Today: Long-run behaviour of continuous time MC
Martingales. Conditional expectation

- Homework 6 is due on Friday, March 4, 11:59 PM
Convergence to the stationary distribution

The exact analog of the convergence theorems for discrete time MC (Cor. 11.1, Thm 11.3, Thm 12.1)

**Thm 22.8** Let \((X_t)\) be an irreducible, continuous time MC with transition rates \(q(i,j)\). Then TFAE:

1. All states are positive recurrent
2. Some state is positive recurrent
3. The chain is non-explosive and there exists a stationary distribution \(\pi\).

Moreover, when these conditions hold, the stationary distribution is given by \(\pi(j) = \frac{1}{\mathbb{E}_j[T_j]}\), where \(T_j\) is the return time to \(j\);

\[ \lim_{t \to \infty} P_t(i,j) = \pi(j) \] for any states \(i,j\).
Convergence to the stationary distribution

Remark: There is no issue with periodicity: if \( p_t(i, j) > 0 \) for some \( t > 0 \), then \( p_t(i, j) > 0 \) for all \( t > 0 \).

Example: M/M/1 queue is positive recurrent if \( \lambda < \mu \)
null recurrent if \( \lambda = \mu \)
transient if \( \lambda > \mu \)

M/M/\infty queue is always positive recurrent

Example: \[ \Theta_j = \frac{\lambda \cdot \lambda \cdot \ldots \cdot \lambda \cdot i \cdot \mu_0}{\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_j} = \left( \frac{\lambda}{\mu} \right)^j \frac{1}{2^j} \] If \( \frac{\lambda}{\mu} \in (1, 2) \), then \( \sum_{j=0}^{\infty} \Theta_j < \infty \) but the explosion occurs.
Martingales
Motivating example

Consider a game: bet 1 dollar and toss a coin.

\[ B_i = \begin{cases} 1, & \text{if you win the } i\text{-th toss} \\ -1, & \text{if you lose the } i\text{-th toss} \end{cases} \]

Let \( X_n \) be your total winning after \( n \) tosses

\[ X_n = B_1 + B_2 + \ldots + B_n \text{ (SSRW on } \mathbb{Z}, \ X_0 = 0) \]

Then for any \( n \in \mathbb{N} \)

\[ E(X_n) = \sum_{i=1}^{n} E(B_i) = 0 \text{ (fair game)} \]

Suppose that you observed \( n \) tosses. What can you say about the expected winnings at time \( n+1 \) given that you know the trajectory of \( X \) up to time \( n \)?
Motivating example

For a SSRW on $\mathbb{Z}$ the answer is trivial:

$$E[X_{n+1} \mid X_0=i_0, \ldots, X_n=i_n] = E[X_{n+1} \mid X_n=i_n]$$

$$= E[X_n + B_{n+1} \mid X_n=i_n] = i_n + E[B_{n+1}] = i_n$$

Similarly, for any $m \in \mathbb{N}$

$$E[X_{n+m} \mid X_0=i_0, \ldots, X_n=i_n] = i_n + E[B_{n+1}] + \ldots + E[B_{n+m}] = i_n$$

or written in a different form

$$E[X_{n+m} - X_n \mid X_0=i_0, \ldots, X_n=i_n] = 0$$

No matter what has happened to the player's fortune so far, the expected net win or loss for any future time is always zero. We call such processes martingales.
Conditional expectation

Let $X$ be a (discrete) random variable, $X \in \mathbb{S} \subseteq \mathbb{R}$, and let $B$ be an event. Then the conditional expectation is given by

$$E[X|B] = \sum_{x \in \mathbb{S}} s \cdot P(X=x|B)$$

Often $B$ has the form $B = \{Y_1 = i_1, Y_2 = i_2, \ldots, Y_n = i_n\}$

We can group all these events into a new random variable

$$E[X|Y_1, \ldots, Y_n] = \sum_{i_1, \ldots, i_n} E[X|Y_1 = i_1, \ldots, Y_n = i_n] \cdot \mathbb{1}\{Y_1 = i_1, \ldots, Y_n = i_n\}$$

Think in the following way: Start with random variable $X$; then we are given some information in the form of random variables $Y_1, \ldots, Y_n$ that we may observe. Then $E[X|Y_1, \ldots, Y_n]$ is our best guess about the value of $X$ given $Y_1, \ldots, Y_n$ (as a function of $Y_1, \ldots, Y_n$).
Examples

Suppose that \( X = F(Y_1, \ldots, Y_n) \). \( X \) is completely determined by \( Y_1, \ldots, Y_n \). What is the best guess for the value of \( X \) given \( Y_1, \ldots, Y_n \)? \( X \) itself.

\[
E[X|Y_1, \ldots, Y_n] = E[F(Y_1, \ldots, Y_n)|Y_1, \ldots, Y_n]
\]

\[
= \sum_{i_1, \ldots, i_n} E[F(Y_1, \ldots, Y_n)|Y_1=i_1, \ldots, Y_n=i_n] \mathbb{1}_{\{y_1=i_1, \ldots, y_n=i_n\}}
\]

\[
= \sum_{i_1, \ldots, i_n} F(i_1, \ldots, i_n) \mathbb{1}_{\{y_1=i_1, \ldots, y_n=i_n\}} = F(Y_1, \ldots, Y_n) = X
\]

When \( X \) is a function of \( Y_1, \ldots, Y_n \), we say that \( X \) is measurable with respect to \( Y_1, \ldots, Y_n \).

