Today: Poisson processes
Birth and death chains
Recurrence and transience

Homework 6 is due on Friday, March 4, 11:59 PM
Poisson processes

The jump chain of a Poisson process has a deterministic trajectory

\[ Y_n = Y_0 + n \]

By Prop. 19.2, given the trajectory the sojourn time are independent exponential r.v. with \( S_k \sim \text{Exp}(q(Y_{k-1})) \)

\[
\mathbb{P}(S_1 > s_1, \ldots, S_n > s_n) = \sum_{i_0, \ldots, i_n} \mathbb{P}(S_1 > s_1, \ldots, S_n > s_n | Y_0 = i_0, \ldots, Y_n = i_n) \mathbb{P}(i_0, \ldots, i_n)
\]

\[
= \mathbb{P}(S_1 > s_1, \ldots, S_n > s_n | Y_0 = i_0, Y_1 = i_0 + 1, \ldots, Y_n = i_0 + n) \mathbb{P}(i_0, i_0 + 1, \ldots, i_0 + n)
\]

\[
= e^{-\lambda s_1} e^{-\lambda s_2} \cdots e^{-\lambda (i_0 + n - 1) s_n} \cdot \mathbb{P}(i_0) = e^{-\lambda s_1} \cdots e^{-\lambda s_n} \mathbb{P}(i_0)
\]

Prop 20.6 If \((X_t)\) is a Poisson process, then \(S_1, S_2, \ldots\) are i.i.d with \( S \sim \text{Exp}(\lambda) \)
Poisson processes

Alternative construction of a Poisson process (with \(X_0 = 0\)):

- take a collection of i.i.d. random variables \(S_k, S_k \sim \text{Exp}(\lambda)\)
- define the jump times \(J_n = S_1 + \cdots + S_n, J_0 = 0\)
- set \(X_t = n\) for \(J_n \leq t < J_{n+1}\)

Then \(X_t\) is a Poisson process with rate \(\lambda\).

You can think about \(J_n\) as the times of some events, and \(X_t\) as the number of events that happen up to time \(t\).

Theorem 20.7 Let \((X_t)_{t \geq 0}\) be a Poisson process of rate \(\lambda, X_0 = 0\). Then for any \(s \geq 0\) the process \(\tilde{X}_t = X_{t+s} - X_s\) is a Poisson process of rate \(\lambda\), independent of \(\{X_u : 0 \leq u \leq s\}\)

No proof.
Independent increments

Given a stochastic process \((X_t)_{t \geq 0}\), its increments are random variables

\[ X_t - X_s, \quad 0 \leq s < t < \infty \]

Suppose that \((X_t)\) is a counting process, i.e.,

\[ P( X_{J_{i+1}} = i+1 \mid X_{J_i} = i ) \]

(jump times = event times,

\(X_t = \# \text{ of events that occurred up to time } t\)). Then for \(s \leq t\)

\[ X_t - X_s = \# \text{ of events that occurred on } (s, t]. \]

Cor. 20.8 If \((X_t)\) is a Poisson process with rate \(\lambda\), then for any \(0 \leq t_0 < t_1 < \ldots < t_n\) the increments \(X_{t_n} - X_{t_{n-1}}, \ldots, X_t - X_{t_0}\) are independent, and each increment \(X_t - X_s\) is a Poisson random variable with rate \(\lambda(t-s)\). These properties uniquely characterize the Poisson process.
Independent increments

**Proof.** • \( X_t - X_s = X_{s+(t-s)} - X_s \sim \text{Pois}(\lambda(t-s)) \) [by Thm 20.7]
• \( X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}} \) are independent

**Induction:** Suppose \( X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}} \) are independent

By Thm 20.7, for any \( t > 0 \) the process

\[
\hat{X}_t := X_{t+n} - X_n
\]

is independent of \( X_s \) for \( s \leq t_n \)

