stationary ergodic.

2. If $B \equiv 1$ and $q \equiv 0$, the condition $\lambda_0 < \lambda_1$ again becomes $\lambda_0 < 2a_-$, which is guaranteed, for instance, when $a_+ < 2a_-$, regardless of the structure of the randomness.

3. It is conceivable that if $\lambda_0 \geq 2a_-$, then transition fronts exist for at least *almost all* ω in the full-measure set where λ_0, λ_1 are constant and (1.2)–(1.4) are satisfied. We do not know the answer to this question at this time and pose it as an open problem.

4. For $B \equiv 1$, $q \equiv 0$, and $f(\omega; x, u) = a(\omega; x)u(1-u)$, propagation speed as $t \to \infty$ for solutions to the Cauchy problem with exponentially decreasing as $x \to \infty$ initial data was studied in [6,7,14]. If the decay rate is large enough, then [6,7] show that solutions propagate almost surely at some deterministic critical speed $c^* \leq c_\lambda$ for all $\lambda \in (\lambda_0, \lambda_1)$ (cf. Remark 5 after Theorem 1.1). If the decay rate is the same as that of ϕ_λ for some $\lambda \in (\lambda_0, \lambda_1)$ (we show in the proof that ϕ_λ almost surely has a deterministic asymptotic exponential decay as $x \to \infty$), then [14] shows that solutions of the Cauchy problem propagate with speed c_λ.

We also provide applications of our method in several spatial dimensions, to the study of solutions of the reaction-diffusion equation

$$u_t = \nabla \cdot (B(x)\nabla u) + q(x) \cdot \nabla u + f(x,u) \tag{1.19}$$

on $\mathbb{R} \times \mathbb{R}^d$, where f, B, q are again as above but with B a matrix field and q a vector field.

Let us start with the special case

$$u_t = \Delta u + f(x,u) \tag{1.20}$$

with $f_u(x,0) \equiv a > 0$ independent of x. The corresponding linear PDE

$$v_t = \Delta v + av \tag{1.21}$$

has "extremal" solutions $v_0(t,x) \equiv e^{at}$ and

$$v_{\gamma\eta}(t,x) \equiv e^{-\gamma\eta\cdot x+(\gamma^2+a)t} = e^{-\gamma(x\cdot\eta-c_{a,\gamma}t)},$$

with $\gamma > 0$, $\eta \in \mathbb{R}^d$ a unit vector, and as before,

$$c_{a,\gamma} = \gamma + a\gamma^{-1} \geq 2\sqrt{a}.$$

From the one-dimensional case mentioned above it immediately follows that each traveling front for (1.20) of the form $u(t,x) = U(x\cdot\eta - ct)$ has the same decay (as $x\cdot\eta \to \infty$) as a multiple of $v_{\gamma\eta}$ for some $\gamma \in (0,\sqrt{a}]$ (with an extra factor $x\cdot\eta - 2\sqrt{a}\,t$ if $\gamma = \sqrt{a}$), and then $c = c_{a,\gamma}$. Both u and $v_{\gamma\eta}$ travel with speed $c_{a,\gamma}$ in the direction η.

We will therefore only consider $\gamma \leq \sqrt{a}$ and let $Y \equiv \overline{B}(0,\sqrt{a})$ be the closed ball in $\mathbb{R}^d$ with radius $\sqrt{a}$ and centered at 0, with topology inherited from $\mathbb{R}^d$. If μ is a finite non-negative non-zero Borel measure on Y, then we let

$$v_\mu(t,x) \equiv \int_Y v_\xi(t,x)d\mu(\xi) = \int_Y e^{-\xi\cdot x+(|\xi|^2+a)t}d\mu(\xi) \tag{1.22}$$

(i.e., $v_{\delta_\xi} = v_\xi$). Notice that $v_\mu(t,x) \leq e^{\sqrt{a}|x|+a(3+\mathrm{sgn}(t))t/2}$ and it is a positive entire solution of (1.21). Also, Y becomes an analog of $[-\lambda_1,-\lambda_0]\cup[\lambda_0,\lambda_1]$ in Theorem 1.1 (the latter set supports measures corresponding to solutions from Remark 4 after Theorem 1.1), after recalling that for homogeneous reactions, $\lambda_0 = a$, $\lambda_1 = 2a$, and $\gamma = \sqrt{\lambda - a}$.

Part (i) of our next result shows that each v_μ gives rise to an entire solution u_μ of (1.20). Moreover, in parts (ii) and (iii) we address the questions when this solution connects 0 and 1 and when does the transition zone between ε and $1-\varepsilon$ have a bounded width (in some sense) for each $\varepsilon > 0$. To this end, let us define the *convex hull* of a measure μ on $\mathbb{R}^d$ to be

$$\mathrm{ch}(\mu) \equiv \{\zeta \in \mathbb{R}^d \,|\, \zeta = \mathbb{E}(\nu) \text{ for some measure } 0 \neq \nu \leq \mu\},$$

with $\mathbb{E}(\nu) \equiv \nu(\mathbb{R}^d)^{-1}\int_{\mathbb{R}^d}\xi d\nu(\xi)$ (here $\nu \leq \mu$ means $\nu(A) \leq \mu(A)$ for any measurable set A). Then $\mathrm{ch}(\mu)$ is convex because

$$\mathbb{E}(\beta\nu + (1-\beta)\nu') = [\beta\nu(\mathbb{R}^d) + (1-\beta)\nu'(\mathbb{R}^d)]^{-1}\left[\beta\nu(\mathbb{R}^d)\mathbb{E}(\nu) + (1-\beta)\nu'(\mathbb{R}^d)\mathbb{E}(\nu')\right]$$

but not necessarily closed. We note that $\mathrm{ch}(\mu)$ is also the intersection of convex hulls of all essential supports of μ, that is, measurable sets $A \subset \mathbb{R}^d$ such that $\mu(A) = \mu(\mathbb{R}^d)$ and $\mu(A') < \mu(A)$ whenever $A' \subset A$ and $A \setminus A'$ has a positive Lebesgue measure (see the remark after the proof of Theorem 1.3), although $\mathrm{ch}(\mu)$ itself need not be an essential support of μ (e.g., if $B \subset \mathbb{R}^d$ is an open ball and μ the uniform measure on the sphere ∂B, then $\mathrm{ch}(\mu) = B$).

Theorem 1.3. *Assume* (1.2)–(1.4) *for $x \in \mathbb{R}^d$ and with $a(x) \equiv a > 0$. Let μ be a finite non-negative non-zero Borel measure with support in the open ball $B(0,\sqrt{a})$ and let v_μ be as in* (1.22).

