Today: Series

> Q&A: February 4

Next: Ross § 17

Week 5:

- Homework 4 (due Sunday, February 6)
Comparison test

**Thm 14.6** Let \((a_n)\) and \((b_n)\) be two sequence, \(\forall n, a_n \geq 0\)

Then

1. \(
\sum_{n=1}^{\infty} a_n \text{ converges } \land \forall n, (|b_n| \leq a_n) \Rightarrow \sum_{n=1}^{\infty} b_n \text{ converges}
\)

2. \(
\sum_{n=1}^{\infty} a_n = +\infty \land \forall n, (b_n \geq a_n) \Rightarrow \sum_{n=1}^{\infty} b_n = +\infty
\)

**Examples**

\[
\]

\[
\]

**Corollary 14.7** Absolutely convergent series are convergent

**Proof:**
**Root Test**

**Thm 14.9** Let $\sum a_n$ be a series, let $\alpha = \limsup \sqrt[n]{|a_n|}$. Then

(i) $\alpha < 1 \Rightarrow \sum a_n$

(ii) $\alpha > 1 \Rightarrow \sum a_n$

(iii) $\alpha = 1$ does not provide information about the convergence of $\sum a_n$

**Proof:**

(i) $\alpha < 1 \Rightarrow \exists$

$\limsup \sqrt[n]{|a_n|} = \alpha$

$\Rightarrow$

Fix $\varepsilon > 0$. Since $\beta < 1$,

Then

(ii) $\exists (n_k)$ s.t.
Ratio Test

Thm 14.8 Let $\sum_{n=1}^{\infty} a_n$ be a series, $\forall n \ (a_n \neq 0)$.

(i) $\limsup_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$

(ii) $\liminf_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$

(iii) $\liminf_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \leq 1 \leq \limsup_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} : \text{not enough information.}$

Proof Let $d = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$. Then by Thm 12.2

$\liminf_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \leq \limsup_{n \to \infty} \sqrt[n]{|a_n|} \leq \limsup_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$.

(i) 

(ii) 

(iii)
Examples

• $\forall \alpha > 1$

Ratio test:

$\Rightarrow$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n}$

Cauchy test:
Integral test

\[
\begin{align*}
  a_n &= \frac{1}{n^2}, \\
  b_n &= \frac{1}{n} \\
  p &> 0
\end{align*}
\]
Examples

\[ a_n = \frac{1}{n}, \quad n \geq 3, \quad \left[ \text{use } \forall n \geq 3 \quad 1 \leq \log n \leq n \right] \]

Root test:
Alternating Series

Thm 15.3 Let \((a_n)\) be a sequence s.t. \(\forall n \ (a_n \geq 0 \land a_n \leq a_{n+1})\). Then

\[
\lim_{n \to \infty} a_n = 0 \implies \text{ PRI.}
\]

Proof. Denote \(\sum_{k=1}^{\infty} a_k =: s_1\), \(\sum_{k=1}^{n} a_k =: s_n\).

1. \((S_{2n})_{n=1}^{\infty}\) is
2. \((S_{2n-1})_{n=1}^{\infty}\) is

2. \(\forall m, n \in \mathbb{N}\)
   Case \(m \leq n\):
   Case \(m \geq n\):

By 2 + Thm 10.2 and

Then \(\forall n \ (S_{2n} \leq s \leq S_{2n+1}) \implies \)

Important example

9. Let $p > 0$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$

Proof. Denote $x_n = \frac{1}{n^p}$, $S_k = \sum_{n=1}^{k} x_n$. $x_1 \geq x_2 \geq \cdots \geq x_n$, $(S_k)$ is increasing.

Consider the sequences:

Then

and $\forall k$

1. $(S_k)$ converges $\iff$

2. $(S_{2^k})$ converges $\iff$

$a_n =$