1. Let \( a, b, c \in \mathbb{R} \) be such that \( a < b < c \) and \((c-a)(c-b) = (b-a)^2\). Show that

\[
\frac{c-a}{b-a} \quad (1)
\]
is not a rational number.

Hint: Show that \( r \) satisfies a polynomial equation with integer coefficients.

2. Using only Definition 9.8 prove that

\[
\lim_{n \to \infty} \log_{10}(\log_{10} n) = +\infty. \quad (2)
\]

Clearly indicate how you chose \( N(M) \) for any \( M > 0 \), and write explicitly \( N(2) \), \( N(5) \), \( N(10) \).

3. Determine if the series

\[
\sum_{n=1}^{\infty} \frac{2^n n!}{n^n} \quad (3)
\]
converges. Justify your answer.

4. Let \( a \in \mathbb{R} \) and let \( f : [a, +\infty) \to \mathbb{R} \) be a function such that

(i) \( f \in C([a, +\infty)) \)

(ii) \( \lim_{x \to +\infty} f(x) = p \in \mathbb{R} \)

Prove that \( f \) is uniformly continuous on \([a, +\infty)\).

5. Compute the derivative of the function \( f : (0, +\infty) \to \mathbb{R} \) given by

\[
f(x) = x + x^x. \quad (4)
\]

Provide all intermediate steps.

6. Prove that the inequality

\[
py^{p-1}(x - y) \leq x^p - y^p \leq px^{p-1}(x - y) \quad (5)
\]
holds for \( 0 < y < x \) and \( p > 1 \).

7. Let

\[
f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}, \quad f(x) = \log(\cos x). \quad (6)
\]

Find a polynomial \( P(x) \) such that

\[
f(x) - P(x) = o(x^3) \quad \text{as} \quad x \to 0. \quad (7)
\]