(i) There is an increasing function $h : [0,\infty) \to [0,1)$ with $h(0) = 0$, $h'(0) = 1$ and $\lim_{v\to\infty} h(v) = 1$, and an entire solution u_μ of (1.20) *such that $(u_\mu)_t > 0$ and* (1.17) *holds.*

In fact, we can choose $h = h_{g,\alpha}$ *from* (2.1) *below, provided* μ *is supported in* $\overline{B}(0, \sqrt{\alpha a})$*. Also,* $u_\mu \not\equiv u_{\mu'}$ *when* $\mu \neq \mu'$*.*

(ii) We have

$$\inf_{x\in\mathbb{R}^d} u_\mu(x,t) = 0 \qquad \text{and} \qquad \sup_{x\in\mathbb{R}^d} u_\mu(x,t) = 1 \tag{1.23}$$

for each $t \in \mathbb{R}$ *(equivalently, for some* $t \in \mathbb{R}$*) if and only if* $0 \notin \mathrm{ch}(\mu)$*.*

(iii) If $0 \notin \mathrm{supp}(\mu)$*, then for each* $\varepsilon, \theta > 0$ *there is* $L_{\varepsilon,\theta}$ *(depending also on* $\mathrm{dist}(0, \mathrm{supp}(\mu))$*,* f*, and* α *from (i)), such that the following holds. If* $u_\mu(t,x) \geq \varepsilon$*, then there is a unit vector* $\eta_{t,x} \in \mathbb{R}^d$ *such that* $u_\mu(t, x+y) \geq 1-\varepsilon$ *whenever* $\eta_{t,x}\cdot y|y|^{-1} \geq \theta$ *and* $|y| \geq L_{\varepsilon,\theta}$*.*

Remark. Regarding the last statement in (i), we note that if $d\nu(\xi) = e^{(|\xi|^2+a)\tau} d\mu(\xi)$ for some $\tau \in \mathbb{R}$, then $v_\nu(t,x) = v_\mu(t+\tau, x)$, and the proof then shows that u_μ and u_ν are also time-shifts of each other.

Part (i) of this result is closely related to a result of Hamel-Nadirashvili [9, Theorem 1.2]. Under the additional assumptions of f being independent of x, concave in u, and $f \in C^2([0,1])$, they prove the existence of an infinite-dimensional manifold of entire solutions of (1.20). These solutions are parametrized by measures supported on the 1-point compactification X of $\mathbb{R}^d \setminus B(0, 2\sqrt{a})$, where distance from origin denotes the front speed $c \geq 2\sqrt{a}$ rather than $\gamma \leq \sqrt{a}$. The mapping $\gamma \mapsto c_{a,\gamma}$ yields a natural identification of Y and X (we consider the former a slightly more natural parameter space for our method than the latter), so one could ask what is the relationship of the two sets of entire solutions.

Under the above additional assumptions on f, it is also shown in [9, Theorem 1.4] that any entire solution $0 < u < 1$ which satisfies

$$\lim_{t\to-\infty} \sup_{|x|<(2\sqrt{a}+\varepsilon)|t|} u(t,x) = 0 \tag{1.24}$$

for some $\varepsilon > 0$, is from their manifold. This gives a characterization of all entire solutions satisfying (1.24). Our u_μ satisfies (1.24) with some $\varepsilon(\alpha) > 0$ as well as the properties of the solution from [9, Theorem 1.2] corresponding to the measure obtained from μ under the above-mentioned identification of Y and X. Since these properties uniquely define a solution in the manifold, it follows that for $f \in C^2([0,1])$, independent of x, and concave in u, the two solutions coincide; and the solutions from Theorem 1.3(i) are all the entire solutions of (1.20) satisfying (1.24).

Moreover, the manifold in [9, Theorem 1.2] also contains solutions corresponding to some measures supported in X but not in its interior (which we do not construct in Theorem 1.3), namely, those whose restriction to $\partial B(0, 2\sqrt{a})$ is a finite sum of Dirac masses.

However, besides proving the existence of this manifold of solutions, [9] only obtains certain claims about the $t \to -\infty$ asymptotic behavior of each of them, with better control only for those corresponding to measures μ which are finite sums of Dirac masses [9, Theorem 1.1]. The contribution of Theorem 1.3(i) is therefore not only in proving the existence of these entire solutions for more general (and even inhomogeneous) KPP reactions, but also in obtaining the explicit estimate (1.17), valid for all times and yielding the new results in (ii) and (iii). Moreover, the usage of our method (from Lemma 2.1 below) makes the proof immediate and elementary, while the proof of [9, Theorem 1.2] is 30 pages long.

In fact, Theorem 1.3 extends to some periodic (a, B, q) (f need not be periodic in x and can even be time-dependent, as mentioned above). Now

$$v_\xi(t,x) \equiv e^{-\xi\cdot x+\kappa_\xi t}\theta_\xi(x),$$

where (θ_ξ, κ_ξ) is the unique solution of

$$\nabla\cdot(B(x)\nabla\theta)+(q(x)-2B(x)\xi)\cdot\nabla\theta+[\xi\cdot B(x)\xi-\nabla\cdot(B(x)\xi)-q(x)\cdot\xi+a(x)]\theta=\kappa\theta \quad (1.25)$$

on the unit cell of periodicity $\mathcal{C}$ (satisfying periodic boundary conditions) with $\theta_\xi > 0$ and $\int_{T^d}\theta_\xi(x)dx = 1$. Again

$$v_\mu(t,x) \equiv \int_Y v_\xi(t,x)d\mu(\xi) \quad (1.26)$$

solves

$$v_t = \nabla\cdot(B(x)\nabla v) + q(x)\cdot\nabla v + a(x)v$$

when μ is as above. Finally, let S_α be the set of all $\xi \in \mathbb{R}^d$ such that

$$\left\| \left(\frac{\nabla\theta_\xi}{\theta_\xi}-\xi\right)\cdot\frac{B}{a}\left(\frac{\nabla\theta_\xi}{\theta_\xi}-\xi\right)\right\|_{L^\infty(\mathcal{C})} \le \alpha. \quad (1.27)$$

Theorem 1.4. *Assume* (1.2)–(1.4) *for* $x \in \mathbb{R}^d$ *and with* (a, B, q) *periodic. Let* μ *be a finite non-negative non-zero Borel measure supported on* S_α *for some* $\alpha < 1$*, and let* v_μ *be as in* (1.26)*. Then Theorem 1.3(i)–(iii) hold with* $h = h_{g,\alpha}$ *from* (2.1) *below, except possibly the last statement in (i).*

