Today: Strong Markov property
Embedded jump chain
Infinitesimal description

- Homework 5 is due on Sunday, February 20, 11:59 PM
Exponential distribution

We write \( T \sim \text{Exp}(q) \). Here are some properties of exponential distribution.

**Prop 18.3** Let \( T_1, T_2, \ldots, T_n \) be independent with \( T_j \sim \text{Exp}(q_j) \)

(a) Density \( f_{T_j}(t) = q_j e^{-q_j t} \), \( \mathbb{E}[T_j] = \frac{1}{q_j} \), \( \text{Var}[T_j] = \frac{1}{q_j^2} \)

(b) \( P(T_j > s+t \mid T_j > s) = P(T_j > t) \)

(c) \( T = \min_{j} T_j \) is exponential with \( T \sim \text{Exp}(q_1 + \cdots + q_n) \). Moreover

\[
P[T = T_j] = \frac{q_j}{q_1 + \cdots + q_n}
\]

**Proof.** (a), (b) are trivial.

(c) \( P[T > t] = \)

\[
P[T = T_j] = P[T_2 > T_1, \ldots, T_n > T_1]
\]

\[
= \frac{q_j}{q_1 + \cdots + q_n}
\]
Transition rates

- Conditioned on $X_0 = i$,
- Denote $p(i,j) = \begin{cases} p(i,i) = \end{cases}$
- Define transition rates

$q(i,j) = \begin{cases} q(i,j) \geq 0, q(i,i) = 0 \end{cases}$

$\sum_j q(i,j) = \begin{cases} \end{cases}$

$p(i,j) = \begin{cases} \end{cases}$
Poisson process

Consider a continuous-time MC on the state space \( S = \{0, 1, 2, \ldots\} \) and transition rates

\[
q(i, i+1) = , \quad q(i, j) = \text{ for } j \neq i+1
\]

We call this process the Poisson process with rate \( \lambda > 0 \).

\[ \text{Start a clock } \text{Exp}(\lambda). \]
\[ \text{When it rings, move up.} \]
\[ \text{Repeat...} \]

Proposition 18.5 Let \( (X_t)_{t \geq 0} \) be a Poisson process with rate \( \lambda \).

The for any \( t > 0 \), conditioned on \( X_0 = 0 \),

\[
P[X_t = k] =
\]
Strong Markov property

Given a MC \((X_t)_{t \geq 0}\), a stopping time \(T\) is a random variable taking values in \([0, +\infty)\) with property that

\[
\text{Thm 19.1 (Strong Markov property)} \text{ Let } (X_t)_{t \geq 0} \text{ be a continuous-time MC with state space } S \text{ and transition rates } q(i,j), i,j \in S. \text{ Let } T \text{ be a stopping time. For some } i > 0, \text{ suppose that } \Pr[X_T = i] > 0. \text{ Then, is a MC with the same}
\]

No proof. Strong Markov property can be used to develop the first step analysis.
First step analysis

For any set $A \subset S$, denote the hitting time

$$\tau_A =$$

For $A, B \subset S$, $\emptyset \neq A \cap B$, what is the probability of reaching $A$ before $B$?

Set $h(i) = \Pr_i[\tau_A < \tau_B]$. Then

$$h(i') = \sum_{j \in S} \Pr_i\left[ X_{j_i} = j \right] \Pr_i\left[ \tau_A < \tau_B \mid X_{j_i} = j \right] =$$

$$\Pr_i\left[ \tau_A < \tau_B \mid X_{j_i} = j \right] =$$
First step analysis

- Expected hitting time: $E_i[\tau_A]$

Denote $g(i) := E_i[\tau_A]$. Then

\[
g(i) = \sum_{j \in S} P_i(X_{J_1} = j) E_i[\tau_A | X_{J_1} = j] =
\]

Define $Y_t = \ldots$. Then

$\tau_A = \min \{ t \geq 0 : X_t \in A \} =

= \ldots$

$E_i[\tau_A | X_{J_1} = j] =

\Rightarrow g(i) = \sum_{j \in S} g(i,j) \left( \frac{1}{q(i)} + g(j) \right) \Rightarrow$
Denote $J_0 = 0$.

