Discussion 10

Zoom OH 6:30 — 8:30 PM Monday

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Theorem 9.2. (Markov's inequality) *Let X be a nonnegative random variable. Then for any $c > 0$*

$$P(X \geq c) \leq \frac{E[X]}{c}. \tag{9.1}$$

Theorem 9.5. (Chebyshev's inequality) *Let X be a random variable with a finite mean μ and a finite variance σ^2. Then for any $c > 0$ we have*

$$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}. \tag{9.2}$$

- Law of large number $E[X_i] = \mu$

$X_1, X_2 \cdots\cdots X_n$ i.i.d.

$$P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) \to 1 \quad n \to \infty \quad \forall \varepsilon > 0$$

- Central limit theorem $Var(X_i) = \sigma^2$

$$\frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{S_n - n\mu}{\sqrt{n} \cdot \sigma} \xrightarrow{d} N(0,1)$$

non negative

Exercise 9.2. Let X be an exponential random variable with parameter $\lambda = \frac{1}{2}$.

(a) Use Markov's inequality to find an upper bound for $\boxed{P(X > 6)}$.

(b) Use Chebyshev's inequality to find an upper bound for $P(X > 6)$.

(c) Explicitly compute the probability above and compare with the upper bounds you derived.

$$\mathbb{E}[X] = \frac{1}{\lambda} = 2$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2} = 4$$

(a) $P(X > 6) \leq \frac{\mathbb{E}[X]}{6}$

(b) $P(|X - 2| \geq c) \leq \frac{4}{c^2}$

$$P(X > 6)$$
$$= P(X - 2 > 4) < P(|X - 2| > 4)$$

(c) $P(X > 6) = \int_6^{+\infty} \frac{1}{2} e^{-\frac{1}{2}x} \, dx$

EX: any random variable X
any real number a

Prove: $$P(X \geq a) \leq e^{-ta} M_X(t)$$ for $\forall\, t \geq 0$

$$P(X \geq a)$$
$$= P(tX \geq ta) = P(e^{tX} \geq e^{ta})$$

$$\overset{\text{Markov}}{\leq} \frac{E[e^{tX}]}{e^{ta}} = e^{-ta} \cdot M_X(t)$$

$$P(X \geq a) \leq e^{-ta} \cdot M_X(t) \quad \forall\, t \geq 0$$

$$P(X \geq a) \leq \inf_{t \geq 0} e^{-ta} \cdot M_X(t)$$

Apply this result to previous $\mathrm{Exp}(\frac{1}{2})$

$$P(X \geq 6) \leq \inf_{t \geq 0} e^{-t6} \cdot M_X(t)$$

$$M_X(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1-2t} \quad \forall\, t \leq \frac{1}{2}$$

$f(t) = \frac{e^{-6t}}{1-2t} \quad \forall_{0 \le t \le \frac{1}{2}} \Rightarrow$ want minimal value

$$f'(t) = \frac{-6e^{-6t}(1-2t) + 2\cdot e^{-6t}}{(1-2t)^2}$$

$$= \frac{2e^{-6t}}{(1-2t)^2}(-3+6t+1) = \frac{4e^{-6t}}{(1-2t)^2}(3t-1)$$

$f'(t) = 0 \Rightarrow t = \frac{1}{3} \rightarrow$ minimal point

$$P(X \geq 6) \leq e^{-2} \cdot \frac{1}{1-\frac{2}{3}} = 3e^{-2}$$

Theorem 9.9. (Law of large numbers with finite variance) *Suppose that we have i.i.d. random variables* $X_1, X_2, X_3, \ldots$ *with finite mean* $E[X_1] = \mu$ *and finite variance* $\mathrm{Var}(X_1) = \sigma^2$. *Let* $S_n = X_1 + \cdots + X_n$. *Then for any fixed* $\varepsilon > 0$ *we have*

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) = 1. \tag{9.4}$$

Theorem 9.11. (Central limit theorem) *Suppose that we have i.i.d. random variables* $X_1, X_2, X_3, \ldots$ *with finite mean* $E[X_1] = \mu$ *and finite variance* $\mathrm{Var}(X_1) = \sigma^2$. *Let* $S_n = X_1 + \cdots + X_n$. *Then for any fixed* $-\infty \le a \le b \le \infty$ *we have*

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - n\mu}{\sigma\sqrt{n}} \le b\right) = \Phi(b) - \Phi(a) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy. \tag{9.6}$$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d.} N(0,1)$$

$$\left.\begin{array}{l} X_1 \sim \mathrm{Exp}(\lambda_1) \\ X_2 \sim \mathrm{Exp}(\lambda_2) \end{array}\right\} \text{independent}$$

$$Y = \min(X_1, X_2) \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$$