Remark. We note that in general, all S_α for $\alpha < 1$ may be empty. However this is not the case when $B - I$ is small in $C^{1,\delta}(\mathbb{T}^d)$ and $a - \bar{a}, q$ (with $\bar{a} \equiv \int_{\mathbb{T}^d} a(x)dx$) are small in $C^\delta(\mathbb{T}^d)$ for some $\delta > 0$. Indeed, in that case we obtain a uniform (in norms of $B - I, a - \bar{a}, q$ in the respective spaces) bound on θ_ξ in $C^{2,\delta}(\mathbb{T}^d)$ for all $|\xi| \le 1$. If now $(a - \bar{a}, B - I, q) \in C^{1,\delta} \times C^\delta \times C^\delta$ is small enough, then $\kappa_\xi - |\xi|^2 - \bar{a}$ is also small, so $a(x) + |\xi|^2 - \kappa_\xi$ is small in C^δ and (1.25) can be rewritten as

$$\begin{aligned}\Delta\theta_\xi + 2\xi\cdot\nabla\theta_\xi = &-\nabla\cdot[(B(x)-I)\nabla\theta_\xi] - [q(x)-2(B(x)-I)\xi]\cdot\nabla\theta_\xi\\ &-[\xi\cdot(B(x)-I)\xi-\nabla\cdot(B(x)\xi)-q(x)\cdot\xi+a(x)+|\xi|^2-\kappa_\xi]\theta_\xi,\end{aligned}$$

with the right-hand side uniformly small in C^δ for all $|\xi| \le 1$. Thus $\theta_\xi - \int_{T^{d-1}}\theta_\xi(x)dx = \theta_\xi - 1$ is uniformly small in $C^{2,\delta}$. This means that for each $\beta < 1$, (1.27) holds for $\alpha \equiv \frac{1}{2}(1+\beta)$ and all $|\xi| \le \beta$ provided $(a - \bar{a}, B - I, q)$ is sufficiently small in $C^{1,\delta} \times C^\delta \times C^\delta$.

We end this introduction with an application of our method to obtaining explicit bounds on certain solutions u of (1.20) with constant $f_u(x, u) = a$, in terms of the solutions of the *heat equation* $\tilde{u}_t = \Delta\tilde{u}$ with the same initial condition (in which case $\tilde{u} \le u \le e^{at}\tilde{u}$). Of course, the latter is just

$$\tilde{u}(t,x) = (4\pi t)^{-d/2}\int_{\mathbb{R}^d} e^{-|x-y|^2/4t}u(0,y)dy. \quad (1.28)$$

Theorem 1.5. *Assume* (1.2)–(1.4) *for* $x \in \mathbb{R}^d$ *and with* $a(x) \equiv a > 0$. *Let* $0 \le u \le 1$ *solve* (1.20) *on* $\mathbb{R}^+ \times \mathbb{R}^d$. *If* $\tilde u$ *from* (1.28) *satisfies*

$$|\nabla \tilde u(t_0, x)| \le \sqrt{\alpha a}\, \tilde u(t_0, x) \tag{1.29}$$

for some $t_0 \ge 0$, $\alpha < 1$, *and all* $x \in \mathbb{R}^d$, *then*

$$h_{g,\alpha}\left(e^{a(t-t_0)}\tilde u(t,x)\right) \le u(t,x) \le \min\{e^{at}\tilde u(t,x), 1\} \tag{1.30}$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, *with* $h_{g,\alpha}$ *from* (2.1) *below (in particular,* $h'_{g,\alpha}(0) = 1 = h_{g,\alpha}(\infty)$*).*

We prove Theorems 1.1–1.5 in the next section, after introducing our main tool, Lemma 2.1.

Finally, we note that existence of transition fronts for (1.1) with very general f (including KPP) is claimed in the paper [17]. This statement is false in the full generality claimed there (in particular, it contradicts the non-existence result in [15]), and its proof is also incorrect. The latter is a direct adaptation of the existence-of-fronts proof for ignition reactions from [13] which, however, does not extend to non-ignition reactions. In particular, various claims in [17], such as the one between (2.22) and (2.23), Corollary 2.6(i), and Proposition 2.7, are made without a proof and are, in fact, false for general non-ignition reactions.

The author would like to thank François Hamel for pointing out the argument in the remark after Theorem 1.4. He also acknowledges partial support by NSF grants DMS-1113017 and DMS-1056327, and by an Alfred P. Sloan Research Fellowship.

2. The Key Lemma and the Proofs of Theorems 1.1–1.5

Our main tool is the following lemma, which constructs sub-solutions $w = h(v)$ of (1.11) from certain solutions v of (1.12) (which are also super-solutions of (1.11)). Here the function $h = h_{g,\alpha} : [0,\infty) \to [0,1)$ depends on $g \in C^1([0,1])$ satisfying (1.3), (1.4) and also on an additional parameter $\alpha \le 1$. Specifically, $h_{g,\alpha}(0) = 0$ and

$$h_{g,\alpha}(v) \equiv U_{g,\sqrt{\alpha}}(-\alpha^{-1/2}\ln v) \tag{2.1}$$

for $v > 0$, where $U_{g,\sqrt{\alpha}}$ is the traveling front profile for the homogeneous PDE

$$u_t = u_{xx} + g(u) \tag{2.2}$$

corresponding to speed $c_{1,\sqrt{\alpha}} \equiv \alpha^{1/2} + \alpha^{-1/2} \ge 2$. That is, $U_{g,\sqrt{\alpha}}(-\infty) = 1$, $U_{g,\sqrt{\alpha}}(\infty) = 0$, $U'_{g,\sqrt{\alpha}} < 0$, and

$$U''_{g,\sqrt{\alpha}} + c_{1,\sqrt{\alpha}}U'_{g,\sqrt{\alpha}} + g(U_{g,\sqrt{\alpha}}) = 0 \tag{2.3}$$

on $\mathbb{R}$. Notice that then $\lim_{v\to\infty} h_{g,\alpha}(v) = 1$ and (2.3) implies

$$\alpha v^2 h''_{g,\alpha}(v) - v h'_{g,\alpha}(v) + g(h_{g,\alpha}(v)) = 0. \tag{2.4}$$

It is well known that $U_{g,\sqrt{\alpha}}$ is unique up to translation and if $\alpha < 1$, then there is a unique translation such that $\lim_{x\to\infty} U_{g,\sqrt{\alpha}}(x)e^{\sqrt{\alpha}x} = 1$ [18]. With this choice of $U_{g,\sqrt{\alpha}}$ we obtain $h'_{g,\alpha}(0) = 1$ for $\alpha < 1$. It then also follows that

$$h_{g,\alpha}(v) \le v \tag{2.5}$$

for $v \in [0,\infty)$ because $h''_{g,\alpha} < 0$ (see the proof of Lemma 2.1 below).

