Today: Martingale convergence theorem

Homework 7 is due on Friday, March 11, 11:59 PM
The martingale convergence theorem

**Theorem 26.1** Let \((X_n)_{n \geq 0}\) be a martingale, and suppose there exists \(C \geq 0\) such that \(\mathbb{P}[X_n \geq -C] = 1\) for all \(n\). Then there is a random variable \(X_\infty\) such that

\[
\mathbb{P} \left( \lim_{n \to \infty} X_n = X_\infty \right) = 1
\]

**Proof** (1) Enough to prove for \(C = 0\).

Consider \(Y_n = X_n + C\). Then \((Y_n)\) is a martingale, \(Y_n \geq 0\), and \(\lim_{n \to \infty} Y_n = Y_\infty\) if and only if \(\lim_{n \to \infty} X_n = Y_\infty - C\).

Assume that \(X_n \geq 0\)

(2) \(\mathbb{P} \left[ \max_{n \geq 0} X_n < \infty \right] = 1\)

\((X_n)\) is a nonnegative martingale, therefore by
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- Doob's Maximal inequality for any $N \in \mathbb{N}$
  
  \[ P\left[ \max_{0 \leq n \leq N} X_n \geq a \right] \leq \frac{E[X_n]}{a} = \frac{E[X_0]}{a} \]

- Take the limit $N \to \infty$ (monotonicity of $P$)
  
  \[ \lim_{N \to \infty} P\left[ \max_{0 \leq n \leq N} X_n \geq a \right] = P\left[ \max_{n \geq 0} X_n \geq a \right] \leq \frac{E[X_0]}{a} \]

- Take the limit $a \to \infty$
  
  \[ \lim_{a \to \infty} P\left[ \max_{n \geq 0} X_n \leq a \right] = P\left[ \max_{n \geq 0} X_n < \infty \right] \geq \lim_{a \to \infty} \left(1 - \frac{E[X_0]}{a}\right) = 1 \]

(3) Each trajectory $(X_n(\omega))$ has a convergent subsequence $(X_{n_k}(\omega))$, denote the limit $X_\infty(\omega)$
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(4) If \((X_n(\omega))\) is not convergent, there are infinitely many terms \(X_n(\omega)\) away from \(X_\infty(\omega)\).

If \((X_n(\omega))\) is not convergent, there are \(A, B \in \mathbb{Q}\), \(A, B \geq 0\), \(A < B\) such that there are infinitely many terms \(X_n(\omega) \geq B\) and infinitely many terms \(X_n(\omega) \leq A\).
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For any \( A, B \in \mathbb{Q}, A, B \geq 0, A < B \) denote

\[ S_1 = \min\{k : X_k \leq a\}, \quad T_n = \min\{k > S_n : X_k \geq b\}, \quad S_n = \min\{k > T_{n-1} : X_k \leq a\} \]

\((S_n, T_n)\) denotes an \((A, B)\)-upcrossing.

(5) If \((X_n(\omega))\) is not convergent, then there exist infinitely many \((A, B)\) upcrossings for some \( A < B \).
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Fix $A, B$. Denote $U_n := \max \{ k : T_k \leq n \}$, number of $(A, B)$-upcrossings before time $n$.

Denote $U = \lim_{n \to \infty} U_n \in \mathbb{N} \cup \{ \infty \}$, total number of $(A, B)$-upcrossings.

(6) $P[U < \infty] = 1$

Consider the following game:

- bet $B_j = \begin{cases} 1, & S_k < j \leq T_k \\ 0, & T_k < j \leq S_{k+1} \end{cases}$, win/lose $B_j (X_j - X_{j-1})$

$\mathbb{C} (X_0, \ldots, X_{j-1})$-measurable

Total winnings: $W_n = \sum_{j=1}^n B_j (X_j - X_{j-1})$
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- \((W_n)_{n \geq 1}\) is a martingale, therefore
  \[ E(W_n) = E(W_1) = 0 \]  
  \[ W_1 = \begin{cases} \emptyset, & \text{if } X_0 > A \\ X_1 \setminus X_0, & X_0 \leq A \end{cases} \]

