1. (5 points) Assume that \( P(A) = \frac{1}{3}, \ P(B) = \frac{1}{3}, \ P(C) = \frac{2}{3} \), and the events \( A, B \) and \( C \) are independent. Find \( P(A \cup B \cup C) \).

   A. \( \frac{4}{3} - \frac{4}{9} \)
   
   B. \( \frac{1}{3} + \frac{1}{3} + \frac{2}{3} \)
   
   C. \( \frac{4}{3} - \frac{5}{9} + \frac{2}{27} \)
   
   D. \( \frac{2}{27} \)
   
   E. None of the above

2. (5 points) If 3 different math books, 3 different history books and 2 different photography books are to be arranged on a shelf, how many arrangements are possible if books of the same subject must be next to each other?

   A. \( 3! \cdot 3! \cdot 2! \)
   
   B. \( \binom{8}{3} \cdot (3! + 3! + 2!) \)
   
   C. \( 3! \cdot (3! + 3! + 2!) \)
   
   D. * \( 3! \cdot 3! \cdot 3! \cdot 2! \)
   
   E. None of the above

3. (5 points) There are 6 urns \( U_1, \ldots, U_6 \). Each urn \( U_i \) has \( i \) red balls and 2 blue balls. We perform the following experiment. We roll a fair six-sided die. If the outcome of the roll is \( i \), we draw a ball from the urn \( U_i \) (having \( i \) red balls and 2 blue balls).

   Given that a blue ball has been drawn, determine the probability it has been drawn from the urn \( U_2 \).

   A. \( \frac{1}{\frac{3}{2} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} + \frac{2}{8}} \)
   
   B. \( \frac{1}{6} \left( \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} + \frac{2}{8} \right) \)
   
   C. \( \frac{1}{6} \)
   
   D. * \( \frac{1}{\frac{3}{5} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} \)
   
   E. None of the above
4. (5 points) In an urn, there are 6 blue balls, 2 red balls, 9 blue cubes and \( x \) red cubes. We select an object from the urn uniformly at random. Denote the events \( R = \{ \text{the selected object is red} \} \) and \( B = \{ \text{the selected object is a ball} \} \).

What is the value of \( x \) for which the events \( R \) and \( B \) are independent?

A. 5 
B. 4 
C.* 3 
D. 6 
E. None of the above 

5. (5 points) The experiment consists of rolling a fair 5-sided die and a fair 12-sided die independently, and computing the sum of the numbers that come up. What is the expected number of times that we have to repeat the experiment to observe 15 (as the sum of numbers that come up) for the first time?

A. 30 
B. 15 
C.* 20 
D. 25 
E. None of the above 

6. (5 points) Let \( X \) be a random variable with uniform distribution on the interval \([0, 2] \), \( X \sim \text{Unif}([0, 2]) \). Compute \( E(X^2) \).

A. 2 
B.* \( \frac{4}{3} \) 
C. \( \frac{5}{3} \) 
D. 3 
E. None of the above 

7. (5 points) The probability density function of the random variable \( X \) is given by 

\[
 f(x) = \begin{cases} 
    \frac{7}{5} + ax^2, & \text{if } 0 \leq x \leq 1, \\
    0, & \text{otherwise}, 
\end{cases}
\]

where \( a \in \mathbb{R} \) is an unknown constant. Find \( a \).

A. 1.4 
B. 1.2 
C. \(-1.4\) 
D.* \(-1.2\) 
E. None of the above
8. (5 points) 8 San Diegans and 6 Angelenos meet on a certain basketball court for a 3-point shot challenge, one shot per person. Each San Diegan has a 10% chance of scoring a 3-pointer, and each Angeleno has a 20% chance of scoring, independently of each other. Find the variance of the total number of 3-pointers scored in this challenge.

A. 1.7
B.* 1.68
C. 1.55
D. 1.625
E. None of the above

9. (5 points) Let \( \Phi \) denote the CDF of the standard normal distribution. Suppose that a marketing consultant wants to estimate the proportion of the population that likes cheesecakes (the proportion is a real number between 0 and 1, inclusive). What is the minimal number of people the marketing consultant must poll to make the fraction \( \hat{p} \) of positive answers in the sample be within 0.04 of the true proportion, with probability at least 0.94? Choose the answer that is the closest.