For $\alpha = 1$ we instead have $\lim_{v\to 0} h_{g,\alpha}(v)(-v\ln v)^{-1} = 1$, provided the first condition in (1.4) is replaced by $\int_0^1 [u - g(u)]|\ln u| u^{-2} du < \infty$ [18].

We state the lemma in a more general form, with time-dependent coefficients.

Lemma 2.1. *With f, B, q Lipschitz and time-dependent (B a matrix and q a vector field) and $a(t,x) \equiv f_u(t,x,u)$, assume (1.2)–(1.5) and (1.13) for $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$, where $-\infty < t_0 < t_1 \le \infty$. Let $v > 0$ be a solution of*

$$v_t = \nabla \cdot (B(t,x)\nabla v) + q(t,x) \cdot \nabla v + a(t,x)v$$

on $(t_0,t_1) \times \mathbb{R}^d$. If for some $\alpha < 1$,

$$\nabla v(t,x) \cdot B(t,x)\nabla v(t,x) \le \alpha a(t,x) v(t,x)^2 \tag{2.6}$$

holds for all $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$, then v and $w \equiv h_{g,\alpha}(v)$ are a super- and sub-solution of

$$u_t = \nabla \cdot (B(t,x)\nabla u) + q(t,x) \cdot \nabla u + f(t,x,u) \tag{2.7}$$

on $(t_0,t_1) \times \mathbb{R}^d$. Therefore, if $0 \le u \le 1$ solves (2.7) with $w(t_0,x) \le u(t_0,x) \le v(t_0,x)$ for all $x \in \mathbb{R}^d$, then for all $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$ we have

$$w(t,x) \le u(t,x) \le \min\{v(t,x), 1\}. \tag{2.8}$$

Remark. Of course, the crucial hypothesis here is (2.6).

Proof. Obviously v is a super-solution of (2.7), giving the second inequality. We also have

$$\begin{aligned} w_t - \nabla \cdot (B\nabla w) - q \cdot \nabla w &= h'(v)[v_t - \nabla \cdot (B\nabla v) - q \cdot \nabla v] - h''(v)\nabla v \cdot B\nabla v \\ &= h'(v)av - h''(v)\nabla v \cdot B\nabla v \\ &\le a[h'(v)v - \alpha h''(v)v^2]. \end{aligned}$$

In the last inequality we used (2.6) and $h'' < 0$. The latter is due to (2.4) and Lemma 3.1 from the Appendix with $\gamma \equiv \sqrt{\alpha}$, which yield

$$\alpha v^2 h''(v) = vh'(v) - g(h(v)) = -\alpha^{-1/2} U'_{g,\sqrt{\alpha}}(-\alpha^{-1/2}\ln v) - g(U_{g,\sqrt{\alpha}}(-\alpha^{-1/2}\ln v)) < 0.$$

Thus (2.4) and (1.2) give

$$w_t - \nabla \cdot (B(t,x)\nabla w) - q(t,x) \cdot \nabla w \le a(t,x)g(h(v)) \le f(t,x,w),$$

so w is a sub-solution of (1.11), and the first inequality in (2.8) follows as well. $\square$

Proof of Theorem 1.5. The comparison principle, together with (1.2) yields the upper bound, as well as $\tilde u \le u$. Then let $v(t,x) \equiv e^{a(t-t_0)}\tilde u(t,x)$ and note that $r \equiv \nabla v v^{-1} = \nabla \tilde u \tilde u^{-1}$ satisfies

$$r_t = \Delta r + \nabla(|r|^2)$$

because

$$(\ln \tilde u)_t = \Delta \tilde u \tilde u^{-1} = \Delta(\ln \tilde u) + |r|^2.$$

Thus $\rho \equiv |r|^2$ satisfies

$$\rho_t = \Delta\rho + 2r \cdot \nabla\rho - 2|\nabla r|^2,$$

so (1.29) and the maximum principle give $\rho(t,x) \le \alpha a$ for $(t,x) \in (t_0,\infty) \times \mathbb{R}^d$. Then Lemma 2.1 yields the lower bound in (1.30). $\square$

Proof of Theorem 1.1. (i) Let us start with the proof of existence of ϕ_λ from (1.15), for $\lambda > \lambda_0$. With $\mathcal{L}$ the operator on the left-hand side of (1.15) and λ_0 from (1.14), we have

$$\int_{\mathbb{R}} \psi(x)[(\lambda - \mathcal{L})\psi](x)dx \geq (\lambda - \lambda_0) \int_{\mathbb{R}} \psi(x)^2 dx$$

for $\psi \in H^2(\mathbb{R})$, after integrating by parts. Thus $(\lambda - \mathcal{L})^{-1} : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ exists and if $0 \not\equiv \psi \in L^2(\mathbb{R})$ is compactly supported in $\mathbb{R}^-$, then $0 \not\equiv \phi \equiv (\lambda - \mathcal{L})^{-1}\psi \in H^2(\mathbb{R})$. Since ϕ also satisfies (1.15) on $\mathbb{R}^+$, Harnack inequality shows that $\lim_{x\to\infty} \phi(x) = 0$. Let $\tilde\phi(x) \equiv \phi(x)$ for $x \geq 0$ and extend it onto $\mathbb{R}^-$ so that it solves (1.15). Then $\tilde\phi$ has no roots because if $\tilde\phi(x_0) = 0$, then plugging the function $\tilde\phi|_{[x_0,\infty)}$, extended by 0 on $(-\infty, x_0)$, into (1.14) would yield $\lambda_0 \geq \lambda$. Thus we have $\phi_\lambda(x) = \tilde\phi(x)\tilde\phi(0)^{-1}$. Uniqueness follows from existence of ψ_λ with the same properties but with $\lim_{x\to-\infty} \psi_\lambda(x) = 0$ (by a reflected argument), from $\lim_{x\to-\infty} \phi_\lambda(x) = \infty$ (by (2.11) below), and the fact that the space of solutions of (1.15) is two-dimensional.

