Today: Hitting times. First step analysis.
Stopping times

- Homework 1 is due on Friday, January 14, 11:59 PM
Expected hitting times

Let \((X_n)_{n \geq 0}\) be a Markov chain with transition probabilities \(p(i,j)\) and state space \(S\).

Notation: \(P_i[\{Y\}] = P[Y|X_0 = i]\), \(E_i[\{Y\}] = E[Y|X_0 = i]\)

Let \(A \subseteq S\), \(\tau_A := \min\{n \geq 0 : X_n \in A\}\)

Q: How long (on average) does it take to reach \(A\)?

Compute \(E_i[\tau_A] =\)

By definition, \(E_i[\{Y\}] = \sum_{k=1}^{\infty} k P[Y=k|X_0 = i]\) \((Y \in \{0,1,2,\ldots\})\)

First step analysis (conditioning on the first step)

\(g(i) = E_i[\tau_A] =\)
Expected hitting times

If $i \in A$, then $g(i) = 0$. Suppose $i \notin A$.

Then

$$P[\tau_A = k \mid X_1 = j, X_0 = i] = P[ X_0 \notin A, X_1 \notin A, \ldots, X_{k-1} \in A, X_k \in A \mid X_1 = j, X_0 = i ]$$

$$= P[ X_0 \notin A, X_1 \notin A, \ldots, X_{k-2} \in A, X_{k-1} \in A \mid X_0 = j ]$$

$$= P[ \tau_A = k-1 \mid X_0 = j ]$$

Compute the expectation

$$g(i) = \sum_{j \in S} E[\tau_A \mid X_1 = j, X_0 = i] \ P[X_1 = j \mid X_0 = i]$$
**Expected hitting times**

**Conclusion:**

\[
\begin{cases}
  g(i) = 1 + \sum_{j \in S} p(i,j) g(j) & \text{if } i \notin A \\
  g(i) = 0 & \text{if } i \in A
\end{cases}
\]

Example 3.2 On average how many times do we need to toss a coin to get two consecutive heads?

Denote by \( X_n \in \{0, 1, 2\} \),

\[
P = \begin{bmatrix}
  0 & 1 & 2 \\
  0.5 & 0 & 0.5 \\
  0.5 & 0 & 0.5
\end{bmatrix}
\]

\( g(2) = 0 \quad g(1) = \quad g(0) = \)

\( g(2) = 0 \quad g(1) = \quad g(0) = \)

Starting from state 0 it takes on average 6 tosses to reach state 2.
**Stopping Times**

**Def 3.3** Let \((X_n)_{n \geq 0}\) be a discrete time stochastic process. A stopping time is a such that for each \(n\) the event \(\{T=n\}\) depends only on

**Examples** \(T_1 = \min\{ n \geq 0 : X_n = i \}\) is a stopping time

\[ \{T_1 = n\} = \]

\(T_2 = \max\{ n \geq 0 : X_n = i \}\) is not a stopping time

\[ \{T_2 = n\} = \]

Recall Markov property: If \((X_n)\) is Markov\((\lambda, P)\), then conditional on \(X_m = \ell\), the process \((X_{\min})_{n \in \mathbb{N}}\) is Markov\((\delta, P)\) independent of \(X_0, X_1, \ldots, X_m\)
**Strong Markov property**

**Proposition 3.6** Let \((X_n)\) be a time-homogeneous Markov chain with state space \(S\) and transition probabilities \(p_{ij}\).

Let \(T\) be a stopping time, \(\ell \in S\) and \(\mathbb{P}[X_T = \ell] > 0\).

Then, conditional on \(X_T = \ell\), \((X_{T+n})_{n \geq 0}\) is a time-homogeneous independent of \(X_0, \ldots, X_T\). In other words, if \(A\) is an event that depends only on \(X_0, X_1, \ldots, X_T\) and \(\mathbb{P}[\mathcal{A} \cap \{X_T = \ell\}] > 0\), then for all \(n \geq 0\) and all \(i_0, i_1, \ldots, i_n \in S\),

\[
\mathbb{P}[X_{T+1} = i_1, X_{T+2} = i_2, \ldots, X_{T+n} = i_n | \mathcal{A} \cap \{X_T = \ell\}] = \\
\mathbb{P}[X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n | \mathcal{A} \cap \{X_0 = i_0\}]
\]

**Proof.** Use the partition \(\{\{T=m\}\}_{m=0}^{\infty}\) and apply Markov property (see the notes).
Classification of states: recurrence and transience

Let \((X_n)\) be a Markov chain with state space \(S\).

