Today: Introduction. Definition of Markov processes

- Test Homework on Gradescope
Def. 1.1 Let $T$ and $S$ be two sets, and let $(\Omega, P)$ be a probability space. We call a collection $(X_t)_{t \in T}$ of random variables that are all defined on the same probability space $(\Omega, P)$ and take values in $S$ a stochastic process indexed by $T$ and taking values in $S$. If $T = [0, +\infty)$, then $(X_t)_{t \geq 0}$ is called a stochastic process. If $T = \mathbb{N}$, then $(X_n)_{n \in \mathbb{N}}$ is a stochastic process.

$T$: index set (time), $S$: state space
Stochastic Processes

Motivation: Mathematical model of phenomena that

Stochastic processes have applications in many disciplines such as
biology, chemistry, ecology, neuroscience, physics, image processing,
signal processing, control theory, control theory, information theory, computer science,
cryptography and telecommunications.

Prices, sizes of populations, number of particles...
Examples

Example 1.2  \( X_1, X_2, \ldots \) are i.i.d. random variables (real-valued) defined on the same probability space. Then \( (X_n)_{n \in \mathbb{N}} \) is a discrete-time stochastic process.

Define

Example 1.3  As above, but

\( (X_n) : \)

\( (S_n) : \)
Examples

Example 1.3 (cont.)

Random walk

Example 1.4
- reflected RW
- absorbing RW
- partially reflected RW

$S_n(\omega)$
Discrete time Markov chain

Suppose that $S$ is a discrete state space, and $(X_1, ..., X_n)$ is a collection of r.v.s with values in $S$.

Q:

$$P[ X_1 = i_1, X_2 = i_2, ..., X_n = i_n ]$$
Def 1.5 Let $X_n$ be a discrete time stochastic process with state space $S$ that is finite or countably infinite. Then $X_n$ is called a \( \text{HMM} \) if for each $n \in \mathbb{N}$ and each $(i_1, \ldots, i_n) \in S^n$

*(M)*

Example 1.2 (Recall $\{X_i\}$ are i.i.d.) Suppose that $S$ is finite or countably infinite. Then (by independence) \[ P[X_n = i_n | X_1 = i_1, \ldots, X_{n-1} = i_{n-1}] \] and \[ P[X_n = i_n | X_{n-1} = i_{n-1}] = \] so (M) is satisfied.
Exemple 1.2 (cont.)  Recall \( S_n = X_1 + \cdots + X_n \), so \( X_n = \) and thus \( \mathbb{P}[S_1 = i_1, \ldots, S_n = i_n] = \) 

Check (M) 

\[ \mathbb{P}[S_n = i_n \mid S_1 = i_1, \ldots, S_{n-1} = i_{n-1}] = \]

\[ = \]

\[ = \]

\[ \mathbb{P}[S_n = i_n \mid S_{n-1} = i_n] = \]

\[ = \]

We conclude that \( S_n \) is
Transition probabilities. Time-homogeneous MC

"Distribution" of a Markov chain is completely described by the collection

Def. 1.6 A Markov chain is called time-homogeneous if for any $i,j \in S$

i.e., there exists a function $p : S \times S \rightarrow [0,1]$ s.t.

We call $P[X_n=j|X_{n-1}=i]$ the "Distribution" of a time-homogeneous MC is determined by the

and
Transition probabilities

If \( p(i,j) \) are the transition probabilities, then

\[
\sum_{j \in S} p(i,j) =
\]

Definition: If \( A \) is an \( n \times n \) matrix s.t. \( \forall i \in \{1, \ldots, n\} \)

\[
\sum_{j=1}^{n} A_{ij} = 1, \text{ then } A \text{ is called }
\]

Suppose \(|S| < \infty\) and let

\[
P = (p(i,j))_{i,j \in S}, \quad P = \begin{bmatrix}
s_1 & s_2 & s_3 & \cdots
\end{bmatrix}
\]

Then
Example 1.7

Markov chain on $S = \{0, 1, 2, \ldots, N\}$

Transition probabilities:

if $i \in \{1, 2, \ldots, N-1\}$ then $p(i,j) = \begin{cases} 
1 & \text{if } j = i+1 \\
0 & \text{otherwise} 
\end{cases}$

Reflecting random walk:

Absorbing random walk:

Partially reflecting walk: