Today: Hitting times. First step analysis

- Test Homework on Gradescope
Initial distribution and transition matrix

Let \((X_n)_{n \geq 0}\) be a (time-homogeneous) Markov chain with finite state space \(S = \{s_1, s_2, \ldots, s_{151}\} (= \{1, 2, 3, \ldots, 151\})\).

Distribution of \(X_n\) is a vector \((P[X_n = 1], P[X_n = 2], \ldots, P[X_n = 151])\).

Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{151})\) be the distribution of \(X_0\), i.e.,

\[
P[X_0 = i] = \lambda_i.\]

Let \(P\) be the transition matrix of \((X_n)\).

Q: What is the distribution of \(X_n\)?

\(X_1: \ P[X_1 = j] = \)

Distribution of \(X_1\) is given by

\(X_n: \ P[X_n = j] = \sum_{i=1}^{151} P[X_n = j \mid X_0 = i] P[X_0 = i] = \)

Distribution of \(X_n\) is given by

We will say that \((X_n)\) is Markov \((\lambda, P)\).
**Markov property**  "future is independent of the past"

**Prop 2.5** Let \((X_n)\) be a time-homogeneous MC with discrete state space \(S\) and transition probabilities \(p(i,j)\). Fix \(m \in \mathbb{N}\), \(i \in S\), and suppose that \(P[X_m = i] > 0\). Then conditional on \(X_m = i\), the process \((X_m)_n \in \mathbb{N}\) is Markov with transition probabilities \(p(i,j)\), initial distribution \((0, \ldots, 0, i, 0, \ldots, 0)\) and independent of the random variables \(X_0, \ldots, X_m\), i.e., if \(A\) is an event determined by \(X_0, X_1, \ldots, X_m\) and \(P[\{X_m = i\}] > 0\) then for all \(n \geq 0\)

\[
P[X_{m+1} = i_{m+1}, \ldots, X_{m+n} = i_{m+n} | A \cap \{X_m = i\}] =
\]

**Proof.** Enough to show that

\[
P[\{X_{m+1} = i_{m+1}, \ldots, X_{m+n} = i_{m+n}, X_m = i\} \cap A] = p(i, i_{m+1}) \cdots p(i_{m+n-1}, i_{m+n}) P[A \cap \{X_m = i\}]
\]
Markov property

- Let \( A = \{ X_0=i_0, \ldots, X_m=e \} \). Then
  \[
  \mathbb{P}[X_0=i_0, \ldots, X_m=e, X_{m+1}=i_{m+1}, \ldots, X_{m+n}=i_{m+n}] =
  \]
  \[
  \mathbb{P}[X_0=i_0, \ldots, X_m=e] = \mathbb{P}[X_0=i_0] \cdot p(i_0,i_1) \cdots p(i_{m-1},e)
  \]

- Any set \( A \) determined by \( X_0, \ldots, X_m \) is a disjoint union of the events of the form \( \{ X_0=i_0, \ldots, X_m=i_m \} \).

  E.g. \( \mathbb{P}[\{X_{m+1}=i_{m+1}, \ldots, X_{m+n}=i_{m+n}\} \cap (A_1 \cup A_2) \cap \{X_m=e\}] = \) 
  
  So \((*)\) holds for any event \( A \).
Hitting times

Q1: When is the first time the process enters a certain set? For ACS, compute

Q2: For A _i^, B_ j^ \in S, A \cap B = \emptyset find the probability

Start with Q2

- trivial:

- take i \in AUB; "first step analysis":

\[ P[\tau_A < \tau_B | X_0 = i] = \]

By the Markov property

\[ P[\tau_A < \tau_B | X_0 = i, X_1 = j] = \]
Hitting times

We conclude that

$$h(i) = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \in B \end{cases} \quad (***)$$

This gives a system of linear equations + boundary conditions

$$h(i) = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \in B \end{cases} \quad (***)$$

If $S$ is finite, denote $\vec{h} = (h(1), h(2), \ldots, h(\|S\|))$. Then ($**$) becomes

Example 2.6 $(X_n)$ random walk on $\{0, 1, 2, \ldots, N\}$, not necessarily symmetric, $p(i, i+1) = q$, $p(i, i-1) = 1-q$, $q \in [0, 1]$

Let $i \in \{1, 2, \ldots, N-1\}$. Compute

$$\mathbb{P}[X_n \text{ reaches } N \text{ before } 0 \mid X_0 = i]$$
Hitting times for random walks

