MATH 142A: Introduction to Analysis

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Today: Subsequences
> Q&A: Jan 28
Next: Ross § 11-12

Week 4:

- Homework 3 (due Sunday, January 30)
Subsequences

An = (-1)^n, n ≥ 1: -1, 1, -1, 1, -1, 1, -1, 1, ...

\( n_k = 2k - 1 \), \( (\alpha_{n_k}) = (-1, -1, -1, -1, ...) \); \( n_k = 2k \), \( (\alpha_{n_k}) = (1, 1, 1, 1, ...) \)

\( b_n = \cos\left(\frac{\pi n}{2}\right), n ≥ 1: 0, -1, 0, 1, 0, 1, 0, -1, 0, 1, ... \)

\( n_k = 2k - 1 \), \( (b_{n_k}) = (0, 0, 0, 0) \); \( n_k = 3k \), \( (b_{n_k}) = (0, 1, 0, 1, 0, 1, ...) \)

\( C_n = n, n ≥ 1: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, ... \)

\( (n_k) = (1, 2, 3, 5, 7, 11, 13, ...) \), \( (C_{n_k}) = (1, 2, 3, 5, 7, 11, 13, ...) \)

\( d_n = \cos(n), n ≥ 1: \cos(1), \cos(2), \cos(3), \cos(4), \cos(5), \cos(6), ... \)

\( (n_k) = (1, 7, 8, 9, 12, 14, 10, 12, 15, ...) \), \( (d_{n_k}) = (\cos(1), \cos(7), \cos(8), ...) \)

Def 11.1 Let \( (S_n) \) be a sequence of real numbers and let

\( 1 ≤ n_1 < n_2 < ... < n_k < ... \) be an increasing sequence of natural numbers.

Then \( (S_{n_k})_{k=1}^\infty = (S_{n_1}, S_{n_2}, S_{n_3}, ...) \) is called a subsequence of \( (S_n)_{n=1}^\infty \).
Subsequences

Thm 11.2 Let \((s_n)\) be a sequence. Let \(t \in \mathbb{R}\).

(i) There exists a (monotonic) subsequence of \((s_n)\) converging to \(t\).

\[ \iff \forall \varepsilon > 0 \text{ the set } \{n \in \mathbb{N} : |s_n - t| < \varepsilon\} \text{ is infinite} \]

Proof. \((\Rightarrow)\) Exercise.

\((\Leftarrow)\) \(\forall \varepsilon > 0 \text{ the set } \{n \in \mathbb{N} : |s_n - t| < \varepsilon\} \text{ is infinite.} \)

Case 1: the set \(\{n! : s_n = t\}\) is infinite, take \((s_{n_k})\) with \(s_{n_k} = t \forall k\).

Case 2: \(\forall \varepsilon > 0 \text{ the set } \{n : 0 < |s_n - t| < \varepsilon\} \text{ is infinite.} \)

Either (a) \(\forall \varepsilon > 0 \text{ the set } \{n : t - \varepsilon < s_n < t\} \text{ is infinite} \)

or (b) \(\forall \varepsilon > 0 \text{ the set } \{n : t < s_n < t + \varepsilon\} \text{ is infinite} \)

Consider Case 2(a). We want to construct an increasing subsequence that converges to \(t\).
Proof of Thm 11.2 (i)

Suppose that \( \forall \varepsilon > 0 \) \( \{ n : t - \varepsilon < S_n < t \} \) is infinite

1. Choose \( n_1 \) such that
\[
1 < S_{n_1} < t
\]

2. Take \( \varepsilon = t - \max \{ S_{n_1}, t - \frac{1}{2} \} \), so that \( t - \varepsilon = \max \{ S_{n_1}, t - \frac{1}{2} \} \)
\[
1 < S_{n_2} < t
\]

Choose \( n_2 \in S \) such that \( n_2 > n_1 \). Then \( \max \{ S_{n_1}, t - \frac{1}{2} \} < S_{n_2} < t \).

