Name (last, first):  

Student ID:  

☐ No calculators, phones, or other electronic devices are allowed.
1. (20 points) You have an urn that initially contains 6 red balls, 2 black balls and 1 green ball. On the first step, you choose one ball uniformly at random from the urn, look at its color, and then return it back to the urn together with one more ball of the same color (e.g., if you pick a red ball, then you put it back to the urn together with another red ball). Then on the second step you choose a ball uniformly at random from the urn (note that on the second step the urn contains the additional ball).

What is the probability that on the second step you choose a red ball?
2. (20 points) You have two urns. The first urn has 3 red balls, 2 blue balls and 2 green balls. The second urn has 2 red balls and 4 blue balls. You choose one of the urns at random (with equal probability), and then sample one ball from that urn. The ball that you picked is blue. What is the probability that the ball was picked from the first urn?
3. (20 points) Let $X$ and $Y$ be independent random variables uniformly distributed on the interval $[0, 1]$, i.e., $X \sim \mathcal{U}[0, 1]$, $Y \sim \mathcal{U}[0, 1]$.

(a) Compute the moment generating function of the sum $X + Y$.

(b) Show that for any $t \in \mathbb{R}$

$$
(e^t - 1)^2 = e^{2t} - 2e^t + 1 = \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k - \sum_{k=2}^{\infty} \frac{2}{k!} t^k.
$$

(c) Use the results of (a) and (b) to compute $E((X + Y)^n)$, moments of the sum, for any $n \in \mathbb{N}$. 

4. (20 points) Suppose that $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(q)$ are independent random variables. Find the probability $P(X < Y)$. 
5. (20 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with $X_1 \sim \text{Unif}[0, 1]$. Let $Y = \min(X_1, \ldots, X_n)$. Find the CDF $F_Y$ and density $f_Y$. 
6. (20 points) Suppose that we choose a number $N$ uniformly at random from the set \{0, \ldots, 4999\}. Let $X$ denote the sum of its digits. For example, if $N = 123$, then $X = 1+2+3 = 6$. Determine $E(X)$. 
7. (20 points) Let $T$ be the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$ (including the interior). Suppose that $P = (X, Y)$ is a point chosen uniformly at random inside of $T$.

(a) What is the joint density function of $(X, Y)$? Use this to compute $\text{Cov}(X, Y)$.

(b) Determine if $X$ and $Y$ are independent.
8. (20 points) Let $X$ and $Y$ be a pair of jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} \ c(x + 3y), & 0 \leq x, y \leq 1, \\ \ 0, & \text{otherwise}, \end{cases}$$

with an unknown parameter $c > 0$.

(a) (5 points) Determine the value of $c > 0$.

(b) (10 points) Compute the marginal densities of $X$ and $Y$.

(c) (5 points) Determine if random variables $X$ and $Y$ are independent.