1. Let \( a, b, c \in \mathbb{R} \) be such that \( a < b < c \) and \((c - a)(c - b) = (b - a)^2\). Show that

\[
r := \frac{c - a}{b - a}
\]
is not a rational number.

Hint: Show that \( r \) satisfies a polynomial equation with integer coefficients.

**Solution.** Since

\[
r = \frac{c - a}{b - a},
\]
we have that

\[
c - a = r(b - a) \quad \text{and} \quad c - b = (c - a) - (b - a) = (r - 1)(b - a).
\]

Plugging the above expressions into the equation \((c - a)(c - b) = (b - a)^2\) we get

\[
(b - a)^2(r - 1)r = (b - a)^2.
\]

Since \( b - a > 0 \), the above equation implies that \( r \) satisfies the equation

\[
r^2 - r - 1 = 0.
\]

By Corollary 2.3, if \( r \) is a rational number, then \( r \in \{-1, 1\} \). Neither \( r = 1 \) nor \( r = -1 \) satisfies Equation (5), therefore we conclude that \( r \) is not a rational number. (Number \( r \) is called the golden ratio)

2. Using only Definition 9.8 prove that

\[
\lim_{n \to \infty} \log_{10}(\log_{10} n) = +\infty.
\]

Clearly indicate how you chose \( N(M) \) for any \( M > 0 \), and write explicitly \( N(2) \), \( N(5) \), \( N(10) \).

**Solution.** Fix \( M > 0 \). Then for any \( n > \lfloor 10^{10M} \rfloor \)

\[
\log_{10}(\log_{10} n) > \log_{10}(\log_{10} 10^{10M}) = M.
\]

Therefore, by Definition 9.8

\[
\lim_{n \to +\infty} \log_{10}(\log_{10} n) = +\infty
\]

with \( N(M) = \lfloor 10^{10M} \rfloor \). In particular, \( N(2) = 10^{100} \), \( N(5) = 10^{100000} \), \( N(10) = 10^{10^{10}} \). (This sequence converges to infinity very slowly)

3. Determine if the series

\[
\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}
\]
converges. Justify your answer.

**Solution.** Denote

\[
a_n := \frac{2^n n!}{n^n}.
\]
Notice that
\[ a_{n+1} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} = \frac{n^n}{2^{n+1}} = \frac{2n^n}{(n+1)^n} = \frac{2}{(1 + \frac{1}{n})^n}. \]

By the Important Example from Lecture 7,
\[ \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e. \]

By Theorem 9.6,
\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{2}{\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n} = \frac{2}{e}. \]

By the Important Example 16, \( e > 2 \), so \( 2/e < 1 \). By Theorem 14.8 (Ratio test) we conclude that the series \( \sum a_n \) converges.

4. Let \( a \in \mathbb{R} \) and let \( f : [a, +\infty) \to \mathbb{R} \) be a function such that
   (i) \( f \in C([a, +\infty)) \)
   (ii) \( \lim_{x \to +\infty} f(x) = p \in \mathbb{R} \)
Prove that \( f \) is uniformly continuous on \([a, +\infty)\).

**Solution.** Fix \( \varepsilon > 0 \).

Since \( \lim_{x \to +\infty} f(x) = p \), by the \( \varepsilon - \delta \) definition of the limit (Lecture 18) there exists \( M > a \) such that for any \( x \in (M, +\infty) \)
\[ |f(x) - p| < \frac{\varepsilon}{2}. \]

Function \( f \) is continuous on \([a, M + 1] \subset [a, +\infty)\), therefore by the Cantor-Heine Theorem (Theorem 19.2) \( f \) is uniformly continuous on \([a, M + 1]\). By definition, this means that there exists \( \delta > 0 \) such that for all \( x, y \in [a, M + 1] \)
\[ |f(x) - f(y)| < \frac{\varepsilon}{2}. \]

Now for any \( x, y \in [a, +\infty), x < y, |x - y| < \min\{\delta, 1\} \), we have
- if \( y \leq M + 1 \), then by (15) \( |f(x) - f(y)| < \varepsilon \).
- if \( y > M + 1 \), then \( x > M \) and by (14) and the triangle inequality
\[ |f(x) - f(y)| \leq |f(x) - p| + |f(y) - p| < \varepsilon. \]

We conclude that \( x, y \in [a, +\infty) \) and \( |x - y| < \min\{\delta, 1\} \) implies \( |f(x) - f(y)| < \varepsilon \). By Definition (Lecture 15) this means that \( f \) is uniformly continuous on \([a, +\infty)\).

5. Compute the derivative of the function \( f : (0, +\infty) \to \mathbb{R} \) given by
\[ f(x) = x + x^x. \]

Provide all intermediate steps.

**Solution.** First, compute the derivative on \( x^x \). For this, rewrite this function as
\[ x^x = e^{\log x^x} = e^{x \log x}. \]
Function \( x \log x \) is differentiable on \((0, +\infty)\), function \( e^x \) is differentiable on \(\mathbb{R} \), therefore by Theorem 28.4 (about the derivative of a composition)

\[
(x^x)' = (e^{x \log x})' = e^{x \log x} (x \log x)' = e^{x \log x} (\log x + 1) = x^x (\log x + 1).
\]

Therefore,

\[
f'(x) = 1 + x^x (\log x + 1).
\]

6. Prove that the inequality

\[
py^{p-1}(x - y) \leq x^p - y^p \leq px^{p-1}(x - y)
\]
holds for \(0 < y < x\) and \(p > 1\).

**Solution.** Consider function \( f(x) = x^p \). Then for any interval \([y, x] \subset (0, +\infty)\), \( f \) is continuous on \([y, x]\) and differentiable on \((y, x)\). Therefore, we can apply Lagrange’s Mean Value Theorem (Theorem 29.3), which gives that there exists a number \( \xi \in (y, x) \) such that

\[
x^p - y^p = p\xi^{p-1}(x - y).
\]

Since \( p > 1, p - 1 > 0 \), and \( y < \xi < x \), we have that

\[
y^{p-1} \leq \xi^{p-1} \leq x^{p-1}.
\]

Together with (22) this implies that

\[
py^{p-1}(x - y) \leq x^p - y^p \leq px^{p-1}(x - y).
\]

7. Let

\[
f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}, \quad f(x) = \log(\cos x).
\]

Find a polynomial \( P(x) \) such that

\[
f(x) - P(x) = o(x^3) \quad \text{as} \quad x \to 0.
\]

**Solution** By the local Taylor’s formula with the remainder in Peano’s form, \( P(x) \) is equal to the Taylor’s polynomial of degree 3 at 0. In order to determine the coefficients of \( P(x) \), compute the derivatives of \( f \)

\[
f'(x) = (\log(\cos x))' = \frac{1}{\cos x} \cdot (-\sin x) = -\frac{\sin x}{\cos x},
\]

\[
f''(x) = \left(-\frac{\sin x}{\cos x}\right)' = -\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = -\frac{1}{\cos^2 x},
\]

\[
f^{(3)}(x) = \left(-\frac{1}{\cos^2 x}\right)' = 2\frac{\sin x}{\cos^3 x}.
\]

Now

\[
f(0) = \log 1 = 0, \quad f'(0) = \tan 0 = 0, \quad f''(0) = -1, \quad f^{(3)}(0) = 0.
\]

We conclude that

\[
f(x) = -\frac{x^2}{2} + o(x^3) \quad \text{as} \quad x \to 0.
\]