Today: Continuous functions
> Q&A: February 7
Next: Ross § 18

Week 6:

- Homework 5 (due Sunday, February 13)
- Homework 3 regrades Tuesday, February 8
Functions

Def. (Function) Let $X$ and $Y$ be two sets. We say that there is a function defined on $X$ with values in $Y$, if via some rule $f$ we associate to each element $x \in X$ an (one) element $y \in Y$. We write $f : X \to Y$, $x \mapsto y$ (or $y = f(x)$). $X$ is called the domain of definition of the function, $\text{dom}(f)$, $y = f(x)$ is called the image of $x$. $f : [0,1) \to [0,1)$, $x \mapsto x^2$

Remarks 1) We consider real-valued functions ($Y \subset \mathbb{R}$) of one real variable ($X \subset \mathbb{R}$).

2) If $\text{dom}(f)$ is not specified, then it is understood that we take the natural domain: the largest subset of $\mathbb{R}$ which the function is well defined ($f(x) = \sqrt{x}$ means $\text{dom}(f) = [0, +\infty)$ $g(x) = \frac{1}{x^2 - x}$ means $\text{dom}(g) = \mathbb{R}\setminus\{0,1\}$).
Continuity of a function at a point

Intuitively: Function $f$ is continuous at point $x_0 \in \text{dom}(f)$ if $f(x)$ approaches $f(x_0)$ as $x$ approaches $x_0$.

Def 17.1 (Continuity). Let $f$ be a real-valued function, $\text{dom}(f) \subset \mathbb{R}$. Function $f$ is continuous at $x_0 \in \text{dom}(f)$ if for any sequence $(x_n)$ in $\text{dom}(f)$ converging to $x_0$, we have $\lim f(x_n) = f(x_0)$

$$\lim f(x_n) = f(\lim x_n)$$

Def 17.6 (Continuity) Let $f$ be a real-valued function. Function $f$ is continuous at $x_0 \in \text{dom}(f)$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \ (x \in \text{dom}(f) \land |x-x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon) \quad (*)$$

Remark Def 17.1 is called the sequential definition of continuity, Def 17.6 is called the $\varepsilon-\delta$ definition of continuity.
Equivalence of sequential and $\varepsilon-\delta$ definitions

**Thm 17.2.** Definitions 17.1 and 17.6 are equivalent

**Proof** ($17.1 \Rightarrow 17.6$). Suppose that (*) fails

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ (x \in \text{dom}(f) \land |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon) \quad (*)$$

This means that

$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in \text{dom}(f) \ (|x - x_0| < \delta \land |f(x) - f(x_0)| \geq \varepsilon)$$

Take $\delta = \frac{1}{n}$: $\exists x_n \in \text{dom}(f) \ (|x_n - x_0| < \frac{1}{n} \land |f(x_n) - f(x_0)| \geq \varepsilon)$

$$\Rightarrow \exists (x_n) \text{ s.t. } \lim x_n = x_0 \land \limsup |f(x_n) - f(x_0)| \geq \varepsilon, \text{ contradiction (\Leftarrow)}.$$ Let $(x_n)$ be such that $\lim x_n = x_0$. Fix $\varepsilon > 0$. By (*)

$$\exists \delta > 0 \ (x \in \text{dom}(f) \land |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)$$

$$\lim x_n = x_0 \Rightarrow \exists N \ A_n > N \ (|x_n - x_0| < \delta) \quad \text{Therefore}

A_n > N \ (x_n \in \text{dom}(f) \land |x_n - x_0| < \delta) \Rightarrow A_n > N \ (|f(x_n) - f(x_0)| < \varepsilon)$$

$$\Rightarrow \lim f(x_n) = f(x_0) \quad \blacksquare$$
Continuity on a set. Examples

**Def.** Let \( f \) be a function, and let \( S \subseteq \text{dom}(f) \). \( f \) is continuous on \( S \) if for all \( x_0 \in S \) \( f \) is continuous at \( x_0 \).

