Discussion 8   8 − 10 AM Mon HSS 5072

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joint density function / joint mass function

continuous   discrete

$$P(X \le t, Y \le s) = \int_{-\infty}^{s}\int_{-\infty}^{t} f_{X,Y}(a,b)\, da\, db$$

$$p_{Y_1,Y_2}(k_1,k_2) = \begin{cases} \frac{1}{16}, & \text{if } 1 \le k_1 \le 4 \text{ and } k_2 = 0 \\ \frac{1}{8}, & \text{if } 1 \le k_1 \le 4,\ 0 < k_2 \le 3 \text{ and } k_1 + k_2 \le 4 \\ 0, & \text{if } 1 \le k_1 \le 4,\ 0 < k_2 \le 3 \text{ and } k_1 + k_2 > 4. \end{cases}$$

We can arrange the values in the following table:

joint mass function

$Y_1 \backslash Y_2$	0	1	2	3
1	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\boxed{\frac{1}{8}}$	0
3	$\frac{1}{16}$	$\frac{1}{8}$	0	0
4	$\frac{1}{16}$	0	0	0

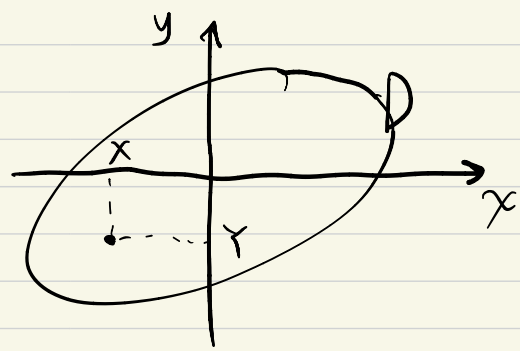
$$P(Y_1 = 2, Y_2 = 2) = \frac{1}{8}$$

Definition 6.18. Let D be a subset of the Euclidean plane $\mathbb{R}^2$ with finite area. Then the random point (X, Y) is **uniformly distributed on** D if its joint density function is

$$f(x,y) = \begin{cases} \frac{1}{\text{area}(D)}, & \text{if } (x,y) \in D \\ 0, & \text{if } (x,y) \notin D. \end{cases} \tag{6.15}$$

Let B be a subset of three-dimensional Euclidean space $\mathbb{R}^3$ with finite volume. Then the random point (X, Y, Z) is **uniformly distributed on** B if its joint density function is

$$f(x,y,z) = \begin{cases} \frac{1}{\text{vol}(B)}, & \text{if } (x,y,z) \in B \\ 0, & \text{if } (x,y,z) \notin B. \end{cases} \tag{6.16}$$



$$\int_{\mathbb{R}^2} f(x,y)\, dx\, dy = 1$$

Exercise 6.36. Suppose that X, Y are jointly continuous with joint probability density function

$$f(x,y) = ce^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}, \quad x, y \in (-\infty, \infty),$$

for some constant c.

(a) Find the value of the constant c. $f_X(x)$ $f_Y(y)$

(b) Find the marginal density functions of X and Y.

(c) Determine whether X and Y are independent.

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f_X(x)$$

$$\int_{\mathbb{R}} f_X(x)\, dx = 1$$

(a) $\displaystyle\iint_{\mathbb{R}\,\mathbb{R}} f(x,y)\,dx\,dy = 1$

$$\boxed{\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C\, e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}\,dx\,dy = 1$$

$$e^{a+b} = e^a e^b$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}-\frac{(x-y)^2}{2}}\,dx\,dy = \frac{1}{C}$$

$$= \iint_{\mathbb{R}\,\mathbb{R}} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(x-y)^2}{2}}\,dx\,dy \qquad \boxed{\begin{cases} z = x-y \\ a = x \end{cases}}$$

$x = a$

$y = a - z$

$$= \iint_{\mathbb{R}\,\mathbb{R}} e^{-\frac{a^2}{2}}\, e^{-\frac{z^2}{2}} \cdot \underbrace{1}_{|\det J|} \cdot\, da\,dz \qquad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = J$$

$$= \left(\int_{\mathbb{R}} e^{-\frac{a^2}{2}}\,da\right) \cdot \left(\int_{\mathbb{R}} e^{-\frac{z^2}{2}}\,dz\right)$$

$$= \left(\sqrt{2\pi}\right)^2 = 2\pi \qquad C = \frac{1}{2\pi}$$

(a) $$f(x,y) = \frac{1}{2\pi}\, e^{-\left(\frac{x^2}{2}+\frac{(x-y)^2}{2}\right)}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(x-y)^2}{2}}\,dy$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} du$$

