Today: Martingales.
Doob's maximal inequality

- Homework 6 is due on Friday, March 4, 11:59 PM
Martingales

Def 24.1 A discrete-time martingale is a stochastic process \((X_n)_{n \geq 0}\) which satisfies \(E[|X_n|] < \infty\) and

\[
E[X_{n+1} | X_0, \ldots, X_n] = X_n \quad \text{for all } n \geq 0
\]

Thm 24.8 (Optional sampling theorem)

Let \((X_n)_{n \geq 0}\) be a martingale, and let \(T\) be a finite stopping time. Suppose that either

1. \(T\) is bounded: \(\exists N < \infty \text{ s.t. } P[T < N] = 1\)

2. \((X_n)_{0 \leq n \leq T}\) is bounded: \(\exists B < \infty \text{ s.t. } P[|X_n| \leq B \text{ for all } n \leq T] = 1\)

Then \(E[X_T] = E[X_0]\).
Example 25.1 Let \((X_n)\) be a SSRW on \(\mathbb{Z}\) conditioned to start at \(X_0 = j\) for some \(j \in \{0, \ldots, N\}\). \((X_n)\) is a martingale.

Denote \(\tau_k := T = \) (stopping times),

We computed using the first-step analysis.

Another approach: use the optional sampling theorem.

- \((X_n)\) is a martingale
- \(0 \leq X_n \leq N\) for all \(0 \leq n \leq T\)

By the Optional sampling theorem
\(X_T\) takes two values, so \(\mathbb{E}[X_T] = \)

so \(\mathbb{P}(X_T = N) = \) , \(\mathbb{P}(X_T = 0) = \) . Finally,

\(\mathbb{P}(X_T = N) = \) , \(\mathbb{P}(X_T = 0) = \)
Example

Let $X_1, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables with $E[|X_n|] < \infty$, $E[X_n] = \mu$ for all $n$, and denote $S_n := X_1 + \cdots + X_n$ and

Then $E[|M_n|] \leq$

$E[ M_{n+1} | M_0, \ldots, M_n ] =$

$E[ M_1 | M_0 ] =$. $(M_n)$ is a martingale.

Let $T$ be a bounded stopping time for $(X_n)$ (and for $(M_n)$).

Then by the Optional sampling theorem

Therefore,
Submartingales/supermartingales

A stochastic process \((X_n)\) is called

a submartingale if \(E[X_{n+1} | X_0, \ldots, X_n] \geq X_n\) for all \(n\)

a supermartingale if \(E[X_{n+1} | X_0, \ldots, X_n] \leq X_n\) for all \(n\)

We use (sub)martingales to establish the maximal inequalities. Recall the Markov's inequality: \(\forall a > 0\)

In particular, if \((X_n)\) is a submartingale and \(X_n \geq 0\), then for any \(i \leq n\)

\[ \mathbb{P}[X_i \geq a] \leq \]

In fact a stronger statement holds.
**Doob's maximal inequality**

**Thm 25.3** Let \((X_n)\) be a non-negative submartingale. Then for any \(a>0\)

**Proof.** Let \(T:=\) a stopping time.

- \(A_k:=\{T=k\}\)
- Since \(X_n\geq 0\), \(E[X_n]\geq\)
- \(E[X_n \mid A_k]=\)
- \(E[X_n]\geq\)
- \(P[T\leq n]=\)
Doob's maximal inequality

Lemma 25.4  Let \((X_n)\) be a martingale, and let \(f: \mathbb{R} \to \mathbb{R}\) be a such that \(E[|f(X_n)|] < \infty\) for all \(n\). Then

Proof  Exercise.

Corollary 25.5  Let \((X_n)\) be a martingale, let \(r \geq 1, a, b > 0\). Then

\[(i) \quad P[\max\{X_0, \ldots, X_n\} \geq a] \leq \]

\[(ii) \quad P[\max\{X_0, \ldots, X_n\} \geq a] \leq \]

Proof. If \(r \geq 1\), then \(f(x) = |x|^r\) is a convex function.

By Lemma 25.4 \((|X_n|^r)\) is a non-negative submartingale.
Doob's maximal inequality

Fix $a > 0$. If $X_k \geq a$, then

$$\mathbb{P}\left[ \max \{X_0, \ldots, X_n \} \geq a \right] \leq \frac{1}{a} \mathbb{E}[e^{\frac{S_n}{\sqrt{n}}}]$$

Therefore,

The second inequality is proven using a similar argument.

Example 25.6 Let $X_1, X_2, \ldots$ be i.i.d. symmetric Bernoulli random variables. $(S_n)$ is a martingale.

Take (ii) in Corollary 25.5 with $b = a = \alpha$, so that

$$\mathbb{P}\left[ \max \{S_0, \ldots, S_n \} \geq \alpha \sqrt{n} \right] \leq \mathbb{E}[e^{\frac{S_n}{\sqrt{n}}}]$$

Now $\mathbb{E}[e^{\frac{S_n}{\sqrt{n}}}] = $