Today: Conditional probability.

Independence

Next: ASV 3.1

Week 2:

- homework 2 (due Wednesday, October 12)
- survey on Canvas Quizzes (due Friday, October 7)
Law of total probability

Let $B_1, B_2, \ldots, B_n$ be a partition of $\Omega$ (i.e., $B_i$ are disjoint, $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$, $P(B_i) > 0$).

Then for every event $A$:

$$P(A) = P(\overline{A} B_1 \cup A B_2 \cup \cdots \cup A B_n)$$

Example 90% of coins are fair, 9% are biased to come up heads 60% of times, 1% are biased to come up heads 80%. You find a coin on the street.

**How likely is it to come up heads?**
Law of total probability

Define $A = \{\text{coin comes up heads}\}, \ B_1 = \{\text{coin is fair}\}$

$B_2 = \{\text{coin is 60\% biased}\} \quad B_3 = \{\text{coin is 80\% biased}\}$

- $B_1, B_2, B_3$ form a
- $P(A | B_1) = \quad P(A | B_2) = \quad P(A | B_3) = $ 

Then using the law of total probability

$P(A) = $ 

Another question: In the same setting, you find a coin and toss it. It comes up heads. How likely is it that this coin is 80\% biased (heavily biased)?
Important remark

We know that $P(A|B_3)=0.8$.

What can we say about $P(B_3|A)$?

Generally speaking,

Example: According to Forbes, there are 2668 billionaires in the world, 2357 of them are men.

Example: Prosecutor's fallacy:

$E = \{ \text{evidence on the defendant} \}$

$I = \{ \text{defendant is innocent} \}$
Bayes' Rule (relation between $P(A|B)$ and $P(B|A)$)

$$P(B|A) =$$

This formula is often used with the law of total probability. Let $B_1, B_2, \ldots, B_n$ be a partition of the sample space. Then for any event $A$ with $P(A) > 0$

$$P(B_k|A) =$$

Example. $A = \{\text{coin comes up heads}\} , B_1 = \{\text{coin is fair}\}$  

$B_2 = \{\text{coin is 60\% biased}\} , B_3 = \{\text{coin is 80\% biased}\}$

Given: $P(B_1) = , P(B_2) = , P(B_3) = , P(A|B_1) = , P(A|B_2) = , P(A|B_3) =$

We have computed that $P(A) = \sum_{i=1}^{3} P(B_i)P(A|B_i) = 0.512$

Then
Bayes' rule

Example

Suppose that a certain test (e.g., virus X test) is 99% accurate (1% false positives, 1% false negatives). 0.25% of the population have this virus.

You test positive. What is the probability you have this virus?

(a) 99%
(b) 20%
(c) 1%
(d) 0.3%
(e) not enough information
Even though only 1% of individuals in $V^c$ get (false) positive test results, it is still 4 times more people than 99% of individuals in $V$ that test positive.

Posterior probabilities are highly sensitive to prior inputs!
The Monty Hall Problem

You play the following game. There are three doors. Each door hides a prize. Behind one door there is a car, behind two other doors - goats. You choose one door. The host opens one of the doors you did not choose, revealing a goat. You are now given a possibility to either stick with your original choice, or switch to the other closed door.

Should you switch? (a) Yes (b) No (c) Doesn't matter
The Monty Hall Problem

Let's call the door you choose #1. In this case Monty will open door #2 or door #3. Suppose Monty opens #2.

\[ B_i = \{ \text{the car is behind door } #i \} \]

\[ A = \{ \text{Monty opens door #2} \} \]

We want to know

\[ P(B_3 | A) = \]
Independence

We have seen how knowing that event B occurred may change the probability of event A, \( P(A) \) vs. \( P(A|B) \)

What if we have two events A and B that have nothing to do with each other?

A and B have nothing to do with each other is not the same as A and B being disjoint!

Example: Flip a coin 3 times. \( A = \{ \text{the first coin is heads} \} \)

\[ A = \{ \text{HHH, HHT, HTH, HTT} \} \]

\( B = \{ \text{the second coin is tails} \} \)

\[ B = \{ \text{TTH, TTT, HTH, HTT} \} \]

The first toss has no influence on the second toss
Independence

**Def** Two events $A$ and $B$ are (statistically) independent if

**Example**

An urn has 4 red and 6 blue balls.

Two balls are sampled.

$A = \{1^{st} \text{ ball is red}\}$

$B = \{2^{nd} \text{ ball is blue}\}$

Are $A$ and $B$ independent?

(a) Yes

(b) No

(c) Not enough information
Independence

Example

An urn has 4 red and 6 blue balls.

Two balls are sampled.

\[ A = \{ \text{1st ball is red} \} \quad B = \{ \text{2nd ball is blue} \} \]

Are \( A \) and \( B \) independent?

1) choose balls with replacement

\[ P(A) = \quad P(A \cap B) = \]

\[ P(B) = \]

2) choose balls without replacement

\[ P(A) = \quad P(A \cap B) = \]

\[ P(B) = \quad A \text{ and } B \text{ are } \]
Independence

A and B are independent

if and only if A and B^c are independent

Proof. (⇒)

(⇐)
Independence for more than two events

Def. A collection of events \( A_1, A_2, \ldots, A_n \) is mutually independent if for any subcollection of events \( A_{i_1}, A_{i_2}, \ldots, A_{i_k} \) with \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \)

Example For \( n=3 \), \( A, B, C \) are mutually independent

\[
\begin{align*}
\Pr(A \cap B) &= \\
\Pr(A \cap C) &= \\
\Pr(B \cap C) &= 
\end{align*}
\]

Suppose that \( A \) and \( B \) are independent. \( A \) and \( C \) are independent, \( B \) and \( C \) are independent.
Important example

Toss a coin

A = \{ \text{there is exactly one tails in the first two tosses} \}

B = \{ \text{there is exactly one tails in the last two tosses} \}

C = \{ \text{there is exactly one tails in the first and last tosses} \}

\[
P(A) = P(A \cap B) .
\]