**Example 1)** \( f(x) = \frac{2x}{x^2 - 1} \) is continuous on \( \mathbb{R} \setminus \{-1, 1\} \)

**Proof.** Let \( x_0 \in \mathbb{R} \setminus \{-1, 1\} \) and let \( (x_n) \) be such that \( \forall n \ x_n \neq \{-1, 1\} \) and \( \lim x_n = x_0 \). Then by Thm 9.2, 9.3, 9.6

\[
\lim f(x_n) = \lim \frac{2x_n}{x_n^2 - 1} = \frac{2 \lim x_n}{(\lim x_n)^2 - 1} = \frac{2x_0}{x_0^2 - 1} = f(x_0)
\]

By Def 17.1 \( f \) is continuous at \( x_0 \) for any \( x_0 \in \mathbb{R} \setminus \{-1, 1\} \)

**2)** \( g(x) = \sin\left(\frac{1}{x}\right) \) for \( x \neq 0 \) and \( g(0) = a \). Then for any \( a \in \mathbb{R} \)

\( g \) is not continuous at 0.

**Proof.** Take \( (x_n) \) with \( x_n = \frac{2}{\pi (2n-1)} \)

Then \( \lim x_n = 0 \) and \( g(x_n) = \sin \left( \frac{\pi (2n-1)}{2} \right) = (-1)^{n+1} \)

\( \forall a \in \mathbb{R} \) \( \lim g(x_n) = a \) fails.
Continuity and arithmetic operations

Thm 17.3 Let $f$ be a real-valued function with $\text{dom}(f) \subseteq \mathbb{R}$. If $f$ is continuous at $x_0 \in \text{dom}(f)$, then $|f|$ and $k \cdot f$, $k \in \mathbb{R}$, are continuous at $x_0$.

Proof. Let $(x_n)$ be a sequence in $\text{dom}(f)$ such that $\lim_{n \to \infty} x_n = x_0$. Then by Thm 9.2 $\lim_{n \to \infty} k \cdot f(x_n) = k \cdot \lim_{n \to \infty} f(x_n) = k \cdot f(x_0)$.

Therefore $k \cdot f$ is continuous at $x_0$.

By the triangle inequality $| |f(x_n)| - |f(x_0)| | \leq |f(x_n) - f(x_0)|$.

Fix $\varepsilon > 0$. Then $\lim_{n \to \infty} f(x_n) = f(x_0) \Rightarrow \exists N \forall n > N |f(x_n) - f(x_0)| < \varepsilon$.

Then $\forall n > N | |f(x_n)| - |f(x_0)| | \leq |f(x_n) - f(x_0)| < \varepsilon$.

This means that $\lim_{n \to \infty} |f(x_n)| = |f(x_0)|$, $|f|$ is continuous at $x_0$. 
Continuity and arithmetic operations

Thm 17.4 Let $f$ and $g$ be real-valued functions that are continuous at $x_0 \in \mathbb{R}$. Then

(i) $f + g$ is continuous at $x_0$  
(ii) $f \cdot g$ is continuous at $x_0$  
(iii) if $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at $x_0$.

Proof: Note that if $x \in \text{dom}(f) \cap \text{dom}(g)$, then $(f+g)(x) = f(x) + g(x)$ and $f \cdot g(x) = f(x) \cdot g(x)$ are well-defined. Moreover, if $x \in \text{dom}(f) \cap \text{dom}(g)$ and $g(x) \neq 0$, then $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ is well-defined.

Let $(x_n)$ be a sequence in $\text{dom}(f) \cap \text{dom}(g)$ s.t. $\lim x_n = x_0$. Then $\lim (f(x_n) + g(x_n)) = \lim f(x_n) + \lim g(x_n) = f(x_0) + g(x_0)$, and $\lim (f(x_n) \cdot g(x_n)) = \lim f(x_n) \cdot \lim g(x_n) = f(x_0) \cdot g(x_0)$. If moreover $\forall n \ g(x_n) \neq 0$, then $\lim \frac{f(x_n)}{g(x_n)} = \frac{\lim f(x_n)}{\lim g(x_n)} = \frac{f(x_0)}{g(x_0)}$, [ $\text{dom}\left(\frac{f}{g}\right) = \text{dom}(f) \cap \{x \in \text{dom}(g) : g(x) \neq 0\}$ ]
Continuity of a composition of functions

Let $f$ and $g$ be real-valued functions. If $x \in \text{dom}(f)$ and $f(x) \in \text{dom}(g)$, then we define $g \circ f(x) := g(f(x))$, $\text{dom}(g \circ f) = \{x \in \text{dom}(f) : f(x) \in \text{dom}(g)\}$

Thm 17.5 If $f$ is continuous at $x_0$ and $g$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at $x_0$.