Conclusion: If \( X \) is measurable with respect to \( Y_1, \ldots, Y_n \), then

\[
E[X|Y_1, \ldots, Y_n] = X
\]
Examples

Another extreme situation. Suppose that $X$ and $Y_1, \ldots, Y_n$ are independent. This means that any information about $Y_1, \ldots, Y_n$ should be essentially useless in determining the value of $X$, the best guess is simply $E[X]$. Indeed for any $i_1, \ldots, i_n$

$$E[X | Y_1 = i_1, \ldots, Y_n = i_n] = \sum_x x P(X = x | Y_1 = i_1, \ldots, Y_n = i_n) = \sum_x x P(X = x) = E[X]$$

Thus

$$E[X | Y_1, \ldots, Y_n] = \sum_{i_1, \ldots, i_n} E[X | Y_1 = i_1, \ldots, Y_n = i_n] \mathbb{1}_{\{Y_1 = i_1, \ldots, Y_n = i_n\}} = \sum_i E[X] \mathbb{1}_{\{Y_1 = i_1, \ldots, Y_n = i_n\}} = E[X]$$

Conclusion: If $X$ and $Y_1, \ldots, Y_n$ are independent, then

$$E[X | Y_1, \ldots, Y_n] = E[X]$$
Examples

Let $X_n$ be a SSRW on $\mathbb{Z}$. Then

$$E[X_{n+m} - X_n \mid X_0, \ldots, X_m] = \mathbb{E}\left[ X_{n+m} - X_n \mid X_0 = i_0, \ldots, X_m = i_m \right] \mathbb{1}_{\{X_0 = i_0, \ldots, X_m = i_m\}}$$

$$= 0$$

Also, $E[X_n \mid X_0, \ldots, X_n] = X_n$. Therefore,

$$E[X_{n+m} - X_n \mid X_0, \ldots, X_n] = E[X_{n+m} \mid X_0, \ldots, X_n] - E[X_n \mid X_0, \ldots, X_n]$$

$$= E[X_{n+m} \mid X_0, \ldots, X_n] - X_n = 0$$

and $E[X_{n+m} \mid X_0, \ldots, X_n] = X_n$

The best guess about our future fortune is our present fortune, the “average fairness” that defines martingales.
Properties of conditional expectation

Prop 23.5
Let $X,X'$ be random variables, and $\bar{Y} = \{Y_1, \ldots, Y_n\}$ a collection of random variables. Then the following holds:

1. For $a,b \in \mathbb{R}$, $E[aX + bX' | \bar{Y}] = a E[X | \bar{Y}] + b E[X' | \bar{Y}]$
2. If $X$ is $\bar{Y}$-measurable, then $E[X | \bar{Y}] = X$
3. If $X$ is independent of $\bar{Y}$, then $E[X | \bar{Y}] = E[X]$
4. (Tower property) Let $\bar{Z} = \{Z_1, \ldots, Z_m\}$ be another collection of random variables, and suppose that $\bar{Y}$ is $\bar{Z}$ measurable, $\bar{Y} = F(\bar{Z})$ (typical situation $\bar{Z} \supseteq \bar{Y}$). Then
   $\{Y_1, \ldots, Y_n, Y_{n+1}\}$ $E[E[X | \bar{Z}] | \bar{Y}] = E[X | \bar{Y}]$
5. (Factoring) If $Y$ is $\bar{Y}$-measurable, then $E[XY | \bar{Y}] = Y E[X | \bar{Y}]$
Properties of conditional expectation

**Cor 23.6** Particular case of the Tower property

\[ \mathbb{E}[\mathbb{E}[X|\bar{Y}]] = \mathbb{E}[X] \]

**Proof.** Take \( \bar{Z} = \emptyset \). Then \( \bar{Z} \) is independent of any collection of random variables, and \( \bar{Z} \subset \emptyset \). Thus by the tower property

\[ \mathbb{E}[\mathbb{E}[X|\bar{Y}]|\emptyset] = \mathbb{E}[X|\emptyset] = \mathbb{E}[X] \]

and

\[ \mathbb{E}[\mathbb{E}[X|\bar{Y}]|\emptyset] = \mathbb{E}[\mathbb{E}[X|\bar{Y}]] \]