Math 180A — Discussion 1

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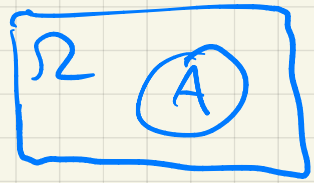
OH: 8–10 AM Mon
HSS 5072

Ω sample space $\#\Omega$



- $\#\Omega$ is finite
  - with replacement $\#\Omega = n^k$
  - without replacement with order $\#\Omega = n(n-1)\cdots(n-k+1) = (n)_k = \frac{n!}{(n-k)!}$
  - without replacement without order

  $$\boxed{P(A) = \frac{\#A}{\#\Omega}}$$

  $A \subseteq \Omega$

  Combinatorics

  1 2 3 …… n $\qquad \#\Omega = \frac{n!}{(n-k)!\cdot \boxed{k!}} = \binom{n}{k}$

- $\#\Omega$ is ∞
  - $\#\Omega$ is countable $\mathbb{N}$

    e.g. flip fair coin until get head

    $\Omega = \{1, 2, 3 \cdots\}$

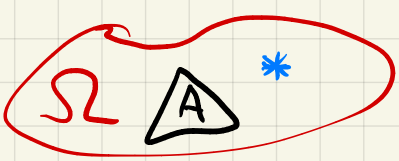
    $$P(k) = \left(\frac{1}{2}\right)^k \quad k \in \mathbb{N}$$

    $$\sum_{k=1}^{+\infty} P(k) = 1$$

  - $\#\Omega$ is uncountable $\Omega = [0,1]$

    calculus

    e.g. Uniformly choose a point in $[a,b]$



$$P(* \in A) = \frac{\text{Area}(A)}{\text{Area}(\Omega)}$$

Exercise 1.7. We have an urn with 3 green and 4 yellow balls. We draw 3 balls one by one without replacement.

(a) Find the probability that the colors we see in order are green, yellow, green.
(b) Find the probability that our sample of 3 balls contains 2 green balls and 1 yellow ball.

(a) $\#\Omega = 7\times6\times5$

$$P(GYG) = \frac{3\times4\times2}{7\times6\times5}$$

(b)

- sampling without replacement but in order

$\#\Omega = 7\times6\times5$

$$\frac{\binom{3}{2}\cdot\binom{4}{1}\times3!}{7\times6\times5}$$

- sampling without replacement and order

2 Green balls & 1 yellows.

$\#\Omega = \binom{7}{3}$

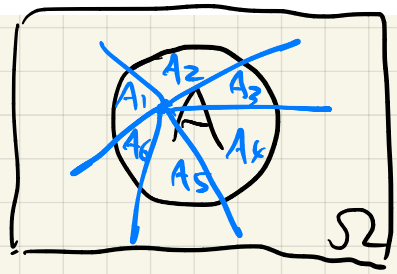
$$= \frac{7\times6\times5}{3!}$$

$$\frac{\binom{3}{2}\cdot\binom{4}{1}}{\binom{7}{3}}$$

Exercise 1.12. We roll a fair die repeatedly until we see the number four appear and then we stop.

(a) What is the probability that we need at most 3 rolls?
(b) What is the probability that we needed an even number of die rolls?

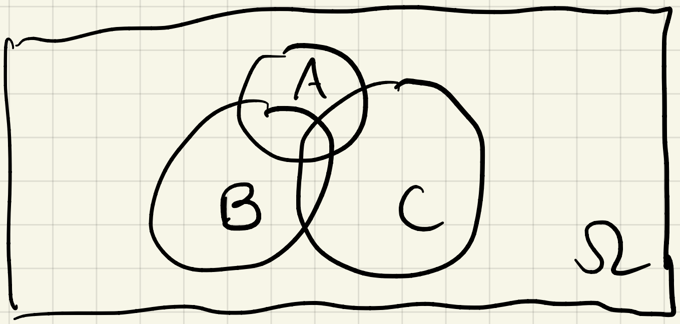
decompose your event "disjoint"



complement of event

$\Omega \setminus A = A^c$ $P(A^c) = 1 - P(A)$

inclusion & exclusion formula $P(A \cup B \cup C)$



$(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

$$P(A \cap B \cap C) = P(A \cup B \cup C) - P(A) - P(B) - P(C) + P(A \cap B) + P(A \cap C) + P(B \cap C)$$

Fact 1.23. (Inclusion-exclusion formulas for two and three events)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{1.16}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ - P(B \cap C) + P(A \cap B \cap C). \tag{1.17}$$

EX: You want to come up with a password with length 8. At each position, you randomly selet one letter from A, B, C.
What is the probability each letter appears at least once in your password? $= 1 - P(X \cup Y \cup Z)$

$\# \Omega = 3^8$

Complement event: at lease one letter not appearing in password.

$\| $

$X \cup Y \cup Z$

$P(X \cup Y \cup Z)$

X = event A is not contained

Y = event B is not contained

Z = event C is not contained

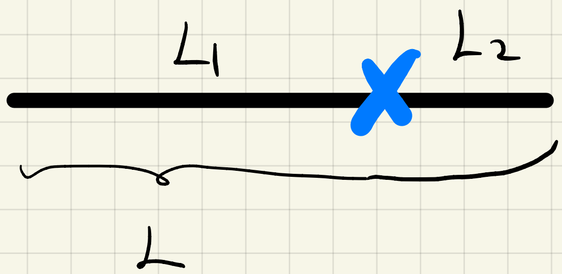
$$P(X) = \frac{2^8}{3^8} = P(Y) = P(Z)$$

$$P(X \cap Y) = \frac{1}{3^8} = P(X \cap Z) = P(Z \cap Y)$$

$$P(X \cup Y \cup Z) = P(X) + P(Y) + P(Z) - P(X \cap Y) - P(Y \cap Z) - P(X \cap Z) + P(X \cap Y \cap Z)$$

$$1 - P(X \cup Y \cup Z) = 1 - \frac{3(2^8 - 1)}{3^8} \approx 0.8$$

Exercise 1.9. We break a stick at a uniformly chosen random location. Find the probability that the shorter piece is less than 1/5th of the original.



Let's assume total length is L.

$$\Omega = \left(0, \frac{L}{2}\right]$$

$$p\left(\min\{L_1, L_2\} < \frac{L}{5}\right) = ?$$