Therefore, \( \hat{X}_t \) is independent of \( X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}} \)

and for any \( t_{n+1} > t_n \) \( \hat{X}_{t_{n+1}} - t_n = X_{t_{n+1}} - X_t \) is independent of \( X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}} \)

• Independent increments uniquely determine the joint distribution of \( (X_{t_0}, \ldots, X_{t_n}) \) for any \( 0 \leq t_0 < \ldots < t_n < \infty \)

\[
P[X_{t_0} = i_0, \ldots, X_{t_n} = i_n] = P[X_{t_0} - X_0 = i_0, X_{t_1} - X_{t_0} = i_1 - i_0, \ldots, X_{t_n} - X_{t_{n-1}} = i_n - i_{n-1}]
\]

\[= P[X_t - X_0 = i_0] \cdots P[X_{t_n} - X_{t_{n-1}} = i_n - i_{n-1}] \]
Birth and death chains

Consider a continuous-time MC with state space

\[ S = \{0, 1, 2, \ldots\} \] and transition rates

\[ q(i, i+1) = \lambda_i \geq 0, \quad q(i, i-1) = \mu_i \geq 0, \quad q(i, j) = 0 \text{ if } j \notin \{i \pm 1\} \]

We call this process the birth and death chain.

- all \( \mu_i = 0 \) pure birth process
- all \( \lambda_i = 0 \) pure death process
- Poisson process is a pure birth process with \( \lambda_i = \lambda \)

Example: Kingman's coalescent

Pure death process with \( \mu_1 = 0, \mu_k = \binom{k}{2} \)

Tracking ancestor lines back in time

\[ E\left[ \min\{t \geq 0 : X_t = 1\} \right] \]
Kingman's coalescent

Denote $T: \min \{ t \geq 0 : X_t = 1 \}$ the time to most recent common ancestor.

Conditioned on $X_0 = N$, $T = S_1 + S_2 + \cdots + S_{N-1}$, where

$S_1 =$ time spent at state $N_1$, $S_2 =$ time spent at $N-1$, ...

$S_1 \sim \text{Exp}((\frac{N}{2}))$, $S_2 \sim \text{Exp}((\frac{N-1}{2}))$ ...

$E[T] = E[S_1 + S_2 + \cdots + S_{N-1}] = \frac{1}{\binom{N}{2}} + \frac{1}{\binom{N-1}{2}} + \cdots + \frac{1}{\binom{2}{2}}$

$= \frac{2}{N(N-1)} + \frac{2}{(N-1)(N-2)} + \cdots + \frac{2}{2} = 2 \left[ \frac{1}{N-1} - \frac{1}{N} + \frac{1}{N-2} - \frac{1}{N-1} + \cdots + \frac{1}{1} - \frac{1}{2} \right] = 2 \left[ 1 - \frac{1}{N} \right]

Denote $L =$ sum of the branch lengths. Compute

Conditioned on $X_0 = N$, $L = N \cdot S_1 + (N-1) \cdot S_2 + \cdots + 2 \cdot S_{N-1}$

$E[L] = N \cdot \frac{2}{N(N-1)} + \frac{2}{(N-1)(N-2)} + \cdots + 2 \cdot \frac{2}{2} = 2 \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N-1} \right) \approx 2 \log N$
Explosion

Let \((X_t)\) be a pure birth process with \(\lambda_i = i^2\).

Condition on \(X_0 = 1\). Denote by \(T_N\) the time to reach \(N\).

Then \(T_N = S_1 + S_2 + \ldots + S_{N-1}\) and

\[
E[T_N] = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(N-1)^2}
\]

Denote \(T := \sum_{i=1}^{\infty} S_i\) the time to reach infinity. Then

\[
E[T] = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} < \infty,
\]

and thus \(P[T < \infty] = 1\).

We call \(T\) the explosion time.

What happens after \(T\)?

We can set \(X_t = \infty\) for \(t \geq T\) (minimal)
or we can restart from another state.