Today: Random variables

Next: ASV 3.2

Week 3:

- homework 3 (due Friday, October 21)
- midterm 1: Wednesday, October 19
Random variables

$(\Omega, \mathcal{F}, P)$ - probability space

A (measurable*) function $X: \Omega \rightarrow \mathbb{R}$ is called a random variable.

Def Let $X$ be a random variable. The probability distribution of $X$ is the collection of probabilities

$P(X \in B)$ for all $B \subset \mathbb{R}$
Probability distribution

If \((\Omega, \mathcal{F}, P)\) is a probability space, and \(X: \Omega \to \mathbb{R}\) is a random variable, we can define a probability measure \(\mu_x\) on \(\mathbb{R}\) given, for any \(A \subset \mathbb{R}\), by

\[
\mu_x(A) = P(X \in A) = P(\{w : X(w) \in A\})
\]

We call \(\mu_x\) the probability distribution (or law) of \(X\).

5) Toss a fair coin 4 times.

\[\Omega = \{(X_1, X_2, X_3, X_4) \in \{H, T\}^4\}\]

\[P = \text{uniform on } \Omega\]

\[P((X_1, X_2, X_3, X_4)) = \]

If \(A \subset \mathbb{R}\) does not contain one of these numbers, then

Enough to know
Toss a fair coin 4 times. Let $X =$ number of tails.

$X \in \{0, 1, 2, 3, 4\}$

$p_X(k) = P(X = k)$

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_X(k)$</td>
<td></td>
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</table>

More generally, if $X =$

$p_X(k) = P(X = k) =$

We call this distribution
4) Choose point \( \omega \) from unit disk uniformly at random.

\[
X(\omega) = \text{dist}(0, \omega)
\]

\[
P(X \leq r) = \begin{cases} 
0, & r < 0 \\
r^2, & 0 \leq r \leq 1 \\
1, & r > 1 
\end{cases}
\]

What can we say about \( P(X \in A) \) for other sets \( A \subseteq \mathbb{R}^2 \)?

Take \( A = (0.3, 0.4] \).
Discrete and continuous random variables

**Discrete**: There are finitely (or countably) many possible outcomes \( \{k_1, k_2, k_3, \ldots \} \) for \( X \).

\( \mu_X \) is described by the

\[
\mu_X(k) = \]

In this case, by the laws of probability.

**Continuous**: For any real number \( t \) \( \in \mathbb{R} \),

\( \mu_X \) is captured by understanding as a function of

For example,
Cumulative Distribution Function (CDF)

For any random variable $X$, define

$$F_X(r) = P(X \leq r)$$

Example: $X \sim \text{Bin}(3, \frac{1}{2})$

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_X(k)$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{1}{8}$</td>
</tr>
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</table>
Properties of the CDF

\( F_X(r) = P(X \leq r) \)

1. Monotone increasing:

2. \( \lim_{r \to -\infty} F_X(r) = 0 \), \( \lim_{r \to +\infty} F_X(r) = 1 \)

3. The function \( F_X \) is right-continuous:

\( \lim_{t \to r^+} F_X(t) = F_X(r) \)

Corollary: If \( X \) is a continuous random variable, \( F_X \) is a

Example: Shoot an arrow at a circular target of radius 1 (choose point from unit disk uniformly at random)

\[ F_X(r) = \begin{cases} 
0, & r \leq 0 \\
1 - (1-r^2), & 0 < r \leq 1 \\
1, & r \geq 1 
\end{cases} \]
Cumulative distribution function (CDF)

Summary: For any random variable $X$, $F_X(r) = P(X \leq r)$

1. Monotone increasing: $s \leq t \Rightarrow F_X(s) \leq F_X(t)$
2. $\lim_{r \to -\infty} F_X(r) = 0$, $\lim_{r \to +\infty} F_X(r) = 1$
3. Right-continuous: $\lim_{t \to r^+} F_X(t) = F_X(r)$

Discrete random variable
Finite or countable set of values with $t_1, t_2, \ldots$, $P(X = t_j) > 0$ and $\sum_j P(X = t_j) = 1$

Continuous random variable
For each real number $t$, $P(X = t) = 0$
Because (1) and (3) this implies that $F_X$ is continuous
no jumps
**Densities (PDF)**

Some continuous random variables have probability densities. This is the infinitesimal version of the probability mass function.

<table>
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<tr>
<th>$X$ discrete, $X \in {t_1, t_2, \ldots }$</th>
<th>$X$ continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_x(t) = P(X = t)$</td>
<td>$P(X = t) = 0$ for all $t \in \mathbb{R}$</td>
</tr>
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**Density mass function**
Example: Shoot an arrow at a circular target of radius 1. 

$X = \text{distance from center}$ 

$$F_X(r) = \begin{cases} 
0, & r \leq 0 \\
\frac{r^2}{2}, & 0 \leq r \leq 1 \\
1, & r \geq 1 
\end{cases}$$
PDF: existence

Thm: If $F_x$ is continuous and (piecewise) differentiable, then $X$ has density

Proof: Follows from FTC

Example: Let $X$ = random number chosen uniformly on $[0, 1]$. We have seen that in this case $P(X \in [s, t]) = t - s$, $0 \leq s \leq t \leq 1$.

\[
F_x(r) = \begin{cases} 
\frac{r^2}{2} & 0 \leq r \leq t \\
1 - \frac{r^2}{2} & t < r 
\end{cases}
\]

$f_x(r) = \begin{cases} 
0 & r < 0 \\
1 & r > t 
\end{cases}$
Example Let \( f(t) = \begin{cases} c \sqrt{1-t^2}, & 1 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad c > 0 \)

Q: (When) Is \( f(t) \) a PDF of some random variable?

- \( f \geq 0 \)
- \( \int_{-\infty}^{\infty} f(t) \, dt = 1 \)

\( f \) is a PDF
Your car is in a minor accident. The damage repair cost is a random number between 100 and 1500 dollars. Your insurance deductible is 500 dollars.

\[ Z = \text{your out of pocket expenses} \]

Question: The random variable \( Z \) is

(a) continuous
(b) discrete
(c) neither
(d) both