Today: Definition of probability

Next: ASV 1.2

Week 1:

- check the course website
- homework 1 (due Friday, September 30)
- join Piazza
Probability theory

The goal of probability theory is to build mathematical models of experiments with random outcomes.

Random outcome = impossible to be predicted with certainty.

1654: starting point, mathematical treatment of gambling problems (Fermat, Pascal)

1933: modern rigorous foundation of probability theory (Kolmogorov)
Warm-up problem

What is the probability that there are at least two student in this room having birthday on the same day (MM/DD)?

100 students, 365 possible birthday dates

\[ P > 99.997\% \]

Moral: Intuition may be misleading
Axioms of probability

How to construct a mathematical model of an experiment with random outcome?

Def. Probability space is the triple $(\Omega, \mathcal{F}, P)$, where
- $\Omega$ is the set of all possible outcomes of the experiment; we call it the sample space
- $\mathcal{F}$ is a collection of subsets of $\Omega$ (events)
- $P$ is a function that assigns to each event a real number and satisfies the following properties:

\begin{align*}
(i) \quad & 0 \leq P(A) \leq 1 \quad \text{for all } A \in \mathcal{F} \\
(ii) \quad & P(\emptyset) = 0, \quad P(\Omega) = 1 \\
(iii) \quad & \text{If } A_1, A_2, \ldots \text{ are disjoint events, then } P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots
\end{align*}
Examples

We call function $P$ that satisfies properties (i)-(iii) a probability measure, or simply probability.

Example 1: Tossing a coin.

$\Omega = \{H,T\}$, $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H,T\}\}$, $P(\emptyset) = 0$, $P(\Omega) = 1$

$P(\{H\}) = \alpha$, $P(\{T\}) = \beta = 1 - \alpha$

$\{H\}$ and $\{T\}$ are disjoint, therefore, by (iii)

$\alpha + \beta = P(\{H\}) + P(\{T\}) = P(\{H\} \cup \{T\}) = P(\{H,T\}) = 1 \Rightarrow \beta = 1 - \alpha$

For any $\alpha \in [0,1]$ we have a different probability measure on $\Omega$ and $\mathcal{F}$.

Fair coin: $\alpha = \frac{1}{2}$, $P(\{H\}) = P(\{T\}) = \frac{1}{2}$
**Examples**

**Example 2:** rolling a fair die

\[ \Omega = \{1, 2, 3, 4, 5, 6 \} , \quad \mathcal{F} = \{ \text{all subsets of } \Omega \} \]

\[ P(\{1\}) = P(\{2\}) = \ldots = P(\{6\}) = \frac{1}{6} \]

What about the events? Take \( A = \{2, 4, 6\} \subset \Omega \).

\[ P(A) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \]

\( B = \{2, 3, 5\} : \quad P(B) = \frac{1}{2} \)

\( C = \{3, 6\} : \quad P(C) = \frac{1}{3} \)

\[ P(A \cup B) = P(\{2, 3, 4, 5, 6\}) = \frac{5}{6} \]

\[ P(B \cap C) = P(\{3\}) = \frac{1}{6} \]

A = "even number"

B = "prime number"

C = "divisible by 3"