Today: Periodic, aperiodic, reducible, irreducible Markov chains with finite state space
Prop. 7.1 Let \((X^n)\) be a MC with finite state space \(S\).

Suppose that there exists \(n_0 \in \mathbb{N}\) s.t. \([P^n]_{ij} > 0\) for all \(i, j \in S\).

Then for each \(j\), \(\pi(j)\) is equal to the asymptotic expected fraction of time the chain spends in state \(j\), i.e.,

\[
\lim_{n \to \infty} E \left[ \frac{1}{n+1} \sum_{k=0}^{n} \mathbb{1}\{X_k = j\} \right] = \pi(j)
\]

Proof.

\[
E \left[ \frac{1}{n+1} \sum_{k=0}^{n} \mathbb{1}\{X_k = j\} \right] = \frac{1}{n+1} \sum_{k=0}^{n} P[X_k = j] = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{i \in S} P[X_k = j|X_0 = i] P[X_0 = i]
\]

By Cor. 6.6, \([\Pi_0 P^k]_j \to \pi(j), k \to \infty\), for all \(j \in S\) and \(\Pi_0\).

Therefore, \(\frac{1}{n+1} \sum_{k=0}^{n} [\Pi_0 P^k]_j \to \pi(j)\) [if \(a_n \to a, n \to \infty\), then \(\frac{1}{n} \sum_{k=1}^{n} a_k \to a\)]
Stationary distribution and expected return times

Recall that $T_{i,k}$ denotes the time of the $k$-th visit to state $i$.

$T_{i,k+1} = T_{i,k}$ are stopping times. Denote

$Y_k = 1$, $k = 1, 2, \ldots$

Then by the strong Markov property

$\{Y_k\}_{k=1}^\infty$ is a collection of i.i.d. random variables

$Y_k \sim$. Notice that $\sum_{k=1}^{m} Y_k = \sum_{k=1}^{m} T_{i,k+1} - T_{i,k} = \frac{1}{m} \sum_{k=1}^{m} T_{i,m+1} = \frac{1}{m} E[T_{i,m+1}]$. When $m \to \infty$, so $T_{i,m+1} \approx \frac{1}{m} E[T_{i}]$. Then

$\sum_{k=0}^{n} \mathbb{1}\{X_k = i\}$. Then $\frac{m+1}{n} \approx$
Periodic and aperiodic chains

Let \((X_n)\) be a MC with state space \(S\) and transition probability \(p(i,j)\).

**Def.** For \(i \in S\), denote \(J_i := \ldots\). We call \(d(i) := \ldots\)

\[
J_i = \begin{cases} \ldots & \text{if } d(i) = 1 \\
\ldots & \text{if } d(i) = \ldots
\end{cases}
\]

**Def.** If \(d(i) = 1\) for all \(i \in S\), then \((X_n)\) is called...
Periodic and aperiodic chains

Lemma 7.2 If $P$ is the transition matrix for an irreducible Markov chain, then for all states $i, j$

Proof. Fix $i \in S$

(1) If $m, n \in J_i$, then

(2) Let $d = d(i)$. Then (definition of $d(i)$)

Take $j \neq i$.

(3) $P$ irreducible $\Rightarrow \exists m, n$ s.t. $p_m(i, j) > 0$, $p_n(j, i) > 0$.

$\Rightarrow p_{m+n}(i, i) > 0 \Rightarrow (2)$

(4) If $e \in J_j$, then $p_e(j, j) > 0$ and thus

$\Rightarrow \Rightarrow \Rightarrow$

$\Rightarrow d$ is a common divisor of $J_j$ $\Rightarrow$

(5) Swap $i$ and $j$: $\exists q_2 \in \mathbb{N}$ s.t. $d(i) = q_2 d(j)$ (4) $\Rightarrow d(i) = d(j)$
Example 7.3 Let $G = (V, E)$ be a finite connected graph.