Proof It is given that $x_0 \in \text{dom}(f)$ and $f(x_0) \in \text{dom}(g)$. Let $(x_n)$ be a sequence such that $\forall n \in \mathbb{N} \ x_n \in \text{dom}(g \circ f)$ and $\lim x_n = x_0$. Denote $y_n = f(x_n)$, $y_0 = f(x_0)$. Since $f$ is continuous at $x_0$, $\lim y_n = \lim f(x_n) = f(x_0) = y_0$. Since $g$ is continuous at $f(x_0) = y_0$, we have $\lim g \circ f(x_n) = \lim g(y_n) = g(y_0) = g \circ f(x_0)$. Therefore, $g \circ f$ is continuous at $x_0$. 


Examples

1) \( \sin(x) \) is continuous on \( \mathbb{R} \)

Proof

\[ \text{Enough to show that } \sin(x) \text{ is continuous at } 0 \]

For any \( x_0 \in \mathbb{R} \) and \( (x_n) \) with \( \lim x_n = x_0 \)

\[
|\sin(x_n) - \sin(x_0)| = \left| 2 \sin \left( \frac{x_n - x_0}{2} \right) \cos \left( \frac{x_n + x_0}{2} \right) \right| \leq 2 \sin \left( \frac{x_n - x_0}{2} \right) - 0
\]

2) Area \( (\triangle) \leq \text{Area } (\bigtriangleup) \)

\[ \forall x \in [0, \frac{\pi}{2}] \quad \frac{1}{2} \sin(x) \leq \pi \cdot \frac{x}{2\pi} = \frac{x}{2} \]

\[ \left| \frac{1}{2} \sin(x) \right| \leq \left| \frac{x}{2} \right| \]

\[ \forall x \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \quad |\sin(x)| \leq |x| \]

3) If \( \lim y_n = 0 \), then \( \exists N \forall n > N \quad |y_n| \leq \frac{\pi}{2} \). Then \( T.9.11(ii) \)

\[ \forall n > N \quad 0 \leq |\sin(y_n)| \leq |y_n| \Rightarrow \lim \sin(y_n) = 0 \]

\[ \sin 0 = 0 \]
**Examples**

2) \( f(x) = \sqrt{x} \) is continuous on \([0, +\infty)\).

1) \( \sqrt{x} \) is continuous at 0

Let \( \lim x_n = 0 \). Fix \( \epsilon > 0 \). Then \( \exists N \in \mathbb{N} \) for all \( n > N \) \( \sqrt{x_n} < \epsilon \)

\[ \implies \forall n > N \quad \sqrt{x_n} < \epsilon \implies \lim \sqrt{x_n} = 0 \]

2) Let \( x_0 \in (0, +\infty) \), \( \{x_n\} \) s.t. \( \forall n \quad (x_n \in [0, +\infty)) \) and \( \lim x_n = x_0 \)

Then \( \lim x_n = x_0 > 0 \implies \exists N_1 \forall n > N_1 \quad (x_n > \frac{x_0}{2}) \)

Fix \( \epsilon > 0 \). Then \( \exists N_2 \forall n > N_2 \quad |x_n - x_0| < \sqrt{x_0} \cdot \epsilon \)

\[ \forall n > \max\{N_1, N_2\} \quad |f(x_n) - f(x_0)| = |\sqrt{x_n} - \sqrt{x_0}| = \left| \frac{x_n - x_0}{\sqrt{x_n} + \sqrt{x_0}} \right| \leq \frac{|x_n - x_0|}{\sqrt{x_0}} < \epsilon \]

3) \( \cos(x) \) is continuous on \( \mathbb{R} \). \( \cos(x) = \sqrt{1 - \sin^2(x)} \), by Thm 17.4

\( 1 - \sin^2(x) \) is continuous on \( \mathbb{R} \). Moreover, \( \forall x \in \mathbb{R} \quad 1 - \sin^2(x) \in [0, 1) \subset [0, +\infty) \)

\[ \implies \text{by example 2) and Thm 17.5} \quad \cos(x) \text{ is continuous on } \mathbb{R} \]