Today: Definition of probability.
Random sampling
Next: ASV 1.3-1.4

Week 1:
- check the course website
- homework 1 (due Tuesday, October 4)
- join Piazza
If Ω is finite, the uniform probability measure is defined by the following property:

for each \( \omega \in \Omega \), \( P(\{\omega\}) = \frac{1}{\#\Omega} \)

From (*) this implies that

for any event \( A \), \( P(A) = \frac{\#A}{\#\Omega} \)
Combinatorics

A collection of $n$ labelled balls $\{1, 2, 3, \ldots, n\}$ are in an urn. $k$ are taken out one by one.

Q: How many ways?

Possible scenarios:

<table>
<thead>
<tr>
<th>Replacement</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>with replacement</td>
<td>order matters</td>
</tr>
<tr>
<td>without replacement</td>
<td>order doesn't matter</td>
</tr>
</tbody>
</table>

$n = 5$, $k = 3$ (choose 3 balls)

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<tr>
<td></td>
<td>$1 \ 2 \ 1 \neq 1 \ 1 \ 2$</td>
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<td></td>
<td>$(b_1, b_2, b_3)$</td>
<td></td>
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<td>$1 \ 2 \ 3 = 3 \ 2 \ 1$</td>
</tr>
<tr>
<td></td>
<td>$(b_1, b_2, b_3), \ b_i \neq b_j$ if $i \neq j$</td>
<td>${b_1, b_2, b_3}$</td>
</tr>
</tbody>
</table>
### Combinatorics

**Sampling with replacement, order matters**

\[ \Omega = \{ (b_1, \ldots, b_k) : 1 \leq b_i \leq n \} = \{1, \ldots, n\}^k \]

**Sampling without replacement, order matters**

\[ \Omega = \{ (b_1, \ldots, b_k) : 1 \leq b_j \leq n, \ b_i \neq b_j \text{ if } i \neq j \} \]

**Sampling without replacement, order does not matter**

\[ \Omega = \{ \{b_1, \ldots, b_k\} : 1 \leq b_i \leq n, \ b_i \neq b_j \text{ if } i \neq j \} \]

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Important remark. Examples

Each of these three models leads to a uniform probability measure!
on the corresponding sample space

Example (sampling with replacement)
Toss a fair coin \( n \) times; record a statistic observing

\[ \#H \text{ vs } \#T \]

Take \( n=10 \). \( \Omega \): compute \( P(\text{odd rolls are all } H) \)

\[ \Omega = \{ (c_1, c_2, \ldots, c_{10}) : c_j \in \{H, T\} \} \quad , \quad \#\Omega = \]
Examples

Example (Sampling without replacement, order matters)

There are 6 labelled balls in an urn. 3 are removed in sequence (without replacement) and lined up in order.

Q: What is the probability that the first two are (4, 5)?

\[ \Omega = \{ (b_1, b_2, b_3) : 1 \leq b_j \leq 6, \ b_1 \neq b_2, \ b_2 \neq b_3, \ b_1 \neq b_3 \} \]
Examples

Example (Sampling without replacement, order does not matter)

An urn contains 10 balls: \( b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10} \)

Two blue, three green, five red

3 balls are chosen without replacement.

Q: Compute \( P(\text{choose 2 green and one red}) \)

\# \Omega =
Combinatorics

- Selecting \( k \) objects among \( n \), with replacement
  \[ \# \text{ ways} = n^k \]

- Selecting \( k \) objects among \( n \), without replacement (order matters)
  \[ \# \text{ ways} = n(n-1)(n-2) \ldots (n-k+1) \quad (k \leq n) \]

- Selecting \( k \) objects among \( n \), without replacement (order doesn't matter)
  \[ \# \text{ ways} = \binom{n}{k} = \frac{n(n-1)(n-2) \ldots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \]

- \# of ways to order \( n \) objects: \( n(n-1) \ldots 1 = n! \)
Warm-up exercise

There are 10 people on a committee.

How many different ways are there to select a subcommittee of 4 people?

(a) \(10 \times 10 \times 10 \times 10 = 10^4 = 10000\)

(b) \(10 \cdot 9 \cdot 8 \cdot 7 = 5040\)

(c) \(\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210\)

(d) \(\frac{10!}{4!} = 151200\)
Infinite sample space
If \( \# \Omega = \infty \), then we need a different notion of uniform probability measure.

Example A random number is chosen in \([0,1]\).

(a) What is the probability that it is \( \geq 0.6 \)

(b) What is the probability that it is \( = \frac{1}{2} \)

\((\Omega, \mathcal{F}, P) : \)

\( \Omega = \mathbb{R} \cap \mathbb{Q} \)

If \( \Omega = [a, b] \), then take
An archery target is a disk 50 cm in diameter.
A blue disk is 25 cm in diameter.
A red disk is 5 cm in diameter.

Given that you hit the target (randomly), what are the chances of hitting the blue disk? The red disk?

$\Omega = \text{target}, \ P(A) = \ F =$

General rule:
Example A fair die is rolled 5 times. What is the probability that you get one at least 3 times?

\[ A = \{ \text{at least 3 ones} \} = \]

\[ P(A) = \]

Consider \( A_3 \), exactly 3 ones. How many ways?

\[
\begin{array}{cccccc}
\_ & \_ & \_ & \_ & \_ & \_ \\
1) & 2)
\end{array}
\]
Decompositions

Example A fair die is rolled 5 times. What is the probability that you get at least one double?

A = \{ \text{some number comes up at least twice} \}

A_k = \{ k \text{ comes up at least two times} \}

A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6

We can further decompose

A_1 = A_1^2 \cup A_1^3 \cup A_1^4 \cup A_1^5 \cup A_1^6 \quad , \quad A^j_i = \{ i \text{ comes up exactly } j \text{ times} \}

Easy solution:
Sometimes we have to take intersections into account.

**Notation:**

- \( A \cup B = \{ \text{all outcomes in either } A \text{ or } B \text{ or both} \} \)
- \( A \cap B = \{ \text{all outcomes in both } A \text{ and } B \} = AB 

\[
A \cup B = A \cap B^c \cup A \cup A^c \cap B
\]
The probability of a union can be computed by adding the probabilities, then subtracting off the intersection overcounted. If you have more sets, you have to keep going and re-add back in over-subtracted pieces etc...
Principle of inclusion/exclusion

Example  Among students enrolled in MATH 180A
  60% are Math majors
  20% are Physics majors
  5% are majoring in both Math and Physics

A student is chosen randomly from the class. What is the probability that this student is neither a Math major nor a Physics major?

\[ A = \{ \text{Math} \} \]
\[ B = \{ \text{Physics} \} \]
\[ C = \{ \text{neither} \} \]