- SRW on $G$ is irreducible (all vertices have the same period) — we call the common period the period of MC.
- For any $i \sim j$, $p(i, j) > 0$, $p(j, i) > 0$, so $p_2(i, i) > 0$, $2 \in \mathbb{Z}_i$
  \[ d(i) \leq 2 \]
- Period is 2 iff:
  \[ V = V_1 \cup V_2, \quad E \subseteq (V_1 \times V_2 \cup V_2 \times V_1) \]
  \[ V = \mathbb{Z}, \quad V_1 = \text{even numbers} \]
  \[ V_2 = \text{odd numbers} \]
Irreducible aperiodic Markov chains

**Theorem 7.4**  Let $P$ be a transition matrix for a finite-state, irreducible, aperiodic Markov chain. Then there exists a unique stationary distribution $\pi$, $\pi = \pi P$, and for any initial probability distribution $\nu$

$$\lim_{n \to \infty} \nu P^n = \pi$$

**Proof.** (1) By PF theorem, enough to show that there exists $n_0 > 0$ s.t. $\forall i,j$.

- Fix $i,j \in S$

(2) $d(i) = 1$ (aperiodic) $\Rightarrow \exists M_i$ s.t. $J_i$ contains all $n \geq M_i$ $\Rightarrow p_n(i,i) > 0$

(3) irreducible $\Rightarrow \exists m_{ij}$ s.t. $p_{m_{ij}}(i,j) > 0$

(2)+(3):

$$\text{Take } n_0 = \max_{i,j} (M_i + m_{ij}) \Rightarrow$$
Reducible Markov chains

Not irreducible MC = reducible MC

Def 7.5 Let \((X_n)\) be a MC with state space \(S\). We say that states \(i\) and \(j\), denoted if there exists \(n,m \in \mathbb{N}_0\) s.t. and

\[
\begin{align*}
    i &\rightarrow j, \quad j \rightarrow k, \quad k \rightarrow i
\end{align*}
\]

Lemma 7.6 Relation \(\leftrightarrow\) on \(S\) is an equivalence relation.

(reflexivity, \(i \leftrightarrow i\)) \(p_0(i,i) = 1\), so \(i \leftrightarrow i\)

(symmetry, \(i \leftrightarrow j \Rightarrow j \leftrightarrow i\)) Follows from Def 7.5

(transitivity, \(i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k\)) \(i \leftrightarrow j: p_n(i,j) > 0, p_m(j,i) > 0\)

\(j \leftrightarrow k: p_n'(j,k) > 0, p_m'(k,j) > 0\). Then

\[
\begin{align*}
    p &\in (0,1)
\end{align*}
\]
Communication classes

Equivalence relation $\leftrightarrow$ splits the state space into communication classes (sets of states that communicate with each other).

MC is irreducible iff it consists of one communication class

Class properties: [proof as in Prop 4.8, Prop. 7.2]

- transience or recurrence: either all states in one class are transient (class) or all are recurrent (class)
- periodicity: all states in one class have the same period
Communication classes

Suppose \( i \) and \( j \) belong to different classes.

- If \( p(i\mid j) > 0 \), then for all \( n \in \mathbb{N} \) (otherwise \( i \leftrightarrow j \)).
- If \( p(i\mid j) > 0 \) and \( p_n(j\mid i) = 0 \) for all \( n \in \mathbb{N} \), then\( p_i(\{ X_n = i \text{ for infinitely many } n \} \leq 1 \), and thus \( i \) is transient.
- Therefore, if \( i \) and \( j \) belong to different classes and \( i \) is recurrent, then (once in a recurrent class, MC stays there forever)

If we split the state space into communication classes, with \( \mathcal{R}_c \) denoting recurrent classes, then the transition matrix has the following form.
General form of transition matrix with finite $S$

\[
P = \begin{bmatrix}
P_1 & 0 & 0 & \cdots & 0 \\
0 & P_2 & 0 & \cdots & 0 \\
0 & 0 & P_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & P_r \\
S & Q & & & \\
\end{bmatrix}
\]

$P_e$ submatrix for the recurrent class $R_e$

$P_e$ is a stochastic matrix, we can consider it as a Markov chain on $R_e$

$[S|Q]$ transition probabilities starting from transient states.