Next, choose $\alpha < 1$ such that

$$m \equiv \inf_{x\in\mathbb{R},\beta\geq\alpha} \left\{ a(x) + \sqrt{\beta(aB)_-}\left[\sqrt{\beta(aB)_-} - |q(x)|\right] B(x)^{-1} \right\} - \mu_1 > 0.$$

Any $\alpha \in (1 - (\lambda_1 - \mu_1)a_+^{-1}, 1)$ works because the derivative of the expression in the brackets with respect to β is bounded above by $(aB)_- B(x)^{-1} \leq a_+$ and is positive for $\beta > 1$ (the latter due to $q_+ \leq 2\sqrt{(aB)_-}$). Now let $w_\mu(t,x) \equiv h_{g,\alpha}(v_\mu(t,x))$ and notice that $w_\mu \leq v_\mu$ by (2.5). Then Lemma 2.1 will be applicable to v_μ, w_μ once we establish

$$B(x)\phi_\lambda'(x)^2 \leq \alpha a(x)\phi_\lambda(x)^2 \tag{2.9}$$

for all $\lambda \in (\lambda_0, \mu_1]$ and $x \in \mathbb{R}$. Indeed, (2.9) and $\phi_\lambda > 0$ then yield (2.6) for v_μ. To this end, we need to show

$$|\psi(x)| \leq \sqrt{\alpha a(x)B(x)} \tag{2.10}$$

for $x \in \mathbb{R}$, with $\psi \equiv B\phi_\lambda'/\phi_\lambda$ and $\lambda \in (\lambda_0, \mu_1]$.

Let us assume that $\psi(x_0) \geq \sqrt{\alpha(aB)_-}$ for some x_0. We have $\psi' = \lambda - a - \psi(\psi + q)B^{-1}$ on $\mathbb{R}$, so $\psi'(x_0) \leq \lambda - m - \mu_1 \leq -m$. But then ψ must be decreasing on $(-\infty, x_0]$ with $\psi' \leq -m$ there. From this and $\psi' = \lambda - a - (\psi^2 + q\psi)B^{-1}$ it follows that ψ must blow up at some $x_1 \in (-\infty, x_0)$, a contradiction. We obtain the same conclusion when assuming $\psi(x_0) \leq -\sqrt{\alpha(aB)_-}$ (because $\psi' = \lambda - a - |\psi|(|\psi| - q)B^{-1}$ when $\psi < 0$), with blowup at some $x_1 \in (x_0, \infty)$. It follows that $\|\psi\|_\infty \leq \sqrt{\alpha(aB)_-}$, which gives (2.10), so Lemma 2.1 applies to v_μ, w_μ, α.

A standard limiting argument (see, for instance, [4]) now recovers an entire solution to (1.11) between $\min\{v_\mu, 1\}$ and w_μ. Indeed, we let u_k be the solution of (1.11) on $(-k, \infty) \times \mathbb{R}$ with initial datum $u_k(-k, x) \equiv w_\mu(-k, x)$. Then by Lemma 2.1 we have

$$w_\mu(t,x) \leq u_k(t,x) \leq \min\{v_\mu(t,x), 1\}$$

on $(-k, \infty) \times \mathbb{R}$. By parabolic regularity, there is a locally uniform (on $\mathbb{R}^2$) limit $u_\mu \in [w_\mu, \min\{v_\mu, 1\}]$ of u_k (along a subsequence if needed), which is an entire solution of (1.11).

Since $(w_\mu)_t \geq 0$, the same is true for u_k and thus u_μ, by the maximum principle. The strong maximum principle then gives $(u_\mu)_t > 0$ because $(u_\mu)_t \not\equiv 0$.

Finally, (1.6) follows from (1.17) and $v_\mu(-\infty) = \infty$, the latter being due to (2.11) below.

(ii) The fact that u_μ is a transition front with a bounded width in the sense of (1.7) when $\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1$ will follow from the existence of $L > 0$ such that

$$\phi_\lambda(c) \geq 2\phi_\lambda(d) \tag{2.11}$$

whenever $\lambda \in [\mu_0, \mu_1]$ and $c \leq d - L$. Indeed, we will show that such L depends only on $a_+, B_\pm, \zeta$, provided $\mu_0 - \lambda_0 \geq \zeta > 0$. Then (2.11) holds with the same L for v_μ in place of ϕ_λ. Therefore, if now $\min\{\mu_0 - \lambda_0, \lambda_1 - \mu_1\} \geq \zeta > 0$, then this and (i) gives (1.7) with L_ε depending only on $g, a_+, B_\pm, \varepsilon, \zeta$.

We are left with proving (2.11). If in (1.14) we take

$$\psi(x) \equiv \begin{cases} \phi_\lambda(x) & x \in (c,d), \\ \phi_\lambda(c)(x - c + 1) & x \in [c-1, c], \\ \phi_\lambda(d)(d + 1 - x) & x \in [d, d+1], \\ 0 & x \in \mathbb{R} \setminus [c-1, d+1] \end{cases}$$

for some $c < d$, then we obtain using (2.9) and $\alpha < 1$,

$$\begin{aligned}
&\int_{\mathbb{R}} [-B(x)\psi'(x)^2 + q(x)\psi'(x)\psi(x) + a(x)\psi(x)^2]dx \\
&\geq \int_c^d [-B(x)\phi_\lambda'(x)^2 + q(x)\phi_\lambda'(x)\phi_\lambda(x) + a(x)\phi_\lambda(x)^2]dx - (B_+ + q_+)(\phi_\lambda(c)^2 + \phi_\lambda(d)^2) \\
&\geq \int_c^d [(B(x)\phi_\lambda'(x))' + q(x)\phi_\lambda'(x) + a(x)\phi_\lambda(x)]\phi_\lambda(x)dx \\
&\qquad - (B_+ + q_+)(|\phi_\lambda'(c)|\phi_\lambda(c) + |\phi_\lambda'(d)|\phi_\lambda(d) + \phi_\lambda(c)^2 + \phi_\lambda(d)^2) \\
&\geq \lambda \int_c^d \phi_\lambda(x)^2 dx - (B_+ + q_+)(1 + a_+^{1/2} B_-^{-1/2})(\phi_\lambda(c)^2 + \phi_\lambda(d)^2).
\end{aligned}$$

This and (1.14) give

$$\lambda_0 \int_c^d \phi_\lambda(x)^2 dx \geq \lambda \int_c^d \phi_\lambda(x)^2 dx - [\lambda_0 + (B_+ + q_+)(1 + a_+^{1/2} B_-^{-1/2})](\phi_\lambda(c)^2 + \phi_\lambda(d)^2),$$

which after setting $M \equiv [\lambda_0 + (B_+ + q_+)(1 + a_+^{1/2} B_-^{-1/2})](\lambda - \lambda_0)^{-1}$ reads

$$\int_c^d \phi_\lambda(x)^2 dx \leq M(\phi_\lambda(c)^2 + \phi_\lambda(d)^2). \tag{2.12}$$

By the Harnack inequality, there is $N > 0$ such that $\phi_\lambda(y) \leq N\phi_\lambda(x)$ if $|x - y| \leq 2M$. Set $L \equiv 6MN^2$ and assume (2.11) is violated for some $c \leq d - L$ (notice that L depends only on $a_+, B_\pm, \zeta$ if $\mu_0 - \lambda_0 \geq \zeta > 0$, because $\lambda_0 \leq a_+$ and $q_+ \leq 2\sqrt{a_+ B_+}$). Then there must be

