SOLUTIONS

Name (last, first): ________________________________

Student ID: ________________________________

☐ Write your name and PID on the top of EVERY PAGE.

☐ Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).

☐ The exam consists of 4 questions. Your answers must be carefully justified to receive credit.

☐ This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.

☐ No calculators, phones, or other electronic devices are allowed.

☐ Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.

☐ You are allowed to use one 8.5 by 11 inch sheet of paper with handwritten notes (on both sides); no other notes (or books) are allowed.

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1. (20 points) Compute the area of the triangle with vertices

\[ P = (1, 0, -1), \quad Q = (0, 1, 0), \quad R = (1, 0, 0). \]

The area of the triangle \( \triangle PQR \) is equal to \( \frac{1}{2} \) of the area of the parallelogram spanned by the vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \).

The area of the parallelogram is \( \| \overrightarrow{PQ} \times \overrightarrow{PR} \| \).

Compute the components of \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \):

\[ \overrightarrow{PQ} = \langle -1, 1, 1 \rangle, \quad \overrightarrow{PR} = \langle 0, 0, 1 \rangle \]

Compute the cross product:

\[
\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-1 & 1 & 1 \\
0 & 0 & 1
\end{vmatrix}
\]

\[
= \hat{i} \cdot 1 - \hat{j} \cdot (-1) + \hat{k} \cdot 0 = \langle 1, 1, 0 \rangle
\]

Compute the magnitude of \( \overrightarrow{PQ} \times \overrightarrow{PR} \):

\[ \| \overrightarrow{PQ} \times \overrightarrow{PR} \| = \| \langle 1, 1, 0 \rangle \| = \sqrt{1^2 + 1^2} = \sqrt{2} \]

Area of the triangle \( \triangle PQR \) is \( \frac{\sqrt{2}}{2} \).
2. (20 points) Let 
\[ P = (1, 0, 2), \quad Q = (1, 1, 4), \quad R = (4, 2, \alpha) \]
be points in \( \mathbb{R}^3 \).
Determine the real number \( \alpha \) such that vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) are orthogonal.

Vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) are orthogonal if \( \overrightarrow{PQ} \cdot \overrightarrow{PR} = 0 \).

Compute the components of \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \):
\[ \overrightarrow{PQ} = \langle 0, 1, 2 \rangle, \quad \overrightarrow{PR} = \langle 3, 2, \alpha - 2 \rangle \]

Compute the dot product with unknown \( \alpha \):
\[ \overrightarrow{PQ} \cdot \overrightarrow{PR} = \langle 0, 1, 2 \rangle \cdot \langle 3, 2, \alpha - 2 \rangle \]
\[ = 2 + 2(\alpha - 2) = 2\alpha - 2 \]

Find \( \alpha \) for which \( \overrightarrow{PQ} \cdot \overrightarrow{PR} \) by solving
\[ 2\alpha - 2 = 0, \quad 2\alpha = 2, \quad \alpha = 1 \]
3. (20 points) Let \( L_1 \) and \( L_2 \) be two lines in \( \mathbb{R}^3 \) given by the parametric equations

\[
L_1 : \begin{cases}
  x = t + 2, \\
  y = t + 3, \\
  z = -3,
\end{cases}
\quad \text{and} \quad
L_2 : \begin{cases}
  x = s + 2, \\
  y = 2s, \\
  z = -s.
\end{cases}
\]

Determine whether the lines \( L_1 \) and \( L_2 \) are equal, parallel but not equal, skew, or intersecting. If the lines intersect, find the point of intersection.

From the parametric equations of the lines we determine the direction vectors for \( L_1 \) and \( L_2 \):

- \( L_1 \) has direction vector \( \vec{p} = \langle 1, 1, 0 \rangle \)
- \( L_2 \) has direction vector \( \vec{q} = \langle 1, 2, -1 \rangle \)

Vectors \( \vec{p} \) and \( \vec{q} \) are not parallel, therefore, \( L_1 \) and \( L_2 \) can be either skew or intersecting.

Determine whether \( L_1 \) and \( L_2 \) have a point in common. \( L_1 \) and \( L_2 \) have a point in common if the system of equations

\[
\begin{align*}
t + 2 &= s + 2 \\
t + 3 &= 2s \\
-3 &= -s
\end{align*}
\]

has a solution.

From the last equation we have \( s = 3 \). Plugging this into the first two equations gives \( t + 2 = 5 \), \( t + 3 = 6 \). Both these equations are satisfied for \( t = 3 \). We conclude that \( L_1 \) and \( L_2 \) intersect.

To find the point of intersection plug \( t = 3 \) into the equation of \( L_1 \) (or \( s = 3 \) into the equation of \( L_2 \)):

\[
x = 5, \quad y = 6, \quad z = -3.
\]

Therefore, the point of intersection is \( (5, 6, -3) \).
4. (20 points) Compute the distance from the point \(O = (0,0,0)\) to the plane passing through the points

\[ P = (1,0,0), \quad Q = (0,2,0), \quad R = (0,0,3). \]

The distance \(d\) from point \(O\) to the plane can be computed as the magnitude of the projection of the vector \(\vec{PO}\) onto the normal vector \(\vec{n}\) of the plane

\[ d = \| \text{proj}_n \vec{PO} \| = \frac{|\vec{n} \cdot \vec{PO}|}{\|\vec{n}\|} \]

Compute the components of \(\vec{PO}\)

\[ \vec{PO} = \langle -1,0,0 \rangle \]

As a normal vector we can take the cross product of \(\vec{PQ}\) and \(\vec{PR}\)

\[ \vec{PQ} = \langle -1,2,0 \rangle, \quad \vec{PR} = \langle -1,0,3 \rangle \]

\[ \vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \vec{i} \cdot 6 - \vec{j} \cdot 3 + \vec{k} \cdot 2 = \langle 6,3,2 \rangle \]

Compute \(\vec{n} \cdot \vec{PO} = \langle 6,3,2 \rangle \cdot \langle -1,0,0 \rangle = -6\)

Compute \(\|\vec{n}\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7\)

Finally, \(d = \frac{|-6|}{7} = \frac{6}{7}\)