Discussion 9 8–10 AM Monday

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MGF

$\mathbb{E}[X^n]$

$M(t)<\infty$

$$M(t) = \sum_{n=0}^{+\infty} \frac{t^n \mathbb{E}[X^n]}{n!}$$

$$M(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} \underbrace{f(x)}_{\text{p.d.f.}}\, dx$$

$$M(0) = 1$$

$\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} = f_Z(t)$

- $Z \sim N(0,1)$ $\quad M_Z(t) = e^{-t^2/2}$
- $Z \sim N(\mu, \sigma^2)$ $\quad \boxed{M_Z(t) = e^{\mu t - \frac{\sigma^2 t^2}{2}}}$
- $X \sim \text{Poisson}(\lambda)$ $\quad M_X(t) = e^{\lambda(e^t - 1)}$

$$\mathbb{E}[X^n] = \frac{d^n}{dt^n}\Big|_{t=0} M(t) \qquad \boxed{\mathbb{E}[Z]=\mu}$$

* $Z \sim N(0,1) \Rightarrow \mathbb{E}[Z^n] = \begin{cases} 0, & n \text{ odd} \\ (n-1)!!, & \text{even} \end{cases}$

$$\begin{tikzcd}
\boxed{\text{distribution (CDF)}} \arrow[d, leftrightarrow] & & \boxed{\text{Moments}} \arrow[d, leftrightarrow] \\
\boxed{\text{density (PDF/PMF)}} \arrow[urr] \arrow[rr] & & \boxed{\text{Moment generating function}}
\end{tikzcd}$$

CDF: $F_X(t) = P(X \leq t)$

p.d.f. $f_X(t) = \frac{d}{dt} F_X(t)$

$$F_X(t) = \int_{-\infty}^{t} f_X(s)\,ds$$

Moments $n \in \mathbb{N}$ $\mathbb{E}[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\,dx$

M.G.F. $M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx$

$$\mathbb{E}[X^n] = M_X^{(n)}(0)$$

$$M_X(t) = \sum_{n=0}^{+\infty} \frac{t^n \mathbb{E}[X^n]}{n!}$$

Ex: Let X be a random variable with MGF

$$M_X(t) = \frac{1}{2}e^{t\cdot 0} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{5t}$$

(a) Compute the distribution of X

(b) Compute $\mathbb{E}[X]$ & $\mathbb{E}[X^2]$

(c) $P(X=5 \mid X \geq 0)$

$P(X=0) = \frac{1}{2}$

$P(X=-4) = \frac{1}{3}$

$P(X=5) = \frac{1}{6}$

(c) $P(X=5 \mid X \geq 0)$

$$= \frac{P(X=5, X \geq 0)}{P(X \geq 0)}$$

$$= \frac{P(X=5)}{P(X \geq 0)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{2}}$$

(b) $\boxed{\mathbb{E}[g(X)] = \sum_k g(k) P(X=k)}$

$\mathbb{E}[X] = 0 \cdot \frac{1}{2} + (-4) \cdot \frac{1}{3} + \frac{1}{6} \cdot 5$

$\mathbb{E}[X^2] = 0 \cdot \frac{1}{2} + (-4)^2 \cdot \frac{1}{3} + 5^2 \cdot \frac{1}{6}$

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$$

$X_1 \cdots X_n$ independent

$$\boxed{\mathrm{Var}[X_1 + \cdots + X_n] = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)}$$

$$+ \sum_{1 \le i \ne j \le n} \mathrm{Cov}(X_i, X_j)$$

$$\mathrm{Var}(X) = \mathrm{Cov}(X, X)$$

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

$$\mathrm{corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}} \in (-1, 1)$$

independent X & Y $\Longrightarrow$ $\mathrm{corr}(X, Y) = 0$

$\Downarrow$ $\Longrightarrow$ $\mathrm{Cov}(X, Y) = 0$

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

If X and Y independent, then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$\mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX} \cdot e^{tY}]$$
$$= \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}]$$

$$\mathbb{E}[W] = 3\mathbb{E}[X] + 4\mathbb{E}[Y] = 3 \cdot 6 + 4 \cdot (-2) = 10$$
$$\mathrm{Var}(W) = \boxed{9}\,\mathrm{Var}(X) + \boxed{16}\,\mathrm{Var}(Y) = 9 \cdot 3 + 16 \cdot 2 = 59$$
$$W \sim N(10, 59)$$

Exercise 7.21. Let X and Y be independent normal random variables with distributions $X \sim \mathcal{N}(6, 3)$ and $Y \sim \mathcal{N}(-2, 2)$. Let $W = 3X + 4Y$.

(a) Identify the distribution of W.

(b) Find the probability $P(W > 15)$.

Exercise 7.22. Let X and Y be two independent normals with the *same* $\mathcal{N}(\mu, \sigma^2)$ distribution, with μ real and $\sigma^2 > 0$. Are there values μ and σ^2 for which $2X$ and $X + Y$ have the same distribution?

Exercise 7.23. Suppose that X and Y are independent standard normals. Find $P(X > Y + 2)$.