$x \in [c,d]$ such that $\phi_\lambda(x) \le N^{-1}\phi_\lambda(d)$ because otherwise

$$\int_c^d \phi_\lambda(x)^2 dx \ge 6M\phi_\lambda(d)^2 > M(\phi_\lambda(c)^2 + \phi_\lambda(d)^2),$$

contradicting (2.12). Let y be the rightmost point such that $y < d$ and $\phi_\lambda(y) = N^{-1}\phi_\lambda(d)$, and z the leftmost point such that $z > d$ and $\phi_\lambda(z) = N^{-1}\phi_\lambda(d)$. Then $y \le d - 2M$, $z \ge d + 2M$, and $\phi_\lambda(x) \ge N^{-1}\phi_\lambda(d)$ for any $x \in [y,z]$. But this contradicts (2.12) with y, z in place of c, d, so (2.11) is proved and we are done. □

Remark. The argument in (i) works even for $\mu_1 = \lambda_1$, with $\alpha = 1$ and $m = 0$. Then $w_\mu \equiv h_{g,1}(v_\mu)$ will again be a sub-solution of (1.11) but this time $w_\mu \not\le v_\mu$ so we cannot recover a solution between them.

Proof of Theorem 1.2. From (1.14) we know that $\lambda_0 : L^\infty_{\rm loc}(\mathbb{R})^3 \to \mathbb{R}$ is lower semi-continuous, which together with measurability of $p : \Omega \to L^\infty_{\rm loc}(\mathbb{R})^3$ means that $A_\zeta \equiv \{\omega \in \Omega \,|\, \lambda_0(\omega) > \zeta\}$ is a measurable set. Obviously $\pi_y A_\zeta = A_\zeta$ for all $y \in \mathbb{R}$, so $\mathbb{P}(A_\zeta) \in \{0,1\}$ for each $\zeta \in \mathbb{R}$. This means that λ_0 is almost constant on Ω. The same follows for λ_1, using its upper semi-continuity as a function on $L^\infty_{\rm loc}(\mathbb{R})^3$, which follows from its definition.

Let us replace Ω by its full-measure subset on which λ_0, λ_1 are constant. Next fix any $\lambda \in (\lambda_0, \lambda_1)$ and let $u_{\delta_\lambda}(\omega; t, x)$ be the corresponding random transition front. The remark after the proof of Theorem 1.1 shows that there is L such that (2.11) holds for any $\omega \in \Omega$ and $c \le d - L$. Therefore also L_ε in that proof is uniform in ω, which means that if $Y(\omega; t)$ is the rightmost point such that $e^{\lambda t}\phi_\lambda(\omega; Y(\omega; t)) = \frac{1}{2}$ and $X(\omega; t)$ the rightmost point such that $u_{\delta_\lambda}(\omega; t, X(\omega; t)) = \frac{1}{2}$, then $|X(\omega; t) - Y(\omega; t)|$ is uniformly bounded on $\Omega \times \mathbb{R}$. Thus we only need to prove (1.18) for Y in place of X.

Notice that if $r_\lambda(\omega) \equiv \phi'_\lambda(0)$, then $r_\lambda : \Omega \to \mathbb{R}$ is measurable because $p : \Omega \to p(\Omega)$ is measurable and $r_\lambda : p(\Omega) \to \mathbb{R}$ is continuous when $p(\Omega)$ is equipped with $L^\infty_{\rm loc}(\mathbb{R})^3$-induced topology. The latter follows from (2.11) and the fact that any solution of (1.15) with $\phi(0) = 1$ and $\phi'(0) \ne r_\lambda(\omega)$ grows exponentially as $x \to \infty$ (by (2.11) applied to the solution ψ_λ converging to 0 as $x \to -\infty$ and the fact that $\phi_\lambda, \psi_\lambda$ are a basis of the set of all solutions).

Therefore $\phi_\lambda(\cdot; x)$ is measurable for any fixed x. Since $\phi_\lambda(\pi_y\omega; \cdot) = \phi_\lambda(\omega; y)^{-1}\phi_\lambda(\omega; y + \cdot)$, we have $\phi_\lambda(\omega; y + x) = \phi_\lambda(\omega; y)\phi_\lambda(\pi_y\omega; x)$. So from ergodicity of $\{\pi_y\}_{y\in\mathbb{R}}$ and Oseledec theorem it follows that for almost all $\omega \in \Omega$,

$$\lim_{x\to\pm\infty} \frac{1}{x} \ln \phi_\lambda(\omega; x) = -\tau_\pm$$

for some $\tau_\pm \in \mathbb{R}$ (and $\tau_\pm > 0$ by (2.11)). Moreover, $\tau_+ = \tau_-$. Otherwise, there exists $\Omega' \subset \Omega$ and $M < \infty$ such that $\mathbb{P}(\Omega') > \frac{1}{2}$ and

$$\left| \frac{1}{\pm M} \ln \phi_\lambda(\omega; \pm M) - \tau_\pm \right| < \frac{|\tau_+ - \tau_-|}{2}$$

for all $\omega \in \Omega'$. But then

$$\left| \frac{1}{M} \ln \phi_\lambda(\pi_{-M}\omega; M) - \tau_- \right| < \frac{|\tau_+ - \tau_-|}{2}$$

for all $\omega \in \Omega'$, so $\Omega' \cap \pi_{-M}\Omega' = \emptyset$, a contradiction with $\mathbb{P}(\pi_{-M}\Omega') = \mathbb{P}(\Omega') > \frac{1}{2}$. Then $\tau_+ = \tau_-$ and (2.11) give

$$\lim_{|t|\to\infty} \frac{Y(\omega;t)}{t} = \frac{\lambda}{\tau_\pm} \equiv c_\lambda$$

and the first claim is proved.

It then immediately follows that any "non-extremal" front also has asymptotic speed c_μ^+ (c_μ^-) as $t \to \infty$ ($t \to -\infty$), which is equal to $\sup c$ ($\inf c$) taken over all c such that there is a Borel set A with $\mu(A) > 0$ and $c_\lambda \ge c$ ($c_\lambda \le c$) for all $\lambda \in A$. Thus $c_\mu^- \le c_\mu^+$. □

Proof of Theorem 1.3. (i) The proof of all the claims, with the exception of the last one, is identical to the proof of Theorem 1.1(i), with $\alpha < 1$ from the statement of Theorem 1.3(i), and (2.9) replaced by

$$|\nabla v_\xi(t,x)|^2 = |\xi|^2 v_\xi(t,x)^2 \le \alpha a v_\xi(t,x)^2$$

for all $|\xi| \le \sqrt{\alpha a}$.

