Today: Continuous time Markov chains

Homework 5 is due on Sunday, February 20, 11:59 PM
Continuous time Markov chains

**Def** Let \( S \) be a finite or countable state space. A stochastic process \((X_t)_{t \geq 0}\) with state space \( S \), indexed by non-negative reals \( t \) (in the interval \([0, \infty)\), or \([a, b]\)) is called a **continuous time Markov chain** if the following two properties hold:

1. **[Markov property]** Let \( 0 \leq t_0 < t_1 < \ldots < t_n < \infty \) be a sequence of times, and let \( i_0, i_1, \ldots, i_n \in S \) be a sequence of states such that \( \mathbb{P}[X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}] > 0 \). Then

\[
\mathbb{P}[X_{t_n} = i_n \mid X_{t_0} = i_0, \ldots, X_{t_{n-1}} = i_{n-1}] = \mathbb{P}[X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}]
\]

2. **[Right-continuity]** For \( t \geq 0 \) and \( i \in S \), if \( X_t = i \) then there is \( \varepsilon > 0 \) such that \( X_s = i \) for all \( s \in [t, t+\varepsilon] \)
Continuous time Markov chains

Moreover, we say that \((X_t)\) is time homogeneous if

(3) For any \(0 \leq s < t < \infty\) and states \(i, j \in S\)

\[
P( X_t = j \mid X_s = i ) = P( X_{t-s} = j \mid X_0 = i )
\]

Recall that the evolution of a discrete time MC can be fully described by the one-step transition probabilities

\[
P( X_1 = j \mid X_0 = i ) = p(i, j)
\]

For the continuous time Markov chains we need to know the transition probabilities for infinitely many times

\[
p_t(i, j) := P( X_t = j \mid X_0 = i ), \ t > 0 \text{ (transition kernel)}
\]

(for any fixed \(i, j\), \(p_t(i, j)\) is a function of \(t\))
Typical trajectory
**Jump times**

Denote

\[ J_1 := \min \{ t \geq 0 : X_t \neq X_0 \} \]

Right-continuity: if \( X_0 = i \) then there exists \( \epsilon > 0 \) s.t.

\[ X_s = i \text{ for } s \in (0, \epsilon), \text{ therefore } P[ J_1 > 0 ] = 1 \]

Suppose we have been waiting for a jump for time \( s \), i.e., \( J_1 > s \). How much longer are we going to wait?

What is the conditional probability of \( J_1 > s + t \) given \( J_1 > s \)?

**Proposition 18.1** For \( s, t > 0 \) and \( i \in S \)

\[ P[ J_1 > s + t \mid J_1 > s ] = P[ J_1 > t ] \]

**Proof.**
Jump times

Suppose \( X_0 = i \).

(1) Denote \( A_k = \{ X_{\frac{s_j}{2^k}} = i \text{ for all } j \in \{0, 1, \ldots, 2^k\} \} \)

Then \( P[J_1 > s] = P[\bigcap_{k=1}^{\infty} A_k] \)

- \( A_k \) and \( J \) are independent.

- If \( J_1 \leq s \), then \( \exists \ s' \in [0, s] \) s.t. \( X_{s'} \neq i \). Since \( X_t \) is right-continuous, there exists \( \varepsilon > 0 \) s.t. \( \forall u \in [s', s' + \varepsilon] \), \( X_u \neq i \). Then there exists \( k' \) and \( j' \) s.t. \( \frac{s_{j'}}{2^{k'}} \in [s', s' + \varepsilon] \), and thus \( A_{k'} \) does not hold. So \( \{ J_1 > s \}^C \subset \{ \bigcap_{k=1}^{\infty} A_k \}^C \)

(2) \( \forall k \in \mathbb{N} \quad A_k \subseteq A_{k+1} \)

For all \( j \in \{0, 1, \ldots, 2^k\} \), \( X_{\frac{s_j}{2^k}} = X_{\frac{s_{2j}}{2^{k+1}}} = i \) and \( 2j \in \{0, 1, \ldots, 2^{k+1}\} \).
(3) By the continuity of the probability measure

\[ P[J_i > s] = P[ \bigcap_{k=1}^{\infty} A_k] = \lim_{k \to \infty} P[A_k] \]