7.22:

$$\left.\begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y \sim N(\mu, \sigma^2) \end{array}\right\} \text{independent.}$$

$$2X \to \mathbb{E}[2X] = 2\mathbb{E}[X] = 2\mu$$
$$\to \mathrm{Var}(2X) = 4\,\mathrm{Var}(X) = 4\sigma^2$$
$$2X \sim N(2\mu, 4\sigma^2)$$

$$X+Y \to \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 2\mu$$
$$\to \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 2\sigma^2$$
$$X+Y \sim N(2\mu, 2\sigma^2)$$

$\mathbb{E}[Z_i]=0 \qquad Var(Z_i)=\mathbb{E}[Z_i^2]-\mathbb{E}[Z_i]^2$

i.i.d. $Z_i \sim N(0,1) \Rightarrow \mathbb{E}[Z_i^2]=1$

Exercise 8.43. Let $Z_1, Z_2, \ldots, Z_n$ be independent normal random variables with mean 0 and variance 1. Let

$Z_i \overset{i.i.d.}{\sim} N(0,1)$

$$Y = Z_1^2 + \cdots + Z_n^2. \geq 0$$

(a) Using that Y is the sum of independent random variables, compute both the mean and variance of Y.

(b) Find the moment generating function of Y and use it to compute the mean and variance of Y.

$\mathbb{E}[Y] = n \cdot \mathbb{E}[Z_1^2] = n \qquad \mathbb{E}[Z_1^4] = 3$

$$Var(Y) = n \cdot Var(Z_1^2) = n \cdot \left(\mathbb{E}[Z_1^4] - (\mathbb{E}[Z_1^2])^2 \right) = 2n$$

$$\mathbb{E}[Z_1^{2k}] = (2k-1)\cdot(2k-3) \cdots (3)\cdot 1$$

$$\mathbb{E}[e^{tY}] = \left(\mathbb{E}[e^{tZ_1^2}]\right)^n = \left(\int_{-\infty}^{+\infty} e^{tx^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx \right)^n$$

Exercise 6.34. Let (X, Y) be a uniformly distributed random point on the quadrilateral D with vertices $(0, 0)$, $(2, 0)$, $(1, 1)$ and $(0, 1)$.

(a) Find the joint density function of (X, Y) and the marginal density functions of X and Y.
(b) Find $E[X]$ and $E[Y]$.
(c) Are X and Y independent?

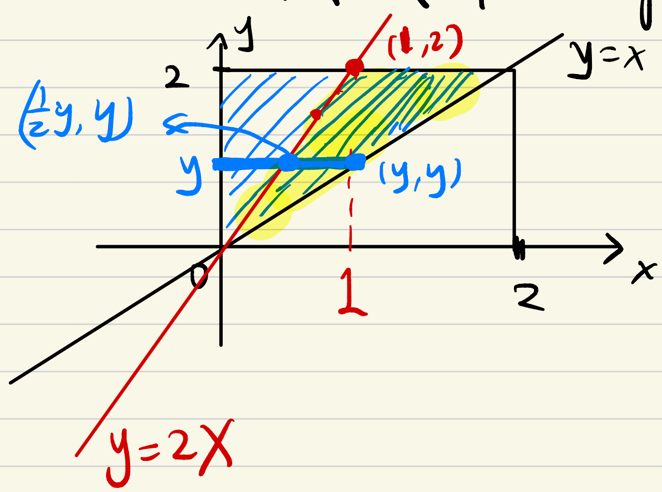
Exercise 6.35. Suppose that X and Y are random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(x+y), & 0 \le x \le y \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Check that $f_{X,Y}$ is a joint density function.
(b) Calculate the probability $P\{Y < 2X\}$.
(c) Find the marginal density function $f_Y(y)$ of Y.

$$\begin{cases} f_{X,Y}(x,y) \ge 0 \\ \int_{\mathbb{R}^2} f_{X,Y}(x,y) = 1 \end{cases}$$

(d) X & Y independent ?



$$\int_{\mathbb{R}^2} f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^2 \int_0^y \tfrac{1}{4}(x+y)\, dx\, dy$$

$$= \int_0^2 \left[\tfrac{1}{8}x^2 + \tfrac{xy}{4}\right]\Big|_0^y dy$$

$$= \int_0^2 \left(\tfrac{1}{8}y^2 + \tfrac{1}{4}y^2\right) dy \quad = \tfrac{3}{8}$$

$$= \tfrac{1}{8}\, y^3 \Big]_0^2$$

$$= 1$$

Fact 7.1. (Convolution of distributions) If X and Y are independent discrete random variables with probability mass functions p_X and p_Y, then the probability mass function of $X+Y$ is

$$p_{X+Y}(n) = p_X * p_Y(n) = \sum_k p_X(k)\, p_Y(n-k) = \sum_\ell p_X(n-\ell)\, p_Y(\ell). \tag{7.2}$$

If X and Y are independent continuous random variables with density functions f_X and f_Y then the density function of $X+Y$ is

$$f_{X+Y}(z) = f_X * f_Y(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = \int_{-\infty}^{\infty} f_X(z-x) f_Y(x)\, dx. \tag{7.3}$$

<u>Ex</u>: independent $\begin{cases} P(X_1=0)=\frac{1}{3}, & P(X_1=1)=\frac{1}{3}, \quad P(X_1=-1)=\frac{1}{3} \\ P(X_2=0)=\frac{1}{2}, & P(X_2=1)=\frac{1}{4}, \quad P(X_2=-1)=\frac{1}{4} \end{cases}$

$\mathbb{E}\left[(X_1+X_2)^4\right]=?$ $\quad \mathbb{E}\left[(X_1+X_2)^2\right]=?$