The last claim is an easy consequence of $u_\mu(t,x)v_\mu(t,x)^{-1} \to 1$ as $v_\mu(t,x) \to 0$ and of

$$\left(\frac{|t|}{\pi}\right)^{d/2} v_\mu(t, 2t\zeta)e^{(|\zeta|^2-a)t}d\zeta \rightharpoonup d\mu(\zeta)$$

as $t \to -\infty$. The latter statement, similar to one in [9], follows from

$$\left(\frac{|t|}{\pi}\right)^{d/2} v_\mu(t, 2t\zeta)e^{(|\zeta|^2-a)t} = \int_Y \left(\frac{|t|}{\pi}\right)^{d/2} e^{-|\xi-\zeta|^2|t|}d\mu(\xi)$$

for $\zeta \in \mathbb{R}^d$ and $t < 0$.

(iii) If $u_\mu(t,x) \ge \varepsilon$, then $v_\mu(t,x) \ge h^{(-1)}(\varepsilon)$ with h from (i). Then there is a unit vector $\eta = \eta_{x,t}$ such that

$$\int_{Y_{\eta,\theta}} e^{-\xi\cdot x+(|\xi|^2+a)t}d\mu(\xi) \ge \frac{\theta}{2\pi}h^{(-1)}(\varepsilon),$$

where

$$Y_{\eta,\theta} \equiv \left\{ \xi \in Y \,\middle|\, \arccos\frac{-\eta\cdot\xi}{|\xi|} \le \frac{\theta}{2} \right\}.$$

If now $\eta \cdot y|y|^{-1} \ge \theta$, then $\arccos(\eta \cdot y|y|^{-1}) \le \frac{\pi}{2} - \theta$, and so $\arccos(-\xi \cdot y|y|^{-1}|\xi|^{-1}) \le \frac{\pi-\theta}{2}$ for any $\xi \in Y_{\eta,\theta}$. Therefore

$$v_\mu(t,x+y) \ge \int_{Y_{\eta,\theta}} e^{-\xi\cdot(x+y)+(|\xi|^2+a)t}d\mu(\xi) \ge \frac{\theta}{2\pi}h^{(-1)}(\varepsilon)|y|\operatorname{dist}(0,\operatorname{supp}(\mu))\cos\frac{\pi-\theta}{2}$$

and the result follows from (1.17) with

$$L_{\varepsilon,\theta} \equiv \left[\frac{\theta}{2\pi}h^{(-1)}(\varepsilon)\operatorname{dist}(0,\operatorname{supp}(\mu))\cos\frac{\pi-\theta}{2}\right]^{-1} h^{(-1)}(1-\varepsilon).$$

(ii) Assume first that $0 \in \operatorname{ch}(\mu)$ and $\nu(Y)^{-1}\int_Y \xi d\nu(\xi) = 0$ for some $0 \ne \nu \le \mu$. Then

$$v_\mu(t,x) \ge \int_Y e^{-\xi\cdot x+a(3-\operatorname{sgn}(t))t/2}d\nu(\xi) \ge \nu(Y)e^{-\nu(Y)^{-1}\int_Y \xi d\nu(\xi)\cdot x}e^{a(3-\operatorname{sgn}(t))t/2} = \nu(Y)e^{a(3-\operatorname{sgn}(t))t/2}$$

by Jensen's inequality. This and (1.17) yield $\inf_{x\in\mathbb{R}^d} u_\mu(x,t) > 0$ for each $t \in \mathbb{R}$.

Now assume that $0 \notin \mathrm{ch}(\mu)$ and define $\hat\mu_d \equiv \mu$. The second claim in (1.23) follows from $\mu > 0$ and (i) so let us prove the first claim. Since $\mathrm{ch}(\mu)$ is a convex set, it must be contained in a closed half-space with 0 on its boundary. Assume without loss it is $\mathbb{R}^{d-1} \times \mathbb{R}_0^+$, and let $\mu_d \equiv \hat\mu_d|_{\mathbb{R}^{d-1}\times\mathbb{R}^+}$ and $\hat\mu_{d-1} \equiv \hat\mu_d|_{\mathbb{R}^{d-1}\times\{0\}} = \hat\mu_d - \mu_d$. Now $\mathrm{ch}(\mu) \cap (\mathbb{R}^{d-1} \times \{0\})$ must be contained in a closed half-space of $\mathbb{R}^{d-1} \times \{0\}$ with 0 on its boundary. Assume without loss it is $\mathbb{R}^{d-2} \times \mathbb{R}_0^+ \times \{0\}$, and let $\mu_{d-1} \equiv \hat\mu_{d-1}|_{\mathbb{R}^{d-2}\times\mathbb{R}^+\times\{0\}}$ and $\hat\mu_{d-2} \equiv \hat\mu_{d-1} - \mu_{d-1}$. Continue in this way until obtaining $\mu_1 = \hat\mu_1$ supported in $\mathbb{R}^+ \times \{0\}^{d-1}$ (because $\hat\mu_0 = \mu|_{\{0\}} = 0$).

Since $\mu = \mu_1 + \cdots + \mu_d$ and $u_\mu \le v_\mu$, it is sufficient to show that for any $\varepsilon > 0$ there is $x \in \mathbb{R}^d$ such that for $k = 1, \dots, d$ we have

$$\int_Y e^{-\xi\cdot x} d\mu_k(\xi) \le \frac{\varepsilon}{d} \tag{2.13}$$

(the extra factor $e^{(|\xi|^2+a)t} \le e^{a(3+\mathrm{sgn}(t))t/2}$ from the definition of v_μ can be absorbed in ε). For $k = 1$, the set of $x \in \mathbb{R}^d$ satisfying (2.13) contains some half-space $[\rho_1, \infty) \times \mathbb{R}^{d-1}$. For each $k = 2, \dots, d$ and any $r_k > 0$, it contains $\bar B_{r_k}(0) \times [\rho_{k,r_k}, \infty) \times \mathbb{R}^{d-k}$ for some $\rho_{k,r_k} > 0$, where $\bar B_{r_k}(0)$ is the closed ball in $\mathbb{R}^{k-1}$ with radius r_k and center 0. If we choose $r_2 \ge \rho_1$ and then recursively $r_k \ge r_{k-1} + \rho_{k-1,r_{k-1}}$ for $k = 3, \dots, d$, the corresponding k sets all contain the point $x = (\rho_1, \rho_{2,r_2}, \dots, \rho_{d,r_d})$. So (2.13) holds for this x and we are done. □