(4) Denote

\[ B_k = \{ X_{\frac{t_j}{2^k}} = i \text{ for all } j \in \{0, 1, \ldots, 2^k\} \} \]

\[ C_k = \{ X_{\frac{s_j}{2^k}} = i \text{ for all } j \in \{0, 1, \ldots, 2^k\} \text{ and } X_{s + \frac{t_i'}{2^k}} = i \text{ for all } j' \in \{0, 1, \ldots, 2^k\} \} \]

Then

\[ B_k \supset B_{k+1}, \quad C_k \supset C_{k+1}, \quad \text{and} \]

\[ P[J_i > t] = P[ \bigcap_{k=1}^{\infty} B_k] = \lim_{k \to \infty} P[B_k], \quad P[J_i > s+t] = P[ \bigcap_{k=1}^{\infty} C_k] = \lim_{k \to \infty} P[C_k] \]
Jump times

\( P[A_k] = \left( P \left[ X_{\frac{s}{2^k}} = i \mid X_0 = i \right]\right)^{2^k} \)

\[
P[A_k] = P \left[ X_0 = i, X_{\frac{s}{2^k}} = i, \ldots, X_{\frac{s}{2^k} - i} = i \right] = \prod P \left[ X_{\frac{s}{2^k}} = i \mid X_0 = i \right] \prod P[A_k] \]

\[
= \left( P \left[ X_{\frac{s}{2^k}} = i \mid X_0 = i \right] \right)^{2^k}
\]

Similarly \( P[B_k] = \left( P \left[ X_{\frac{s}{2^k}} = i \mid X_0 = i \right]\right)^{2^k} \) and

\[
P[C_k] = \left( P \left[ X_{\frac{s}{2^k}} = i \mid X_0 = i \right]\right)^{2^k} \left( P \left[ X_{\frac{s}{2^k}} = i \mid X_0 = i \right]\right)^{2^k}
\]

\( \forall k \quad P[C_k] = P[A_k] P[B_k] \Rightarrow \lim_{k \to \infty} P[C_k] = \lim_{k \to \infty} P[A_k] \lim_{k \to \infty} P[B_k] \)

Finally \( P[J_1 > s + t] = P[J_1 > s] P[J_1 > t] \)
Exponential distribution

\[ P[J_i > s + t \mid J_i > s] = P[J_i > t] \]

is called the memoryless property. There is a unique one-parameter family of distributions on \((0, \infty)\) that possesses the memoryless property.

**Prop. 18.2** If \( T \) is a random variable taking values in \((0, \infty)\) and if \( T \) has the memoryless property \( P[T > s + t \mid T > s] = P[T > t] \) for all \( s, t > 0 \), then \( T \) is an exponential random variable with some intensity \( q > 0 \): \( P[T > t] = e^{-qt}, t > 0, \quad (f_T(t) = q e^{-qt}) \)

**Proof.** Denote \( G(t) = P[T > t] \) and \( G(1) = e^{-q} \). Then \( G(t+s) = G(t)G(s) \)

- \( \exists n_0 \) s.t. \( G(1/n_0) > 0 \) \( \Rightarrow \) \( G(1) = (G(1/n_0))^{n_0} > 0 \) \( \Rightarrow \exists q > 0 \) s.t. \( G(1) = e^{-q} \)
- \( \forall n \in \mathbb{N} \) \( G(1/n) = e^{-q/n} \), \( \forall m \in \mathbb{Q}^+ \) \( G(m/n) = e^{-qm/n} \) \( G(t) = e^{-qt} \) for \( t \in \mathbb{Q}^+ \)
- \( G(t) \) is decreasing, so if \( (t_n), (t'_n) \in \mathbb{Q}^+, t_n \uparrow t, t'_n \uparrow t \)

\[ e^{-qt} = \lim_{n \to \infty} e^{-qt_n} \leq G(t) \leq \lim_{n \to \infty} e^{-qt'_n} = e^{-qt} \]
We write $T \sim \text{Exp}(q)$. Here are some properties of exponential distribution.

Prop 18.3 Let $T_1, T_2, \ldots, T_n$ be independent with $T_j \sim \text{Exp}(q_j)$.

(a) Density $f_{T_j}(t) = q_j e^{-q_j t}$, $E[T_j] = \frac{1}{q_j}$, $\text{Var}[T_j] = \frac{1}{q_j^2}$

(b) $P(T_j > s + t | T_j > s) = P(T_j > t)$

(c) $T = \min_{j} T_j$ is exponential with $T \sim \text{Exp}(q_1 + \cdots + q_n)$. Moreover

$$P(T = T_j) = \frac{q_j}{q_1 + \cdots + q_n}$$