1. Let \((X_t)_{t \geq 0}\) be a Markov chain, with state space \(S\), transition rates \(q(i, j)\) and \(q(i) = \sum_{j \neq i} q(i, j) > 0\), and transition kernel \(p_t(i, j) = \mathbb{P}_i(X_t = j)\) for \(t \geq 0\) and \(i, j \in S\).
   (a) For a state \(i \in S\), show that
   \[
   \int_0^\infty p_t(i, i) \, dt = \frac{1}{q(i)} \sum_{n=0}^\infty \tilde{p}_n(i, i)
   \]
   where \(\tilde{p}_n(i, j)\) denotes the \(n\)-step transition matrix for the jump chain; i.e. \(\tilde{p}_n(i, j) = \mathbb{P}_i(X_{J_n} = j)\).
   (b) Conclude that, for any state \(i \in S\),
   \[
   i \text{ is recurrent if } \int_0^\infty p_t(i, i) \, dt = \infty,
   \]
   \[
   i \text{ is transient if } \int_0^\infty p_t(i, i) \, dt < \infty.
   \]

2. Let \((X_t)_{t \geq 0}\) be a Markov chain with state space \(S\), and transition kernel \(p_t(i, j)\) for \(t \geq 0\) and \(i, j \in S\). For given states \(i, j \in S\), show that if there exists some time \(t > 0\) so that \(p_t(i, j) > 0\), then in fact \(p_t(i, j) > 0\) for all \(t > 0\). (I.e. there is never periodicity for continuous-time Markov chains.)

3. Consider a queueing system in which customers arrive according to a Poisson process of rate \(\lambda\), are served independently with \(\text{Exp}(\mu)\) service times, and there are \(k \geq 1\) servers (so up to \(k\) customers can be served at a given time). This model is called an \(\text{M/M/}k\) queue.
   (a) Model this process as a birth and death chain.
   (b) Show that the chain is recurrent if and only if \(\lambda \leq k\mu\).
   (c) Show that the chain is positive recurrent if and only if \(\lambda < k\mu\).

4. Consider a modification of the \(\text{M/M/1}\) queue in which, when there are \(n\) customers already in the queue, a new potential customer decides to join the queue with probability \(r_n \in (0, 1)\), and goes away with probability \(1 - r_n\). This queue can be modeled by a birth and death chain in which \(\lambda_n = r_n \lambda\) for some \(\lambda > 0\) and \(\mu_n = \mu\) for all \(n \geq 1\). Show that, if \(\lim_{n \to \infty} r_n = 0\), this chain is positive recurrent for any \(\mu, \lambda > 0\).