MATH 180A (Lecture C00)
Introduction to Probability

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Today: Conditional probability

Next: ASV 2.2

Week 2:

- homework 1 (due Tuesday, October 4)
- survey on Canvas Quizzes (due Friday, October 7)
Conditional probability

Example 1. Your friend rolls two fair dice. What is the probability the sum is 10.

\[
\Omega = \{ (i,j) : 1 \leq i, j \leq 6 \} \quad A = \{ (i,j) : i+j=10 \} = \{ (4,6), (5,5), (6,4) \}
\]

\[
P(A) = \frac{\#A}{\#\Omega} = \frac{3}{36} = \frac{1}{12}
\]

Example 2. Your friend rolls two fair dice and tells you that the sum that came up is a two digit number. What is the probability that the sum is 10?

“Updated” \(\tilde{\Omega} = \{ \text{sum has 2 digits} \} = \{ i+j=10 \} \cup \{ i+j=11 \} \cup \{ i+j=12 \}
\]

\[
= \{ (4,6), (5,5), (6,4), (5,6), (6,5), (6,6) \}
\]

\[
\tilde{P} = \frac{\#A}{\#\tilde{\Omega}} = \frac{3}{6} = \frac{1}{2}
\]

(sometimes need to update A)
### Conditional probability

If we know that the event $B$ happened

- keep the same $\Omega$ and $\mathcal{F}$
- define new probability $\tilde{P}$ on $(\Omega, \mathcal{F})$ that takes into account the additional information
  - “probability of everything that is not part of $B$ is set to zero”
  - if $C \cap B$, then $\tilde{P}(C) = \frac{P(C)}{P(B)}$

**Def.** Let $B \in \mathcal{F}$ satisfy $P(B) > 0$. Then for all $A \in \mathcal{F}$

the conditional probability of $A$ given $B$ is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
Conditional probability

If $P(B)>0$, then the conditional probability $P(\cdot \mid B)$ satisfies all the properties of probability:

- Axioms of probability,
- Probability of the complement,
- Monotonicity,
- Inclusion-exclusion...

**Remark** If $\Omega$ is finite and $P$ is uniform, then

$$P(A \mid B) = \frac{\#A \cap B}{\#B}$$

**Example** Roll two fair dice. $A=\{\text{sum is 10}\}$, $B=\{\text{sum is a two-digit number}\}$

- $\#B = 6$, $\#A \cap B = \#A = 3$
- $P(A \mid B) = \frac{3}{6} = \frac{1}{2}$
Examples

An urn contains 4 red ball and 6 blue balls. Three balls are sampled without replacement. What is the probability that exactly two are red?

\[ A = \{ \text{exactly 2 red} \} \]

\[ \# \Omega = \binom{10}{3} = 120 \quad , \quad \# A = \binom{4}{2}\binom{6}{1} = 36 \quad , \quad P(A) = \frac{36}{120} = \frac{3}{10} \]

Suppose we know a priori that at least one red ball is sampled. What is the conditional probability of \( A \)?

\[ B = \{ \text{at least one red ball} \} \]

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\binom{4}{2}\binom{6}{1}}{\binom{6}{3}} = \frac{18}{20} = 0.36 \quad P(B) = 1 - P(B^c) = \frac{5}{6} \]

\[ P(B^c) = \frac{\binom{6}{3}\binom{4}{0}}{\binom{10}{3}} = \frac{20}{120} = \frac{1}{6} \]
Multiplication rule

By definition \( P(B \mid A) = \frac{P(A \cap B)}{P(A)} \)

\[ \Rightarrow P(A \cap B) = P(A) \cdot P(B \mid A) \leftarrow \text{multiplication rule} \]

For \( A \cap B \cap C \) : \( P(A \cap B \cap C) = P(A \cap B) \cdot P(C \mid A \cap B) = P(A) \cdot P(B \mid A) \cdot P(C \mid A \cap B) \)

Example

An urn contains 4 red balls and 6 blue balls. Two balls are sampled without replacement. What is the probability that both are red?

\( B_1 = \{1^{st} \text{ is red}\} \)

\[ P(B_1 \cap B_2) = P(B_1) \cdot P(B_2 \mid B_1) = \frac{4}{10} \cdot \frac{3}{9} = \frac{2}{15} \]

\( B_2 = \{2^{nd} \text{ is red}\} \)

\[ P(B_2) = \frac{4}{10}, \quad P(B_2 \mid B_1) = \frac{3}{9} \quad \left(\text{old way :} \quad \frac{\binom{4}{2} \binom{6}{0}}{\binom{10}{2}} \right) \]
Two-stage experiments

- perform an experiment, measure a random outcome
- perform a second experiment whose setup depends on the outcome of the first!

Example

![Diagram of two urns with balls: urn A with blue, red, and yellow balls; urn B with red and yellow balls.]

Q: What is the probability that the sampled ball is red?

\[ R = \{\text{sample red ball}\}, \ A = \{\text{choose urn A}\}, \ B = \{\text{choose urn B}\} \]

\[ P(R) = P((R \cap A) \cup (R \cap B)) \]

\[ = P(R \cap A) + P(R \cap B) \]

\[ = P(A) \cdot P(R | A) + P(B) \cdot P(R | B) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{6} + \frac{1}{5} = \frac{11}{30} \]
**Law of total probability**

Let $B_1, B_2, ..., B_n$ be a partition of $\Omega$ (i.e., $B_i$ are disjoint, $B_1 \cup B_2 \cup ... \cup B_n = \Omega$, $P(B_i) > 0$).

Then for every event $A$:

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

Example 90% of coins are fair, 9% are biased to come up heads 60% of times, 1% are biased to come up heads 80%. You find a coin on the street.

How likely is it to come up heads?