- If $P_e$ is aperiodic, then $P_e^n \rightarrow \left[ \begin{array}{c} \pi^{(c)} \\ \vdots \\ \pi^{(u)} \end{array} \right]$, $n \rightarrow \infty$

- What about transient states?

- What if $P_e$ is not aperiodic?
Transient states

Suppose there exists one transient class \( T \).

- If \( S = 0 \), then \( T \) is recurrent.
- If \( S \neq 0 \), then \( Q \) is substochastic, i.e., \( \exists i \in T \) s.t. \( \sum_{j \in T} Q_{ij} < 1 \).

If \( Q \) is substochastic, then for all eigenvalues \( \lambda \) of \( Q \) \( |\lambda| < 1 \Rightarrow Q^n \to 0, \; n \to \infty \), i.e., for \( i, j \in T \) \( \mathbb{P}(X_n = j) \to 0, \; n \to \infty \).

\[ I + Q + Q^2 + \cdots = I + VDV^{-1} + VD^2V^{-1} + \cdots = V(I + D + D^2 + \cdots)V^{-1} \]

For \( i, j \in T \),
\[ E_i \left[ \sum_{n=0}^{\infty} 1_{X_n = j} \right] = \]
Transient states

Conclusion: if TCS is a transient class, then \( \forall i, j \in T \)

\[
\lim_{n \to \infty} P_i \left[ X_n = j \right] = \]

\[
E \left[ \sum_{k=0}^{n} \mathbb{1}_{\{X_n = j\}} \right] = \text{expected number of visits to } j \text{ starting from } i
\]

Example 8.1

\[
Q = \begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0
\end{bmatrix}
\]

\[
(I - Q)^{-1} = \begin{bmatrix}
\frac{3}{2} & 1 & \frac{1}{2} & 0 & 0 \\
1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\
\frac{1}{2} & 1 & \frac{3}{2} & 0 & 0 \\
1 & \frac{3}{2} & 1 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 1 & \frac{3}{2} & 0
\end{bmatrix}
\]

Expected number of visits to 2 starting from 0 is 1

Expected number of steps before absorption starting from 0 is \( \frac{3}{2} + 1 + \frac{1}{2} = 3 \)
Transient states

Recall, First step analysis for the mean hitting time

\[ g_i = E_i [ \tau_A ] = \begin{cases} 0, & i \in A \\ 1 + \sum_{j \in S} p(i, j) g_j, & i \notin A \end{cases} \]

\[ \tau_A = \sum_{n=0}^{\infty} 1_{\{X_n \notin A\}} \]

Instead of adding 1 for each step, add 1 only when \( X_n \) visits \( j \):

Denote \( S \setminus A =: T \), and for \( i, j \in T \)

Then \( \text{FSA} \)

\[ g_{ij} = \begin{cases} \text{if } i \in A \\ \end{cases} \]

\[ g_{ij} = \]

\[ G = [ g_{ij} ] \]
**Transient states**

Starting from $T_1$, in which class will $(X_n)$ end up?

Collapse each $R_e$ into one state $r_e$, keep transient states $t_e$, $T = \{t_e\}$, $(\tilde{X}_n)$ new MC on the reduced state space, and transition matrix $\tilde{P}$, with $\tilde{s}(t_i, r_j) = \lambda(t_i, r_j)$

Denote $\tilde{A} = [\lambda(t_i, r_j)]$ with

Then $\tilde{A} = \begin{bmatrix} \tilde{r}_1 & \tilde{r}_2 & \tilde{r}_3 & \cdots & \tilde{r}_n \end{bmatrix}$
Transient states

Example 8.2

What is the probability that starting from a transient state $i$ we end up in a recurrent state $j$?

Use $\tilde{A} =$ (nothing to collapse in this case)

\[
\tilde{A} = \begin{bmatrix} \frac{3}{4} & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}
\]

Expected transit times from $i$ to $j$ (think about $j$ as absorbing)