Remark. We have $\mathrm{ch}(\mu) \subseteq \mathrm{chess}(\mu)$, the intersection of convex hulls of all essential supports of μ. This is because if A is an essential support of μ and $\mathrm{ch}(A)$ its convex hull, then $\mathbb{E}(\nu) = \nu(\mathbb{R}^d)^{-1} \int_A \xi d\nu(\xi) \in \mathrm{ch}(A)$ when $0 \ne \nu \le \mu$. The opposite inclusion follows from the construction at the end of the previous proof applied to any $\zeta \notin \mathrm{ch}(\mu)$ instead of 0. Indeed, for any such ζ, one can again find open half-spaces $S_d, \dots, S_1$ of dimensions $d, \dots, 1$ whose boundaries contain ζ (without loss these can be assumed to be $S_k = \zeta + \mathbb{R}^{k-1} \times \mathbb{R}^+ \times \{0\}^{d-k}$) and measures μ_k on S_k ($k = d, \dots, 1$) such that $\mu = \mu_1 + \cdots + \mu_d$. Thus $S \equiv \bigcup_{k=1}^d S_k$ is an essential support of μ and $\zeta \notin S$, which yields $\mathrm{ch}(\mu) \supseteq \mathrm{chess}(\mu)$. Therefore $\mathrm{ch}(\mu) = \mathrm{chess}(\mu)$.

Proof of Theorem 1.4. This is identical to the previous proof, using that (1.27) yields (2.6) for v_ξ when $\xi \in S_\alpha$, and thus also for v_μ because $v_\xi > 0$. □

Appendix.

Lemma 3.1. *Assume that $g \in C^1([0,1])$ satisfies (1.3) and $g'(u) \le 1$ for $u \in (0,1)$. Let $U : \mathbb{R} \to (0,1)$ be a traveling front profile for (2.2) corresponding to speed $\gamma + \gamma^{-1} \ge 2$ with $\gamma \in (0,1]$, that is, $U(-\infty) = 1$, $U(\infty) = 0$, $U'(x) < 0$ for all $x \in \mathbb{R}$, and U satisfies*

$$U'' + (\gamma + \gamma^{-1})U' + g(U) = 0$$

on $\mathbb{R}$. Then

$$0 < -U' < \gamma g(U).$$

Proof. Let $V \equiv U'$ and consider the curve $\{(U(x), V(x))\}_{x\in\mathbb{R}}$ in $\mathbb{R}^2$. It connects $(1, 0)$ to $(0, 0)$ and lies in the fourth quadrant $U > 0 > V$. We need to show that it lies in the domain

$$D \equiv \{(u, v) \,|\, u \in (0, 1) \text{ and } v \in (-\gamma g(u), 0)\}.$$

We have $(U', V') = (V, -\gamma V - \gamma^{-1}V - g(U))$ and the condition $g' \le 1$ ensures that the vector $(v, -\gamma v - \gamma^{-1}v - g(u))$ points inside D (or is parallel to ∂D) when $v = -\gamma g(u)$. This means that $(U(y), V(y)) \in D$ for all $y \ge x$ whenever $(U(x), V(x)) \in D$. Thus if $(U(x), V(x)) \notin D$ for some $x \in \mathbb{R}$, then $(U(y), V(y)) \notin D$ for all $y \le x$. But then $V(y) \le -\gamma g(U(y))$ for $y \le x$, so $-\gamma V(y) - \gamma^{-1}V(y) - g(U(y)) \ge -\gamma V(y) > 0$ for $y \le x$. Since $V(-\infty) = 0$, it follows that $V(x) > 0$, a contradiction. □

References

[1] H. Berestycki, *The influence of advection on the propagation of fronts in reaction-diffusion equations,* Nonlinear PDEs in Condensed Matter and Reactive Flows, NATO Science Series C, 569, H. Berestycki and Y. Pomeau eds, Kluwer, Doordrecht, 2003.

[2] H. Berestycki and F. Hamel, *Front propagation in periodic excitable media,* Comm. Pure and Appl. Math. **55** (2002), 949–1032.

[3] H. Berestycki and F. Hamel, *Generalized traveling waves for reaction-diffusion equations*, In: Perspectives in Nonlinear Partial Differential Equations. In honor of H. Brezis, Contemp. Math. **446**, Amer. Math. Soc., 2007.

[4] P.C. Fife and J.B. McLeod, *The approach of solutions of non-linear diffusion equations to traveling front solutions*, Arch. Ration. Mech. Anal. **65** (1977), 335-361.

[5] R. Fisher, *The wave of advance of advantageous genes,* Ann. Eugenics **7** (1937), 355–369.

[6] M. Freidlin, *Functional Integration and Partial Differential Equations,* Ann. Math. Stud. 109, Princeton University Press, Princeton, NJ, 1985.

[7] J. Gärtner and M. Freidlin, *The propagation of concentration waves in periodic and random media,* Dokl. Acad. Nauk SSSR **249** (1979), 521–525.

[8] F. Hamel and N. Nadirashvili, *Entire solutions of the KPP equation,* Comm. Pure Appl. Math. **52** (1999), 1255–1276.

[9] F. Hamel and N. Nadirashvili, *Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$*, Arch. Ration. Mech. Anal. **157** (2001), 91–163.

[10] A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov, *Étude de l'équation de la chaleur de matière et son application à un problème biologique,* Bull. Moskov. Gos. Univ. Mat. Mekh. **1** (1937), 1–25.

[11] H. Matano, various conference talks.

[12] A. Mellet, J. Nolen, J.-M. Roquejoffre and L. Ryzhik, *Stability of generalized transition fronts,* Commun. PDE **34** (2009), 521–552.

[13] A. Mellet, J.-M. Roquejoffre and Y. Sire, *Generalized fronts for one-dimensional reaction-diffusion equations,* Discrete Contin. Dyn. Syst. **26** (2010), 303–312.

[14] J. Nolen, *A central limit theorem for pulled fronts in a random medium,* Networks and Heterogeneous Media **6** (2011), 167–194.

[15] J. Nolen, J.-M. Roquejoffre, L. Ryzhik, and A. Zlatoš, *Existence and non-existence of Fisher-KPP transition fronts,* Arch. Ration. Mech. Anal., to appear.

[16] J. Nolen and L. Ryzhik, *Traveling waves in a one-dimensional heterogeneous medium,* Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 1021–1047.

[17] Y. Shu, W.-T. Li, and N.-W. Liu, *Generalized fronts in reaction-diffusion equations with mono-stable nonlinearity,* Nonlinear Anal. **74** (2011), 433–440.

[18] K. Uchiyama, *The behavior of solutions of some non-linear diffusion equations for large time,* J. Math. Kyoto Univ. **18** (1978), 453–508.

[19] A. Zlatoš, *Generalized traveling waves in disordered media: Existence, uniqueness, and stability,* preprint.

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA
Email: zlatos@math.wisc.edu