Discussion 4 zhw036@ucsd.edu

Zhichao Wang 8-10 AM Monday

HSS 5072

Properties of Random Variables	
Discrete	Continuous
Probability mass function p.m.f. $p_X(k) = P(X = k)$	Probability density function p.d.f. $f_X(x)$
$P(X \in B) = \sum_{k: k \in B} p_X(k)$	$P(X \in B) = \int_B f_X(x) dx$
Cumulative distribution function $F_X(a) = P(X \leq a)$	
$F_X(a) = \sum_{k: k \leq a} p_X(k)$ F_X is a step function.	$F_X(a) = \int_{-\infty}^{a} f(x)\, dx$ F_X is a continuous function.
$P(X < a) = \lim_{t \to a^-} F(t) = F(a-)$ $P(X = a) = F(a) - \lim_{t \to a^-} F(t) = F(a) - F(a-)$	
$E(X) = \sum_k k\, p_X(k)$	$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$
$E(aX + b) = aE[X] + b$	
$E[g(X)] = \sum_k g(k)\, p_X(k)$	$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$
$\mathrm{Var}(X) = E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2$	
$\mathrm{Var}(aX + b) = a^2\, \mathrm{Var}(X)$	

expectation
mean
average

$E[aX+bY] = aE[X] + bE[Y]$

a, b constant

$\mathrm{Var}(\text{constant}) = 0$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x)\cdot f(x)\,dx$$

p.d.f.

$$\mathbb{E}[X] = \sum_k k\,P(X=k)$$

jump of C.D.F.

$$\mathbb{E}[X^2] = \sum_k k^2 P(X=k)$$

$$\mathbb{E}[g(X)] = \sum_{k \text{ jumps}} g(k)\,P(X=k)$$

mean

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right]$$

not random

$$= \mathbb{E}\left[X^2 + (\mathbb{E}[X])^2 - \boxed{2\,\mathbb{E}[X]}\cdot X\right]$$

$$= \mathbb{E}[X^2] + \mathbb{E}\left[(\mathbb{E}[X])^2\right] - 2\,\mathbb{E}[X]\cdot\mathbb{E}[X]$$

$$= \mathbb{E}[X^2] + (\mathbb{E}[X])^2 - 2(\mathbb{E}[X])^2$$

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

2nd moment

EX: $X =$ # number of rolls of a 10-sided fair die until a 3 or 5 or 7 appears

① find p.m.f. of X

② $E[X]$, $Var(X)$ and $\boxed{\text{standard deviation}}$ $\sqrt{Var(X)}$

$X = 1, 2, 3, \ldots$

$P(\text{roll this die once and get 3 or 5 or 7}) = \frac{3}{10}$

$$P(X=k) = \left(1-\frac{3}{10}\right)^{k-1} \cdot \frac{3}{10} \qquad (k = 1, 2, 3 \ldots)$$

$p = \frac{3}{10}$

$$\boxed{\begin{array}{ll} \text{Geom}(p) & P(X=k) = (1-p)^{k-1} \cdot p \quad \text{p.m.f.} \\ & k = 1, 2, 3 \ldots \end{array}}$$

p: probability of success for one trial

$$f(a) = \sum_{k=0}^{+\infty} a^k = \frac{1}{1-a} \qquad (|a|<1)$$

$a = 1-p$

$$f'(a) = \sum_{k=0}^{+\infty} k a^{k-1} = \left(\frac{1}{1-a}\right)' = \frac{1}{(1-a)^2}$$

$$\boxed{\sum_{k=0}^{+\infty} k \cdot a^{k-1} = \frac{1}{(1-a)^2}}$$

$a = 1-p$

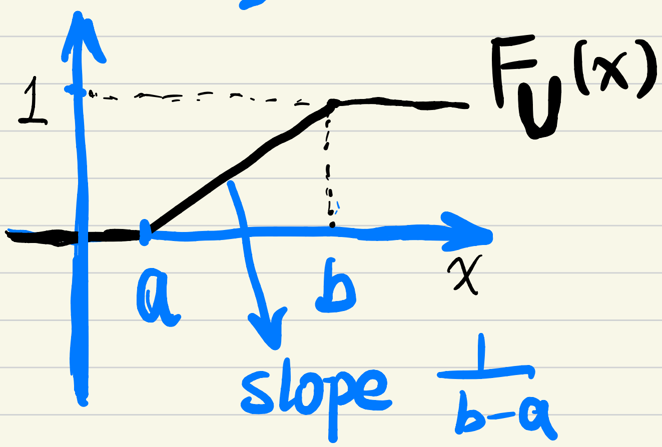
$$\mathbb{E}[X] = \sum_{k=1}^{+\infty} k \cdot P(X=k) = \frac{1}{p}$$