Denote \( A = \{N\} \), \( B = \{0\} \). Need \( \mathbb{P}[\tau_A < \tau_B \mid X_0 = i] = h(i) \)

- boundary conditions

Consider \( 0 < i < N \)
  - recall \( p(i,j) = \begin{cases} q, & j = i+1 \\ 1-q, & j = i-0 \\ 0, & \text{otherwise} \end{cases} \), so \((**\) becomes

\[
\begin{align*}
h(i) &= \sum_{j \in S} p(i,j) h(j) \\
\forall i \in \{1, \ldots, N-1\}
\end{align*}
\]

- if \( q = 0 \), then \( h(i) = \)
- if \( q = 1 \), then \( h(i) = \)
- if \( q \in (0,1) \), denote \( \Delta h(i) = h(i) - h(i-1), \Theta = \frac{1-q}{q} \)
Hitting times for random walks

\[
\begin{cases}
\Delta h(1) = \\
\Delta h(2) = \\
\Delta h(N) = \\
\vdots \\
\end{cases}
\]

Take the sum of the first \(i\) equations

LHS: \(\Delta h(1) + \Delta h(2) + \cdots + \Delta h(i) = \)

RHS:

\[
\Rightarrow \forall i \in \{2, 3, \ldots, N\} \quad h(i) = \\
h(N) = 1 = \Rightarrow \Delta h(1) = \\
\Rightarrow h(i) = \\
\text{for } i \in \{1, \ldots, N-1\} \]
Gambler's ruin

Suppose you have $100, at each game you bet $1, and you stop either when your fortune reaches $200 or when you lose everything. \[ N = 200, \ h(100) = ? \]

(fair game) If probability of winning is 0.5 \( (q = 0.5) \)
then \( \theta = \frac{0.5}{0.5} = 1 \), \( h(100) = \frac{100}{200} = \frac{1}{2} = 0.5 \)

(real gambling) If probability of winning is \( \frac{18}{38} \) \( (q = 0.474) \)
then \( h(100) = \frac{1 - \theta^{100}}{1 - \theta^{200}} = \)
**Expected hitting times**

Let \((X_n)_{n \geq 0}\) be a Markov chain with transition probabilities \(p(i,j)\) and state space \(S\).

**Notation:** \(P_i[Y] = P[Y|X_0 = i]\), \(E_i[Y] = E[Y|X_0 = i]\)

Let \(A \subseteq S\), \(\tau_A := \min \{n \geq 0 : X_n \in A\}\)

**Q1:** How long (on average) does it take to reach \(A\)?

Compute \(E_i[\tau_A] = \)

By definition, \(E_i[Y] = \sum_{k=1}^{\infty} k P[Y = k|X_0 = i]\) \((Y \in \{0,1,2,\ldots\})\)

**First step analysis** (conditioning on the first step)

\(g(i) = E_i[\tau_A] = \)
**Expected hitting times**

If \( i \in A \), then \( g(i) = 0 \). Suppose \( i \notin A \).

Then

\[
P[\tau_A = k \mid X_1 = j, X_0 = i] = P[ X_0 \notin A, X_1 \notin A, \ldots, X_{k-1} \notin A, X_k \in A \mid X_1 = j, X_0 = i ]
\]

\[
= P[ X_0 \notin A, X_1 \notin A, \ldots, X_{k-2} \notin A, X_{k-1} \in A \mid X_0 = j ]
\]

\[
= P[ \tau_A = k-1 \mid X_0 = j ]
\]

Compute the expectation

\[
g(i) = \sum_{j \in S} E[ \tau_A \mid X_1 = j, X_0 = i ] P[ X_1 = j \mid X_0 = i ]
\]

\[
= \quad \quad \quad
\]

\[
= \quad \quad \quad
\]
**Expected hitting times**

**Conclusion:**

\[
\begin{aligned}
g(i) &= 1 + \sum_{j \in S} p(i, j) g(j) \quad \text{if } i \notin A \\
g(i) &= 0 \quad \text{if } i \in A
\end{aligned}
\]

**Example 3.2**  On average how many times do we need to toss a coin to get two consecutive heads?

Denote by \( X_n \) the number of consecutive heads after \( n \)th toss.

\( X_n \in \{0, 1, 2\} \), \[
P = \begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 0 \\
2 & 0 & 0 \end{bmatrix}
\]

\[
\begin{aligned}
g(2) &= \\
g(1) &= \\
g(0) &= \\
\end{aligned}
\]

Starting from state 0 it takes on average 6 tosses to reach state 2.