Today: Conditional probability

Next: ASV 1.3-1.4

Week 1:

- check the course website
- homework 1 (due Tuesday, October 4)
- join Piazza
Example: A fair die is rolled 5 times. What is the probability that you get one at least 3 times?

\[ A = \{ \text{at least 3 ones} \} = \]

\[ P(A) = \]

Consider \( A_3 \), exactly 3 ones. How many ways?

\[ \square \square \square \square \square \square \]

1) \[ \] 2) \[ \]
Example A fair die is rolled 5 times. What is the probability that you get at least one double?

$A = \{\text{some number comes up at least twice}\}$

$A_k = \{k \text{ comes up at least two times}\}$

$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$

We can further decompose

$A_1 = A_1^2 \cup A_1^3 \cup A_1^4 \cup A_1^5 \cup A_1^6$, $A_1^j = \{1 \text{ comes up exactly } j \text{ times}\}$

Easy solution:
Intersections and unions

Sometimes we have to take intersections into account.

Notation: \( A \cup B = \{ \text{all outcomes in either } A \text{ or } B \text{ or both} \} \)

\( A \cap B = \{ \text{all outcomes in both } A \text{ and } B \} = AB \)

\( A \cup B = A \cap B \cup A \cap B \cap B \cup A \cap B \cap \bar{B} \)

\( A \cap B \cup A \cap \bar{B} \cup \bar{A} \cap B \cup \bar{A} \cap \bar{B} \)

\( A \cup B = A \cap B \cup A \cap B \cup A \cap B \cap B \cup A \cap B \cap \bar{B} \cup A \cap \bar{B} \cap B \cup A \cap \bar{B} \cap \bar{B} \cup \bar{A} \cap B \cap B \cup \bar{A} \cap B \cap \bar{B} \cup \bar{A} \cap \bar{B} \cap B \cup \bar{A} \cap \bar{B} \cap \bar{B} \)
Principle of inclusion/exclusion

The probability of a union can be computed by adding the probabilities, then subtracting off the intersection overcounted. If you have more sets, you have to keep going and re-add back in over-subtracted pieces etc...
Principle of inclusion/exclusion

Example: Among students enrolled in MATH 180A

- 60% are Math majors
- 20% are Physics majors
- 5% are majoring in both Math and Physics

A student is chosen randomly from the class.
What is the probability that this student is neither a Math major nor a Physics major?

\[ A = \{ \text{Math} \} \]
\[ B = \{ \text{Physics} \} \]
\[ C = \{ \text{neither} \} \]
Monotonicity

If $A \subseteq B$, then

Indeed, $P(B) = \quad \geq \quad \text{max} \{P(A), P(B)\}$

In particular, $P(A \cup B) \geq \text{max} \{P(A), P(B)\}$

$P(A \cap B) \leq \text{min} \{P(A), P(B)\}$

Useful tools

- $P(A) + P(A^c) = 1$ (events and their complements)
- $A \subseteq B$ implies $P(A) \leq P(B)$ (monotonicity)
- $P(A \cup B) = P(A) + P(B) - P(AB)$ (inclusion-exclusion)
Conditional probability

Example 1. Your friend rolls two fair dice. What is the probability the sum is 10.

$$\Omega =$$

$$A =$$

$$P(A) =$$

Example 2. Your friend rolls two fair dice and tells you that the sum that came up is a two digit number. What is the probability that the sum is 10?
Conditional probability

If we know that the event $B$ happened

- keep the same $\Omega$ and $\mathcal{F}$
- define new probability $\tilde{P}$ on $(\Omega, \mathcal{F})$ that takes into account the additional information

Def. Let $B \in \mathcal{F}$ satisfy $P(B) > 0$. Then for all $A \in \mathcal{F}$,

the conditional probability of $A$ given $B$ is defined as
Conditional probability

If $P(B)>0$, then the conditional probability $P(\cdot \mid B)$ satisfies all the properties of probability: axioms of probability, probability of the complement, monotonicity, inclusion-exclusion...

Remark: If $\Omega$ is finite and $P$ is uniform, then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Example: Roll two fair dice. $A = \{\text{sum is 10}\}$, $B = \{\text{sum is a two-digit number}\}$
Examples

An urn contains 4 red ball and 6 blue balls. Three balls are sampled without replacement. What is the probability that exactly two are red?

Suppose we know a priori that at least one red ball is sampled. What is the conditional probability of A?
**Multiplication rule**

By definition \( P(B|A) = \frac{P(A \cap B)}{P(A)} \)

\[ \Rightarrow \]

For \( A \cap B \cap C \): \( P(A \cap B \cap C) = \)

**Example**

An urn contains 4 red balls and 6 blue balls. Two balls are sampled without replacement. What is the probability that both are red?