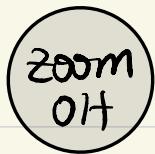


Discussion 10

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6:30 — 8:30 PM Monday

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Theorem 9.2. (Markov's inequality) Let X be a nonnegative random variable. Then for any $c > 0$

$$P(X \geq c) \leq \frac{E[X]}{c}. \quad (9.1)$$

Theorem 9.5. (Chebyshev's inequality) Let X be a random variable with a finite mean μ and a finite variance σ^2 . Then for any $c > 0$ we have

$$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}. \quad (9.2)$$

- Law of large number $E[X_i] = \mu$

X_1, X_2, \dots, X_n i.i.d.

$$P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) \rightarrow 1 \quad n \rightarrow \infty$$

$\forall \varepsilon > 0$

- Central limit theorem $\text{Var}(X_i) = \sigma^2$

$$\frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{S_n - n\mu}{\sqrt{n} \cdot \sigma} \xrightarrow{d.} N(0, 1)$$

non negative

Exercise 9.2. Let X be an exponential random variable with parameter $\lambda = \frac{1}{2}$.

- (a) Use Markov's inequality to find an upper bound for $P(X > 6)$.
- (b) Use Chebyshev's inequality to find an upper bound for $P(X > 6)$.
- (c) Explicitly compute the probability above and compare with the upper bounds you derived.

$$(a) P(X > 6) \leq \frac{\mathbb{E}[X]}{6}$$

$$\mathbb{E}[X] = \frac{1}{\lambda} = 2$$

$$\text{Var}(X) = \frac{1}{\lambda^2} = 4$$

$$(b) P(|X - 2| \geq c) \leq \frac{4}{c^2}$$

$$\begin{aligned} P(X > 6) \\ = P(X - 2 > 4) < P(|X - 2| > 4) \end{aligned}$$

$$(c) P(X > 6) = \int_6^{+\infty} \frac{1}{2} e^{-\frac{1}{2}x} dx$$

Ex: Any random variable X
any real number a

Prove: $P(X \geq a) \leq e^{-ta} M_X(t)$ for $\forall t \geq 0$

$$P(X \geq a) = P(tX \geq ta) = P(e^{tX} \geq e^{ta})$$

Markov

$$\leq \frac{E[e^{tX}]}{e^{ta}} = e^{-ta} \cdot M_X(t)$$

$$P(X \geq a) \leq e^{-ta} \cdot M_X(t) \quad \forall t \geq 0$$

$$P(X \geq a) \leq \inf_{t \geq 0} e^{-ta} \cdot M_X(t)$$

Apply this result to previous $\text{Exp}(\frac{1}{2})$

$$P(X \geq 6) \leq \inf_{t \geq 0} e^{-tb} \cdot M_X(t)$$

$$M_X(t) = \frac{\lambda}{\lambda-t} = \frac{1}{1-2t} \quad \forall t \leq \frac{1}{2}$$

$$f(t) = \frac{e^{-6t}}{1-2t} \quad \forall 0 < t \leq \frac{1}{2} \Rightarrow \text{want minimal value}$$

$$f'(t) = \frac{-6e^{-6t}(1-2t) + 2 \cdot e^{-6t}}{(1-2t)^2}$$

$$= \frac{2e^{-6t}}{(1-2t)^2} (-3 + 6t + 1) = \frac{4e^{-6t}}{(1-2t)^2}(3t-1)$$

$$f'(t)=0 \Rightarrow t=\frac{1}{3} \rightarrow \text{minimal point}$$

$$P(X \geq 6) \leq e^{-2} \cdot \frac{1}{1-\frac{2}{3}} = 3e^{-2}$$

Theorem 9.9. (Law of large numbers with finite variance) Suppose that we have i.i.d. random variables X_1, X_2, X_3, \dots with finite mean $E[X_1] = \mu$ and finite variance $\text{Var}(X_1) = \sigma^2$. Let $S_n = X_1 + \dots + X_n$. Then for any fixed $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) = 1. \quad (9.4)$$

Theorem 9.11. (Central limit theorem) Suppose that we have i.i.d. random variables X_1, X_2, X_3, \dots with finite mean $E[X_1] = \mu$ and finite variance $\text{Var}(X_1) = \sigma^2$. Let $S_n = X_1 + \dots + X_n$. Then for any fixed $-\infty \leq a \leq b \leq \infty$ we have

$$\lim_{n \rightarrow \infty} P\left(a \leq \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq b\right) = \Phi(b) - \Phi(a) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \quad (9.6)$$

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$$\frac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1)$$

$$X_1 \sim \text{Exp}(\lambda_1) \\ X_2 \sim \text{Exp}(\lambda_2) \quad \rightarrow \text{independent}$$

$$Y = \min(X_1, X_2) \sim \text{Exp}(\lambda_1 + \lambda_2)$$