Today: Conditional probability. Independence
Next: ASV 3.1

Week 2:

- homework 2 (due Wednesday, October 12)
- survey on Canvas Quizzes (due Friday, October 7)
Law of total probability

Let $B_1, B_2, \ldots, B_n$ be a partition of $\Omega$
(i.e., $B_i$ are disjoint, $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$, $P(B_i) > 0$).

Then for every event $A$:

$$P(A) = P(A \cap B_1 \cup A \cap B_2 \cup \ldots \cup A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$= \sum_{i=1}^{n} P(B_i)P(A \mid B_i)$$

Example 90% of coins are fair, 9% are biased to come up heads 60% of times, 1% are biased to come up heads 80%. You find a coin on the street.

How likely is it to come up heads?
**Law of total probability**

Define $A = \{\text{coin comes up heads}\}$, $B_1 = \{\text{coin is fair}\}$, $B_2 = \{\text{coin is 60\% biased}\}$, $B_3 = \{\text{coin is 80\% biased}\}$.

- $B_1, B_2, B_3$ form a partition and
  
  $P(B_1) = 0.9$, $P(B_2) = 0.09$, $P(B_3) = 0.01$

- $P(A|B_1) = 0.5$, $P(A|B_2) = 0.6$, $P(A|B_3) = 0.8$

Then using the law of total probability

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + P(B_3) \cdot P(A|B_3)$$

$$= 0.9 \cdot 0.5 + 0.09 \cdot 0.6 + 0.01 \cdot 0.8 = 0.512$$

Another question: In the same setting, you find a coin and toss it. It comes up heads. How likely is it that this coin is 80\% biased (heavily biased)?
We know that \( P(A \mid B_3) = 0.8 \).

What can we say about \( P(B_3 \mid A) \)?

Generally speaking, \( P(A \mid B) \neq P(B \mid A) \).

**Example** According to Forbes, there are 2668 billionaires in the world, 2357 of them are men.

\[
P(M \mid B) = \frac{2357}{2668} \approx 88\% \neq P(B \mid M)
\]

**Example** Prosecutor's fallacy:

\( E = \{ \text{evidence on the defendant} \} \)
\( I = \{ \text{defendant is innocent} \} \)

Usually \( P(E \mid I) \) is small, but this does not imply that \( P(I \mid E) \) is small.
Bayes’ Rule (relation between \( P(A|B) \) and \( P(B|A) \))

\[
P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B) \cdot P(B)}{P(A)}
\]

This formula is often used with the law of total probability.

Let \( B_1, B_2, \ldots, B_n \) be a partition of the sample space. Then for any event \( A \) with \( P(A) > 0 \)

\[
P(B_k|A) = \frac{P(A \cap B_k)}{P(A)} = \frac{P(A|B_k) \cdot P(B_k)}{\sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)}
\]

Example. \( A = \{ \text{coin comes up heads} \} \), \( B_1 = \{ \text{coin is fair} \} \)

\( B_2 = \{ \text{coin is 60\% biased} \} \), \( B_3 = \{ \text{coin is 80\% biased} \} \)

Given: \( P(B_1) = 0.9 \), \( P(B_2) = 0.09 \), \( P(B_3) = 0.01 \), \( P(A|B_1) = 0.5 \), \( P(A|B_2) = 0.6 \), \( P(A|B_3) = 0.8 \)

We have computed that \( P(A) = \sum_{i=1}^{3} P(B_i)P(A|B_i) = 0.512 \)

Then \( P(B_3|A) = \frac{0.8 \cdot 0.01}{0.512} \approx 0.0156 \)
Bayes' rule

Example

Suppose that a certain test (e.g., virus X test) is 99% accurate (1% false positives, 1% false negatives). 0.25% of the population have this virus.

You test positive. What is the probability you have this virus?

(a) 99%

\( T = \{ \text{positive test} \} \)

\( P(T \cap V^c) = 0.01 = P(T^c \mid V) \)

(b) 20%

\( V = \{ \text{has virus} \} \)

\( P(V) = 0.0025 \)

(c) 1%

\( \Omega = V \cup V^c \)

(d) 0.25%

\( P(T \cap V) = 1 - P(T^c \mid V) = 0.99 \)

(e) not enough information

\[
P(V \mid T) = \frac{P(T \mid V) \cdot P(V)}{P(T)} = \frac{0.99 \cdot 0.0025}{0.99 \cdot 0.0025 + 0.01 \cdot 0.9975} \approx 0.1987 \approx 20%\]
What if

\( \Omega \)

\( V \)

\( V^c \)

Even though only 1\% of individuals in \( V^c \) get (false) positive test results, it is still 4 times more people than 99\% of individuals in \( V \) that test positive.

Posterior probabilities are highly sensitive to prior inputs!
The Monty Hall Problem

You play the following game. There are three doors. Each door hides a prize. Behind one door there is a car, behind two other doors - goats.

You choose one door. The host opens one of the doors you did not choose, revealing a goat. You are now given a possibility to either stick with your original choice, or switch to the other closed door.

Should you switch?  (a) Yes  (b) No  (c) Doesn't matter