$u = y - x$

$du = dy$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Rightarrow \text{Gaussian.}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} e^{-\frac{(x-y)^2}{2}} dx$$

$$-\frac{1}{2}x^2 - \frac{1}{2}(x-y)^2$$

$$= -\frac{1}{2}(x^2 + x^2 + y^2 - 2xy)$$

$a^2 - 2ab + b^2 = (a-b)^2$

$$= -\frac{1}{2}y^2 - \frac{1}{2}(2x^2 - 2xy)$$

$$= -\frac{1}{2}y^2 - \left(x^2 - xy + \frac{y^2}{4} - \frac{y^2}{4}\right)$$

$$= -\frac{1}{2}y^2 - \left(x^2 - xy + \frac{y^2}{4}\right) + \frac{y^2}{4}$$

$$= -\frac{1}{4}y^2 - \left(x - \frac{y}{2}\right)^2$$

$$= \frac{1}{2\pi} e^{-\frac{1}{4}y^2} \int_{-\infty}^{+\infty} e^{-(x-\frac{y}{2})^2} dx$$

$\frac{u}{\sqrt{2}} = x - \frac{y}{2}$

$\frac{1}{\sqrt{2}} du = dx$

$$= \frac{1}{2\pi} e^{-\frac{1}{4}y^2} \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} \frac{1}{\sqrt{2}} du$$

$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}y^2}$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{(x-y)^2}{2}}$$

$$f_X(x)\, f_Y(y)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}y^2}$$

$$= \frac{1}{2\sqrt{2}\pi} e^{-\frac{x^2}{2} - \frac{1}{4}y^2}$$

Independence of X & Y

$\Longleftrightarrow \quad f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)$

Exercise 6.33. Let the random variables X, Y have joint density function

$$f(x,y) = \begin{cases} 3(2-x)y, & \text{if } 0 < y < 1 \text{ and } y < x < 2-y \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal density functions f_X and f_Y.

(b) Calculate the probability that $X + Y \leq 1$.

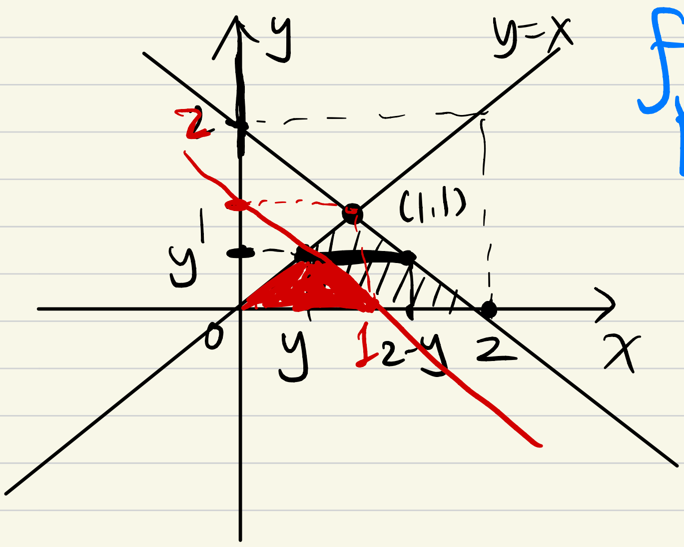
(c) X & Y independent?

Exercise 6.34. Let (X, Y) be a uniformly distributed random point on the quadrilateral D with vertices $(0, 0)$, $(2, 0)$, $(1, 1)$ and $(0, 1)$.

(a) Find the joint density function of (X, Y) and the marginal density functions of X and Y.

(b) Find $E[X]$ and $E[Y]$.

(c) Are X and Y independent?



$$f_Y(y) = \int_y^{2-y} f(x,y)\,dx$$

$$= \int_y^{2-y} 3(2-x)y\,dx$$

$$= \int_y^{2-y} (6y - 3xy)\,dx$$

$$= 6yx - \tfrac{3}{2}x^2y \Big]_y^{2-y}$$

$$= 6y(2-y) - \frac{3}{2}(2-y)^2 y$$

$$- 6y^2 + \frac{3}{2}y^3$$

$$= 12y - 6y^2 - \frac{3}{2}\cdot 4 \cdot y$$

$$- \frac{3}{2}y^2 \cdot y + \frac{3}{2}\cdot 4 \cdot y^2$$

$$- 6y^2 - \frac{3}{2}y^3$$

$$P(X+Y \leq 1) = \int_{\{x+y\leq 1\}} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^{\frac{1}{2}} \int_y^{1-y} 3(2-x)\,y\,dx\,dy$$