- \[ W_n = \sum_{k: T_k \leq n} \sum_{s \leq j \leq T_k} 1 \cdot (X_j - X_{j-1}) = \sum_{k=1}^{n} \sum_{s \leq j \leq T_k} (X_j - X_{j-1}) + \sum_{j = S_{n+1}}^{n} (X_j - X_{j-1}) \]
  \[ \geq U_n \cdot (B - A) + X_n - X_{S_{n}} \geq U_n (B - A) - A \]
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1. \( \mathbb{E}[W_n] = 0 \geq (B - A) \mathbb{E}[U_n] - A \implies \mathbb{E}[U_n] \leq \frac{A}{B - A} < \infty \)

2. \( \lim_{n \to \infty} \mathbb{E}[U_n] = \mathbb{E}[U] < \infty \implies \mathbb{P}[U < \infty] = 1 \)

(7) For any \( A, B \in \mathbb{Q}, A, B \geq 0, A < B \)

\[ \mathbb{P}[\text{infinitely many (A,B)-upcrossings}] = 0 \]

(8) \( \mathbb{P}[\exists A, B \in \mathbb{Q}, A, B \geq 0, A < B \text{ s.t. the exists } \infty \text{-many (A,B)-upcrossings}] = 0 \)

\[ \mathbb{P}\left( \lim_{n \to \infty} X_n = X_\infty \right) = 1 \]
Example

\((X_n)_{n \geq 0} \) SSRW on \( \mathbb{Z} \), \( X_0 = 1 \). \( T = \min \{ n \geq 0 : X_n = 0 \} \)

Consider \( M_n := X_{T \wedge n} \). \( M_n \) is a nonnegative martingale. Therefore, by the Martingale convergence thm
there exists r.v. \( M_\infty \) s.t. \( \Pr \left[ \lim_{n \to \infty} M_n = M_\infty \right] = 1 \).

What is \( M_\infty \)? \( M_n(\omega) \) is eventually constant for any \( \omega \).

Since \( \{ M_n(\omega) = k, M_{n+1}(\omega) = k \} \) is not possible for any \( k \geq 1 \), \( M_\infty = 0 \) with probability 1.

Remark \( E[M_n] = E[M_\infty] = E[X_0] = 1 \), but \( M_\infty = 0 \).

In particular, \( \lim_{n \to \infty} E[M_n] \neq E[\lim_{n \to \infty} M_n] \)
Example. Polya Urns

An urn initially contains a red balls and b blue balls. At each step, draw a ball uniformly at random and return it with another ball of the same color. Denote by $X_n$ the number of red balls in the urn after $n$ turns. Then $(X_n)$ is a Markov chain (time inhomogeneous)

$$P[X_{n+1} = k+1 \mid X_n = k] = \frac{k}{n+a+b}, \quad P[X_{n+1} = k \mid X_n = k] = 1 - \frac{k}{n+a+b}$$

Long-run behavior of the process? Techniques developed for time-homogeneous MC cannot be applied.

Let $M_n := \frac{X_n}{n+a+b}$ be the fraction of red ball after $n$ turns. Then $0 \leq M_n \leq 1$, $E[|M_n|] \leq 1$.
Example. Polya Urns

Next,
\[ E[X_{n+1} | X_0, \ldots, X_n] = E[X_{n+1} | X_n] \quad (X_n \text{ is Markov}) \]

and
\[ E[X_{n+1} | X_n] = (X_{n+1}) \cdot \frac{X_n}{n+a+b} + X_n \left(1 - \frac{X_n}{n+a+b}\right) \]
\[ = \frac{X_n}{n+a+b} + X_n = X_n \cdot \frac{n+1+a+b}{n+a+b} \]

\[ E[M_{n+1} | M_0, \ldots, M_n] = E\left[\frac{X_{n+1}}{n+1+a+b} \mid X_0, \ldots, X_n\right] = \frac{X_n}{n+a+b} = M_n \]

\( (M_n) \) is a nonnegative martingale. Therefore, by the Martingale convergence theorem \( M_n \to M_\infty, n \to \infty \, \text{a.s.} \)

One can show that \( M_\infty \) has beta distribution
\[ f_{M_\infty}(x) = \frac{(a+b-1)!}{(a-1)! (b-1)!} x^{a-1} (1-x)^{b-1}, 0 < x < 1 \]