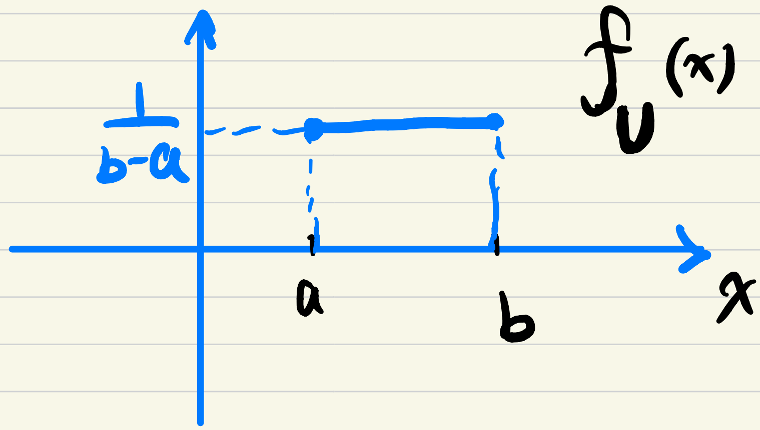
$$f''(a) = \sum_{k=0}^{+\infty} k(k-1) a^{k-2} = \frac{2}{(1-a)^3}$$

$$\boxed{\sum_{k=0}^{+\infty} k(k-1)\, a^{k-2} = \frac{2}{(1-a)^3}}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= \mathbb{E}[X(X-1)] + \mathbb{E}[X] - \mathbb{E}[X]^2$$
$$= \sum_{k=0}^{+\infty} k(k-1) \cdot P(X=k) + \frac{1}{p} - \left(\frac{1}{p}\right)^2$$
$$= \sum_{k=0}^{+\infty} k(k-1)\,(1-p)^{k-1} \cdot p + \frac{1}{p} - \left(\frac{1}{p}\right)^2$$
$$= \left(\sum_{k=0}^{+\infty} k(k-1)\,(1-p)^{k-2}\right)(1-p) \cdot p + \frac{1}{p} - \left(\frac{1}{p}\right)^2$$

$U \sim \text{Unif}([a,b])$





$$E[g(U)] = \int_{-\infty}^{+\infty} g(x) \cdot f_U(x)\, dx$$

$$= \int_a^b g(x) \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b g(x)\, dx$$

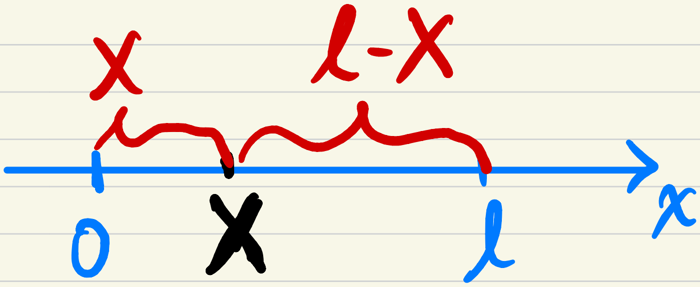
Example 3.36. Let $c > 0$ and let U be a uniform random variable on the interval $\boxed{[0, c]}$. Find the nth moment of U for all positive integers n.

$$E[U^n] = \frac{1}{c} \int_0^c x^n\, dx$$

$$= \frac{1}{c} \left. \frac{x^{n+1}}{n+1} \right]_0^c$$

$$= \frac{1}{c} \cdot \frac{c^{n+1}}{n+1} = \frac{c^n}{n+1}$$

Example 3.34. A stick of length ℓ is broken at a uniformly chosen random location. What is the expected length of the longer piece?



$X \sim \mathrm{Unif}([0,\ell])$

$g(X)$ = length of the longer piece

$X \in [0,\ell]$

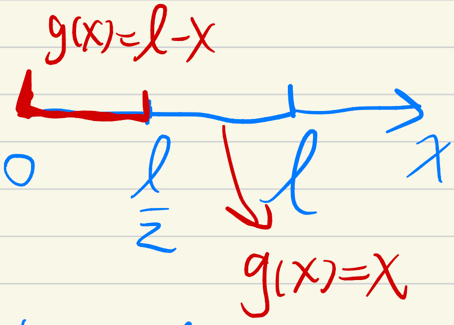
$$g(X) = \begin{cases} \underline{\ell - X}, & X \le \ell - X \iff X \le \frac{\ell}{2} \\ X, & X > \ell - X \iff X > \frac{\ell}{2} \end{cases}$$

$$E[g(X)] = \frac{1}{\ell}\int_0^{\ell} g(x)\,dx$$

$$= \frac{1}{\ell}\left(\int_0^{\ell/2} (\ell - x)\,dx + \int_{\ell/2}^{\ell} x\,dx\right)$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x)\cdot f_X(x)\,dx$$

$$= \int_0^{\ell} \frac{1}{\ell}\cdot g(x)\,dx$$



$$= \boxed{\int_0^{\frac{\ell}{2}}} \frac{1}{\ell}\underline{(\ell-x)}\,dx + \boxed{\int_{\frac{\ell}{2}}^{\ell}} \frac{1}{\ell}\, x\,dx$$

$$= -\frac{1}{\ell}\,\frac{(\ell-x)^2}{2}\Bigg]_0^{\frac{\ell}{2}} + \frac{1}{\ell}\,\frac{x^2}{2}\Bigg]_{\frac{\ell}{2}}^{\ell}$$

$$= -\frac{1}{\ell}\,\frac{\left(\frac{\ell}{2}\right)^2}{2} + \frac{1}{\ell}\,\frac{\ell^2}{2} + \frac{1}{\ell}\,\frac{\ell^2-\left(\frac{\ell}{2}\right)^2}{2}$$

$$= \frac{3\ell}{4}$$

<u>EX</u>: X is a continuous r.v.
with cumulative distr. function

$$F(x) = \begin{cases} 0, & \text{if } x < \sqrt{2} \\ x^2 - 2, & \text{if } \sqrt{2} \le x < \sqrt{3} \\ 1, & \text{if } x \ge \sqrt{3} \end{cases}$$

find ① $P(X = 1.6)$

② $P(1 \le X \le \frac{3}{2})$

③ $P(1 < X \le \frac{3}{2})$

④ probability density function

⑤ smallest interval $[a, b]$ s.t.

$$P(a \le X \le b) = 1$$