A. \( \left( \frac{\Phi^{-1}(0.94)}{0.08} \right)^2 \)
B. \( \frac{\Phi^{-1}(0.97)}{0.08} \)
C.* \( \left( \frac{\Phi^{-1}(0.97)}{0.08} \right)^2 \)
D. \( \frac{\Phi^{-1}(0.94)}{0.08} \)
E. None of the above

10. (5 points) 6 fair coins are rolled 320 times. Which of the following is the closest to the probability that we get 6 heads for at least 2 times. (Do not consider continuity correction.)

A. \( \Phi \left( \frac{3}{\sqrt{64}} \right) \)
B.* \( 1 - 6e^{-5} \)
C. \( 1 - \Phi \left( \frac{3}{\sqrt{64}} \right) \)
D. \( 6e^{-5} \)
E. None of the above
11. (5 points) Suppose $X \sim \text{Exp}(1)$. Let $Y = -X$. Compute the moment generating function of $Y$.

A. $M_X(t) = \begin{cases} -\frac{1}{1-t}, & t < -1, \\ \infty, & t \geq -1. \end{cases}$

B. $M_X(t) = \begin{cases} \frac{1}{1-t}, & t < 1, \\ \infty, & t \geq 1. \end{cases}$

C. $M_X(t) = \begin{cases} \frac{1}{t-1}, & t > 1, \\ \infty, & t \leq 1. \end{cases}$

D.* $M_X(t) = \begin{cases} \frac{1}{1+t}, & t > -1, \\ \infty, & t \leq -1. \end{cases}$

E. None of the above

12. (5 points) Suppose that $X \sim \text{Ber} \left( \frac{1}{2} \right)$ and $Y \sim \text{Poisson}(1)$ are two independent random variables. Let $Z = X + Y$. Which of the following is the moment generating function of $Z$?

A. $\frac{1}{2} + \frac{1}{2} e^t + e^{t-1}$

B. $e^t \left( \frac{1}{2} + e^{-t} \right)$

C. $e^{t\left(\frac{1}{2}+1\right)}$

D.* $\left( \frac{1}{2} + \frac{1}{2} e^t \right) e^{t-1}$

E. None of the above

13. (5 points) Suppose that a random variable $X$ has PDF

$$f(x) = \frac{1}{2} e^{-|x|}.$$ 

Compute the moment generating function of $X$.

A.* $M_X(t) = \begin{cases} \frac{1}{1-t^2}, & |t| < 1 \\ \infty, & |t| \geq 1 \end{cases}$

B. $M_X(t) = \begin{cases} \frac{1}{2} \cdot \frac{1}{1-t}, & t < 1 \\ \infty, & t \geq 1 \end{cases}$

C. $M_X(t) = \begin{cases} \frac{1}{1-|t|}, & |t| < 1 \\ \infty, & |t| \geq 1 \end{cases}$

D. $M_X(t) = \begin{cases} \frac{1}{(1-t)^2}, & t < 1 \\ \infty, & t \geq 1 \end{cases}$

E. None of the above
14. (5 points) The joint probability mass function of the discrete random variables $X$ and $Y$ is given by the following table

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0</td>
<td>0.2</td>
</tr>
</tbody>
</table>

In this table, 2, 4 and 6 are the values of $X$, 1, 2 and 3 are the values of $Y$, and the numbers between 0 and 1 are the probabilities $P(X = k, Y = l)$. The random variables $X$ and $Y$ are:

A. Positively correlated
B. * Uncorrelated
C. Negatively correlated
D. Independent
E. None of the above

15. (5 points) Let $R$ be the rectangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$, $(2,1)$ and $(2,0)$ (including the interior). Suppose that $P = (X,Y)$ is a point chosen uniformly at random inside of $R$. Compute $E(X^2Y)$.

A. 4
B. $\frac{9}{8}$
C. * $\frac{2}{3}$
D. $\frac{1}{2}$
E. None of the above

16. (5 points) Suppose $X$ and $Y$ are independent random variables with $E(X) = 2$, $E(Y) = -1$, $\text{Var}(X) = 2$ and $\text{Var}(Y) = 4$. Compute $\text{Var}(-2X + Y + 2)$.

A. 2
B. 14
C. * 12
D. 8
E. None of the above