**Def 4.1** A state \(i \in S\) is called recurrent if

A state \(i \in S\) is called transient if

**Remark**

Let \(T_{i,k} = \) time \(X_n\) (starting from \(i\)) visits state \(i\) \(k^{th}\) time

\[ T_{i,1} = 0, \quad T_{i,k+1} = \]

Then, for \(k \geq 2\), \(T_{i,k}\) are stopping times. Indeed,

\[ \{T_{i,2} = m\} = \]

\[ \{T_{i,k} = m\} = \bigcup_{l=k-2}^{m-1} \{T_{i,k-1} = l, T_{i,k} = m\} = \bigcup_{l=k-2}^{m-1} \{T_{i,k-1} = l, X_{e+1} \neq i, \ldots, X_{m-1} \neq i, X_m = i\} \]

\(^\uparrow\) depends on \(X_0, \ldots, X_e\).
Classification of states: recurrence and transience

Denote $T_i := T_i, 2 = \Gamma_i := \Gamma_i$.

**Theorem 4.2**

Let $i \in S$. Then

1. $i$ is recurrent $\iff$ $\langle \rangle$
2. $i$ is transient $\iff$ $\langle \rangle$

**Proof. Step 1:** By the Strong Markov property

\[ P_i[T_{i, k+1} < \infty | T_{i, k} < \infty] = P_i[T_{i, k+1} < \infty] = \]

**Step 2:** Denote $N_i := \langle \rangle$ # times $(X_n)$ visits state $i$

\[ \forall \ k \geq 1, \ \{N_i \geq k\} = \langle \rangle, \ \text{so} \ P_i[N_i \geq k] = P_i[T_{i, k} < \infty] = \]
Classification of states: recurrence and transience

Thus

\[ \mathbb{E}_i[N_i] = \sum_{k=1}^{\infty} P_i[N_i \geq k] = \]

\[ \mathbb{E}_i[N_i] = \mathbb{E}_i[\sum_{n=0}^{\infty} 1_{X_n = i}] = \sum_{n=0}^{\infty} P_i[X_n = i] = \]

Since \( r_i \in [0,1] \), \( \sum_{k=0}^{\infty} r_i^k = \infty \iff \sum_{k=0}^{\infty} r_i^k < \infty \iff \)

Step 3: \( r_i = 1 \iff \forall k \ P_i[N_i \geq k] = 1 \), i.e., \( i \) is

Step 4: \( r_i < 1 \iff P_i[N_i \geq k] = r_i^k \to 0, k \to \infty \),

so \( P_i[N_i = \infty] = 0 \), i.e., \( i \) is

\[ \sum_{n=0}^{\infty} \rho_n(i;i) = \sum_{k=0}^{\infty} r_i^k = \]

\[ \square \]
Recurrence and transience of RW

Example 4.5

Let \((X_n)\) be a random walk on \(\mathbb{Z}\), \(p(i,j) = \begin{cases} p, & j = i + 1 \\ 1-p, & j = i - 1 \\ 0, & \text{otherwise} \end{cases}\)

Fix \(i \in \mathbb{Z}\). Is \(i\) recurrent or transient?

Use the \(\sum_{n=0}^{\infty} p_n(i,i)\) criterion.

Notice that \(p_n(i,i) = 0\) if \(n\) is odd

Goal: compute \(\sum_{n=0}^{\infty} p_{2n}(i,i)\)

\[ p_{2n}(i,i) = \quad (\text{trivial for } p = 0 \text{ or } p = 1) \]

Case 1: \(p \in (0,1), p \neq \frac{1}{2}\). Then \(p(1-p) < \frac{1}{4}\)

\[ \sum_{n=0}^{\infty} p_{2n}(i,i) = \sum_{n=0}^{\infty} \binom{2n}{n} (p(1-p))^n \]

\[ \binom{2n}{n} < 4^n \Rightarrow \text{all states are} \]
Recurrence and transience of RW

Case 2: \( p = \frac{1}{2} \)

\[
\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!} \leftarrow \text{use Stirling's approximation}
\]

\[
n! \sim \sqrt{2\pi n} \cdot \frac{n^n}{e^n}
\]

\[
\binom{2n}{n} \sim \quad \sum_{n=0}^{8} \quad p_n(i,i) = \quad \Rightarrow \text{all states are}
\]