By the strong Markov property, for any $i_0, \ldots, i_n \in \mathbb{S}$

$$
\mathbb{P}(X_{J_n} = i_n, \ldots, X_{J_{i_1}} = i_{i_1}, X_0 = i_0) = \mathbb{P}(X_{J_n} = i_n \mid X_{J_{i_1}} = i_{i_1}, \ldots, X_0 = i_0) \mathbb{P}(X_{J_{i_1}} = i_{i_1}, \ldots, X_0 = i_0)
$$

Denote $Y_n := \ldots$, the embedded jump chain of $(X_t)_{t \geq 0}$. 
Embedded jump chain

The embedded jump chain \((Y_n)_{n \geq 0}\) is a discrete-time MC with state space \(S\) and transition probabilities

\[
P[Y_1 = j | Y_0 = i] = P[X_{J_1} = j | X_0 = i] =
\]

What is the distribution of the time between two consecutive jumps? Denote by \(S_k := \) the sojourn times.

We know that \(S_1 = J_1\). Denote \(\tilde{X}_t := \). Given \(Y_{k-1} = i_{k-1}\) (and \(J_{k-1} < \infty\)) by the SMP for \((X_t)\) and \(J_{k-1}\), the first jump time of \(\tilde{X}_t\) has exponential distribution \(\tilde{J}_1 =
\]

\[
P[\tilde{X}_{J_1} = i_k] = P[Y_k = i_k] =
\]

, \(S_k, Y_k\) are indep. and indep. of \(S_1, \ldots, S_{k-1}, Y_0, \ldots, Y_{k-1}\)

Prop. 19.2 Conditioned on \(Y_0, \ldots, Y_{n-1}\), the sojourn times \(S_1, \ldots, S_n\) are independent exponential random variables with
**Embedded jump chain**

Jump and hold construction

- embedded jump chain \( (Y_n) \)
  - with \( \Pr[Y_{n+1} = j \mid Y_n = i] = \frac{q_{ij}}{q_i} \)
- exponential sojourn times \( S_n \)
  - with \( S_n \sim \text{Exp}(q(Y_{n-1})) \)

- start from \( X_0 = Y_0 = i_0 \)
- wait at \( i_0 \), \( S_1 \sim \text{Exp}(q(i_0)) \)
- \( J_1 = S_1, \quad X_{J_1} = Y_1 = i_1 \)
- wait at \( i_1 \), \( S_2 \sim \text{Exp}(q(i_1)) \)
- \( J_2 = S_1 + S_2, \quad X_{J_2} = Y_2 = i_2 \)
  
  \[ ... \]
Infinitesimal description

Transition rates completely determine the Markov chain.

Q: What is the distribution of $X_t$? $P_i [X_t = j] = p_t (i,j) = ?$

Thm 19.3 Let $(X_t)_{t \geq 0}$ be a MC with state space $S$ and transition rates $q(i,j)$. Then the transition probabilities satisfy

$$p_t (i,i) =$$

Proof.

$$p_t (i,j) =$$

(1) $p_t (i,i) = P_i \{ X_t = i \}$

(2) $p_t (i,j)$

$$p_t (i,j) = P_i \{ X_t = j \} \geq$$

= 

= 
Infinitesimal description

(3) We can write (1) and (2) as

\[ p_t(i,i) = 1 - q(i) t + \xi_{ii}(t), \quad \xi_{ii}(t) = O(t) \]
\[ p_t(i,j) = q(i,j) t + \xi_{ij}(t), \quad \xi_{ij}(t) = O(t) \]

Then

\[ p_t(i,i) = 1 - q(i) t + \xi_{ii}(t) \]
\[ p_t(i,j) = q(i,j) t + \xi_{ij}(t) \]

Take the sum

\[ p_t(i,i) + \sum_{j \neq i} p_t(i,j) = \]

\[ \Rightarrow \quad \Rightarrow \quad = \]

Remark In order to identify a Markov chain it is enough to compute \( p_t(i,j) \) to first order in \( t \) as \( t \to 0 \).