Discussion 7 zhw036@ucsd.edu

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No Discussion Next Week

Online OH 6—8 PM on Monday

Exercise 4.51. Suppose customer arrivals at a post office are modeled by a Poisson process N with intensity $\lambda > 0$. Let T_1 be the time of the first arrival. Let $t > 0$. Suppose we learn that by time t there has been precisely one arrival, in other words, that $N_t = 1$. What is the distribution of T_1 under this new information? In other words, find the conditional probability $P(T_1 \le s \,|\, N_t = 1)$ for all $s \ge 0$.

T_1 = time of first arrival

$N_t = N(0,t) = 1$

$P(T_1 \le s \mid N(0,t) = 1) \quad 0 < s \le t$

$$= \frac{P(T_1 \le s,\ N(0,t)=1)}{P(N(0,t)=1)}$$

$N(0,s) \sim \text{Poisson}(\lambda s)$

$N(s,t) \sim \text{Poisson}(\lambda(t-s))$

$P(N(s,t)=0) = e^{-\lambda(t-s)}$

$P(N(0,t)=1)$

$P(T_1 \le s,\ N(0,t)=1)$

$= P(N(0,s)=1,\ N(s,t)=0) = P(N(0,s)=1) \cdot P(N(s,t)=0)$

Poisson Process λ

$N(t) = \#$ calls in $[0, t]$

$I = (a, b]$

$N(I) = \#$ calls in $(a, b]$

- $N(I) \sim \text{Poisson}((b-a)\lambda)$
- $I_1 \; I_2 \; .. \; I_k$ disjoint

  $N(I_1) \cdots N(I_k)$ independent

$X \qquad M_X(t) = \mathbb{E}[e^{tX}] \qquad t \in \mathbb{R}$

$$M_X(0) = 1$$

Prop: If X $M_X(t)$, Y $M_Y(t)$ $\Rightarrow$ $\boxed{M_X(t) = M_Y(t)} < \infty$ $t \in (-\delta, \delta)$

then $X = Y$ in distribution

Ex: $X \sim \text{Uniform}[0,1]$

$$\mathbb{E}[e^{tX}] = \int_0^1 e^{tx}\,dx = \frac{1}{t} e^{tx}\Big]_0^1 = \frac{e^t - 1}{t} \qquad t \neq 0$$

$$\lim_{t \to 0} \frac{e^t - 1}{t} = 1 \qquad M_X(t) = \begin{cases} \frac{e^t - 1}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Exercise 5.3. Let $X \sim \text{Unif}[0,1]$. Find the moment generating function $M(t)$ of X. Note that the calculation of $M(t)$ for $t \neq 0$ puts a t in the denominator, hence the value $M(0)$ has to be calculated separately.

Exercise 5.4. In parts (a)–(d) below, either use the information given to determine the distribution of the random variable, or show that the information given is not sufficient by describing at least two different random variables that satisfy the given condition.

(a) X is a random variable such that $M_X(t) = e^{6t^2}$ when $|t| < 2$. $\Rightarrow X \sim N(0, 12)$

(b) Y is a random variable such that $M_Y(t) = \frac{2}{2-t}$ for $t < 0.5$. $\Rightarrow X \sim \text{Exp}(2)$

(c) Z is a random variable such that $M_Z(t) = \infty$ for $t \geq 5$.

(d) W is a random variable such that $M_W(2) = 2$.

$$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = \boxed{e^{\mu t + \sigma^2 t^2/2}}$$

$$X \sim \text{Exp}(\lambda) \Rightarrow M_X(t) = \begin{cases} \infty, & t \geq \lambda \\ \frac{\lambda}{\lambda - t}, & t < \lambda \end{cases}$$

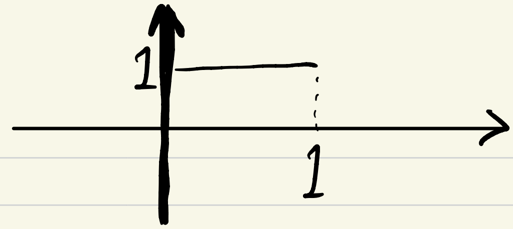
$$X \sim \text{Poisson}(\lambda) \Rightarrow M_X(t) = e^{\lambda(e^t - 1)}$$

(c) $\left.\begin{array}{l} Z \sim \text{Exp}(4) \\ Z' \sim \text{Exp}(5) \end{array}\right\} \Rightarrow$ both satisfy $M_Z(t) = \infty$, $t \geq 5$

(d) $e^{\lambda(e^2-1)} = 2 \Rightarrow \lambda = \frac{\ln 2}{e^2 - 1} \Rightarrow \text{Poisson}\left(\frac{\ln 2}{e^2-1}\right)$

$$\frac{\lambda}{\lambda - 2} = 2 \Rightarrow \lambda = 4 \Rightarrow \text{Exp}(4)$$

$X \sim \text{Unif}\,[0,1]$



$$\mathbb{E}[e^{tX}] = \int_0^1 e^{tx}\,dx$$

$$= \frac{1}{t} e^{tx}\Big]_0^1 = \frac{e^t - 1}{t} \qquad t \neq 0$$

$$\lim_{t \to 0} \frac{e^t - 1}{t} = 1 \qquad M_X(0) = 1$$

$X \sim \text{Bin}(n,p)$ $\qquad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$0 \leq k \leq n$

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$= \sum_{k=0}^{n} e^{tk} P(X=k) \qquad e^{tk} = (e^t)^k$$

$$= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (p \cdot e^t)^k (1-p)^{n-k} = (pe^t + 1 - p)^n$$

(6) Over the course of 365 days, 1 million radioactive atoms of Cesium-137 decayed to 977,287 radioactive atoms. Use the Poisson distribution to estimate the probability that on a given day, 50 radioactive atoms decayed. *Hint: how many atoms decay on average every day?*

(7) Telephone calls enter a college switchboard on the average of two every three minutes. What is the probability of 5 or more calls arriving in a 9-minute period?

(6) $\text{Poisson}(\lambda) \sim X = \#$ decayed atoms

$$P(X=50) \qquad \mathbb{E}[X] = \frac{10^6 - 977287}{365}$$

(7) $$N_3 \sim \text{Poisson}(3\lambda) \quad \mathbb{E}[N_3] = 2 \Rightarrow \lambda = \frac{2}{3}$$

$$X = N_9 \sim \text{Poisson}(9\lambda) \Rightarrow P(X \geq 5)$$