3. Suppose we have numbers \( n_1 < n_2 < \cdots < n_{k-1} \) such that
\[
\forall j \left( \max \{ S_{n_j-1}, t - \frac{1}{j} \} < S_{n_j} < t \right)
\]
\( \{ n : t - \varepsilon < S_n < t \} \) is infinite \( \Rightarrow \) \( \exists k \) \( n_k > n_{k-1} \) s.t. \( \max \{ S_{n_k-1}, t - \frac{1}{k} \} < S_{n_k} < t \)
\[
(S_{n_k})_{k=1}^\infty \text{ is a subsequence of } (S_n)_{n=1}^\infty \text{, and } \forall k \frac{t}{k} < S_{n_k} < t \Rightarrow \lim_{k \to \infty} S_{n_k} = t
\]
**Subsequences**

**Thm 11.2** Let \((s_n)\) be a sequence.

(ii) \((s_n)\) has a (monotonic) subsequence that diverges to \(+\infty\)

\[ \iff (s_n) \text{ is unbounded above} \]

(iii) \((s_n)\) has a (monotonic) subsequence that diverges to \(-\infty\)

\[ \iff (s_n) \text{ is unbounded below} \]

**Proof (ii) \((\Rightarrow)\)** Exercise.

\((\Leftarrow)\) Suppose that \((s_n)\) is unbounded above.

1. Let \(n_1=1\), so that \(s_{n_1}=S_1\)
2. \((s_n)\) unbounded above \(\Rightarrow T_2:=\{n: \max\{s_i,2\} < s_n\} \text{ is infinite} \)
   
   choose \(n_2 \in T_2\) s.t. \(n_2 > n_1\)
3. \(T_k:=\{n: \max\{s_{n_{k-1}}, k\} < s_n\} \text{ is infinite} \), choose \(n_k \in T \text{ s.t. } n_k > n_{k-1}\)

Then \((s_{n_k})\) is a subsequence, \(\forall k\) \(s_{n_k} > k \Rightarrow \lim_{k \to \infty} s_{n_k} = +\infty\)
**Subsequences**

**Thm 11.3** If \((s_n)\) converges, then any subsequence of \((s_n)\) converges to the same limit.

**Proof.** Let \((s_{n_k})\) be a subsequence of \((s_n)\).

1. \(\forall k \in \mathbb{N} \quad (n_k \geq k)\)
   
   Proof by induction: \(n_1 \geq 1\)

   if \(n_{k-1} \geq k-1\), then \(n_k \geq n_{k-1} + 1 \geq k\)

2. Suppose \((s_n)\) converges to \(s \in \mathbb{R}\). Fix \(\varepsilon > 0\). Then
   
   \[\exists N \in \mathbb{N} \quad \forall n > N \quad (|s_n - s| < \varepsilon)\]. But since \(\forall k \quad n_k \geq k\)

   \[\forall k \geq N \quad (n_k > N)\] and thus \(\forall n_k - s| < \varepsilon\)
Subsequences

Thm 11.4 Every sequence has a monotonic subsequence.

Proof Let \((s_n)\) be a sequence of real numbers. We say that \(s_n\) is dominant if \(\forall m > n \ (s_n > s_m)\)

Denote \(D = \{n : s_n \text{ is dominant}\}\)

Case 1: \(D\) is infinite. Take \(n_1 = \min D, \ldots, n_k = \min \{n \in D : n > n_{k-1}\}\)

Then \(n_1 < n_2 \implies s_{n_1} > s_{n_2}, \ n_{k-1} < n_k \implies s_{n_{k-1}} > s_n_k =: (s_{n_k})\) is decreasing

Case 2: \(D\) is finite. Take \(n_1 = \max D + 1\). Then \(s_{n_1}\) is not dominant

\[ \implies \exists n_2 > n_1 \text{ s.t. } s_{n_2} \geq s_{n_1}. \text{Term } s_{n_2} \text{ is not dominant} \implies \exists n_3 > n_2 \ (s_{n_3} \geq s_{n_2}) \]

If we have \(n_{k-1}\), then \(s_{n_{k-1}}\) is not dominant \(\implies \exists n_k > n_{k-1} \ (s_{n_k} \geq s_{n_{k-1}}) \implies (s_{n_k})\) is increasing.
Bolzano-Weierstrass Theorem

Thm 11.5 Every bounded sequence has a convergent subsequence.

Proof Let \((s_n)\) be a bounded sequence.

By Thm 11.4 \((s_n)\) has a monotonic subsequence \((s_{n_k})\)

Since \((s_n)\) is bounded, \((s_{n_k})\) is also bounded.

\((s_{n_k})\) is monotonic and bounded, therefore by Thm 10.2 \n
\[(s_{n_k})_{k=1}^\infty \text{ converges.}\]