Today: Poisson distribution.

Expectation

Next: ASV 3.4

Week 5:

• homework 3 (due Wednesday October 26)

• Midterm 1 regrades open until October 30
**Expectation**

**Example** Toss a fair coin 1000 times, and record the sequence of outcomes 1101001000110.

Average them \( \frac{1}{1000} (1+1+0+1+0+0+\cdots) \approx \frac{1}{2} = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 \)

What if the coin is biased \( P(X_j=1)=p, P(X_j=0)=1-p \)?

Then the average (random) is approximately \( p = p \cdot 1 + (1-p) \cdot 0 \).

**Def.** Let \( X \) be a discrete random variable with possible values \( t_1, t_2, t_3, \ldots \). The expectation or expected value or mean of \( X \) is

\[
E(X) = \sum_j t_j \cdot P(X=t_j)
\]

weighted average
Q: Is the expectation $E(X)$ the value that $X$ is equal to most often?

(a) Yes, always
(b) No, not generally

Example Let $X$ be the number rolled on a fair die.

Example Let $Y$ be $\text{Ber}(p)$. 
Expectation

Example: You toss a biased coin repeatedly until the first heads. How long do you expect it to take?
Examples. Binomial

\( S_n \sim \text{Bin}(n,p) \) \((S_n = X_1 + X_2 + \ldots + X_n \text{ for } X_j \text{ independent } \Ber(p))\)

\( E(S_n) = \)
Examples. Poisson

\( X \sim \text{Poisson}(\lambda) \)

\[ E(X) = \]

Example A factory has, on average, 3 accidents per month. Estimate the probability that there will be exactly 2 accidents this month.
Examples

Toss a fair coin until tails comes up. If this is on the first toss, you win 2 dollars and stop. If heads comes up, the pot doubles and you continues. That is, if the first tails is on the k-th toss, you win $2^k$ dollars. What is your expected winnings?
Expectation of continuous random variables

$X$ discrete, $X \in \{t_1, t_2, \ldots \}$

$E(X) = \sum_k t_k \ P(X = t_k)$

$X$ continuous, $P(X = t) = 0$ for each $t \in \mathbb{R}$

with density $f_X(t)$

Example: Let $U \sim \text{Unif}([a, b])$, $f_U(t) = \begin{cases} \frac{1}{b-a}, & a \leq t \leq b \\ 0, & \text{otherwise} \end{cases}$
Example

Q: Shoot an arrow at a circular target of radius 1. What is the expected distance of the arrow from the center?

a) 1

b) \( \frac{2}{3} \)

c) \( \frac{1}{2} \)

d) \( \frac{1}{4} \)

e) 0