Today: Stationary distribution

- Homework 1 is due on Friday, January 14, 11:59 PM
SSRW on $\mathbb{Z}^d$, $d \in \{1, 2, 3\}$
Simple symmetric random walk on $\mathbb{Z}^2$

$Y_n = \text{projection of } X_n \text{ on } y=x$

$Z_n = \text{projection of } X_n \text{ on } y=-x$

$X_n = (i,j) \iff Y_n = i+j, \ Z_n = j-i$

$X_n = (0,0) \iff Y_n = 0, \ Z_n = 0$

Let $(Y_n)$ and $(Z_n)$ be two independent SSRW on $\mathbb{Z}$

Define $\tilde{X}_n = Y_n - Z_n$

Then $(\tilde{X}_n)$ is a

$\tilde{p}_n(0,0) = p_n(0,0)p_n(0,0) \sim \sum_{n=0}^{\infty} \tilde{p}_n(0,0) \sim$
Markov processes

Let \((X_n)\) be a Markov chain with initial distribution \(\lambda\) and transition matrix \(P\).

- Distribution of \(X_n\): \(\lambda P^n\)
- First step analysis:
  - absorption probabilities (gambler’s ruin)
  - mean hitting times (two consecutive heads)
- Class structure: recurrence / transience
  - criteria
  - SSRW on \(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3\)
- Irreducibility
Long-run behavior of Markov chains

Denote by $\pi_n$ the distribution of $X_n$, i.e.,

$$\pi_n = (P[X_n = 1], P[X_n = 2], \ldots, P[X_n = 15])$$

$\pi_n = \pi_0 P^n$ (follows from the Chapman-Kolmogorov eqs.)

What happens with $\pi_n$ as $n \to \infty$?

for a stochastic matrix $P$

Examples:

① $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

② $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

③ $P^n = \begin{cases} P & n \text{ odd} \\ I & n \text{ even} \end{cases}$

$\pi_n = \pi_0 P^n$
Stationary distribution

Def 6.1 Let \((X_n)_{n \geq 0}\) be a Markov chain with state space \(S\) and transition matrix \(P\). A vector \(\pi = (\pi(i))_{i \in S}\) is called a stationary distribution if for all \(i \in S\),

\[ (*) \]

If \(\pi\) is the stationary distribution and \(\pi_0 = \pi\), then \(\forall n\)

In order to find the stationary distribution we have to solve the linear system \((*)\):

- \(\pi\) is the left eigenvector of \(P\) with e.v. 1
Stationary distribution

Q1: Existence of the stationary distribution
Q2: Uniqueness of the stationary distribution
Q3: Convergence to the stationary distribution

Examples 6.3. (1) $S = \mathbb{Z}$, $p(i, i+1) = 1 \ \forall i \in \mathbb{Z}$ (deterministic). Then $\forall i$, so st. distr.

(2) $S = \{1, 2, 3, 4\}$, $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Then $\pi = \ldots$ and $\pi'$ are both stationary distributions.

(3) SSRW on $\mathbb{L} = \{1, 2, 3\}$ if $X_0 = 1$, then

$\mathbb{P}(X_{n+1} \in \{1, 3\}) = 0$

$\mathbb{P}(X_{2n} \in \{1, 3\}) = 1$

$\pi = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$
**General 2-state Markov chain**

\[
P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, \quad p, q \in [0,1]
\]

\[
\text{det}(P - \lambda I) = (1-p-\lambda)(1-q-\lambda) - pq = \lambda^2 + \lambda (p+q-2) + 1-p-q = 0
\]

\[\Rightarrow \text{eigenvalues} \]

\[
P - I = \begin{bmatrix} -p & p \\ q & -q \end{bmatrix}
\]

\[
P_e \{ [[1,0]], [[0,1]], [[1,0]], [[0,1]] \} =
\]

**Case 1:**

\[
p, q \in \{0,1\}
\]

\[
\pi = \pi = \pi = \pi =
\]

\[
P^n = P^n = P^n = P^{2n+1}, \quad P^{2n}
\]
General 2-state Markov chain

Case 2: \( p \in (0, 1) \) or \( q \in (0, 1) \)

\[
\begin{align*}
\begin{cases}
- \pi(1)p + \pi(2)q = 0 \\ 
\pi(1) + \pi(2) = 1
\end{cases} \Rightarrow \\
\pi(1) = \frac{-q}{p+q}, \quad \pi(2) = \frac{p}{p+q}
\end{align*}
\]

\[
(x, y) \begin{pmatrix} q \\ p \end{pmatrix} = (0, 0) \Rightarrow \\
Q^{-1} = \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ 1 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} q & p \\ p+q & p+q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

\[
P = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \Rightarrow P^n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}
\]

\[
\lim_{n \to \infty} P^n = \begin{pmatrix} 1 & \frac{p}{p+q} \\ 1 & -\frac{q}{p+q} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q & p \\ p+q & p+q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q & p \\ p+q & p+q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
\lim_{n \to \infty} \pi_n = \pi \text{ regardless of initial distribution.}
\]
General Markov chain with finite state space

Let \((X_n)\) be a MC with finite state space \(S\).

Suppose that \(\pi = P\pi\), \(P = QDQ^{-1}\) such that

\[
Q = \begin{bmatrix} \ldots \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} \ldots \end{bmatrix}, \quad D = \begin{bmatrix} \ldots \end{bmatrix},
\]

Then \(\lim_{n \to \infty} P^n = \lim_{n \to \infty} QD^nQ^{-1} = \begin{bmatrix} \ldots \end{bmatrix}\begin{bmatrix} \ldots \end{bmatrix}\begin{bmatrix} \ldots \end{bmatrix} = \begin{bmatrix} \ldots \end{bmatrix}\)

Enough to have the following: (use Jordan normal form)

1) \(1\) is a simple eigenvalue \((1\) is always an eigenvalue since \((P1)_i = \sum_j \pi(j) = 1\), so \(P1 = 1\), \(1 = \begin{pmatrix} \ldots \end{pmatrix}\) is an e.v.\)

2) There is a left eigenvector of \(1\) with all nonnegative entries

3) If \(\lambda\) is an eigenvalue of \(P\) and \(\lambda = 1\), then \(|\lambda| < 1\)
Perron-Frobenius theorem

Theorem 6.5 Let $M$ be an $N \times N$ matrix all of whose entries are strictly positive. Then

Moreover, eigenspace contains a vector with

Finally, let $P$ be a stochastic matrix with all strictly positive entries. Then, therefore $1$ is the PF eigenvalue:

with (left) eigenvector $\pi$ with

If $(X_n)$ is a MC with transition matrix $P$, then

(Enough to have $P$ s.t. has strictly positive entries for some)