Today: Equations of a plane

Next: Strang 3.1

Week 3:

- homework 3 (due Monday, October 17)
- Midterm 1: Wednesday, October 19 (vectors, dot product, cross product, equations of lines and planes)
Planes

Two points determine a line: for any two points $P, Q$ (in $\mathbb{R}^2$ or $\mathbb{R}^3$) there exists a unique line passing through $P$ and $Q$. A point $X$ is in the line through $P$ and $Q$ if $\overrightarrow{PX}$ is a multiple of $\overrightarrow{PQ}$, i.e., $\overrightarrow{PX} = t \overrightarrow{PQ}$ for some $t \in \mathbb{R}$.

Three points (that do not all lie on the same line) determine a plane: for any three points $P, Q$ and $R$ in $\mathbb{R}^3$ that do not all lie on the same line, there exists a unique plane that passes through these three points. A point $X$ is in the plane passing through $P, Q$ and $R$ if $\overrightarrow{PX}$ is a linear combination of vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

\[ \overrightarrow{PX} = t \overrightarrow{PQ} + s \overrightarrow{PR} \quad \text{for some} \quad t, s \in \mathbb{R} \]
Another way to describe a plane is by identifying a point in the plane and a vector that is perpendicular (orthogonal) to the plane. If $P$ is a point in the plane and vector $\vec{n}$ is orthogonal to the plane (called the normal vector) then point $X$ is in this plane if and only if

$$\vec{n} \perp \vec{PX}, \quad \vec{n} \cdot \vec{PX} = 0 \text{ (vector equation of a plane)}$$
Equation of a plane

Consider a plane containing point \( P = (x_0, y_0, z_0) \) with normal vector \( \vec{n} = \langle a, b, c \rangle \). Then point \( X = (x, y, z) \) belong to this plane if and only if

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

scalar equation of a plane

If we denote \( d := -ax_0 - by_0 - cz_0 \), then \((*)\) becomes

\[ ax + by + cz + d = 0 \]

general form of the equation of a plane

Suppose that we know the coordinates of three points \( P, Q, R \) in the plane. How can we find a normal vector to this plane? \( \vec{n} = \vec{PQ} \times \vec{PR} \)
Example
Write the vector equation for the plane containing points \( P = (1, 1, 0), \ Q = (-2, 1, 1), \ R = (0, 0, 1) \)
\[ \vec{PQ} = \langle -3, 0, 1 \rangle, \ \vec{PR} = \langle -1, -1, 1 \rangle \]

Compute the normal vector to the plane
\[ \vec{n} = \vec{PQ} \times \vec{PR} = \langle -3, 0, 1 \rangle \times \langle -1, -1, 1 \rangle = \langle 1, 2, 3 \rangle \]

Point \( X = (x, y, z) \) is in the plane if
\[ \vec{n} \cdot \vec{PX} = 0 \], or equivalently [ vector equation of the plane ]
\[ \langle 1, 2, 3 \rangle \cdot \langle x-1, y-1, z-0 \rangle = 0 \]

Equivalent
\[ 1 \cdot (x-1) + 2 \cdot (y-1) + 3 \cdot z = 0 \] [ scalar equation ]
\[ x + 2y + 3z - 3 = 0 \] [ general form ]
Distance between a plane and a point

Consider a plane with point P and normal vector \( \vec{n} \). Suppose that point X does not belong to this plane. The distance \( d \) between X and the plane is the smallest distance between X and points in the plane. If \( \vec{R}X \) is orthogonal to the plane (parallel to \( \vec{n} \)), then \( \vec{R}X \perp R\vec{Q} \) for any point \( Q \neq R \) in the plane, \( ||\vec{R}X|| < ||QX|| \).
Distance between a plane and a point

Conclusion: \( d = \| \overrightarrow{RX} \| \), where \( R \) is in the plane and \( \overrightarrow{RX} \) is perpendicular to the plane (\( \overrightarrow{RX} \) is parallel to \( \hat{n} \)).

How to find \( \overrightarrow{RX} \) (and \( \| \overrightarrow{RX} \| \)) if \( P \) and \( \hat{n} \) are given?

\[
\overrightarrow{RX} = \text{proj}_\hat{n} \overrightarrow{PX} = \frac{\overrightarrow{PX} \cdot \hat{n}}{\| \hat{n} \|^2} \hat{n}, \quad \| \overrightarrow{RX} \| = \frac{\| \overrightarrow{PX} \cdot \hat{n} \|}{\| \hat{n} \|^2} \cdot \| \hat{n} \| = \frac{\| \overrightarrow{PX} \cdot \hat{n} \|}{\| \hat{n} \|} = d
\]
Distance between a plane and a point

Example

Find the distance between the point $X = (0, 0, 0)$ and the plane given by $x + 2y + 3z - 3 = 0$.

This is the equation in the general form. First find the normal vector $\hat{n} = <1, 2, 3>$

Next we need a point in the plane (any point), i.e., any numbers $x_0, y_0, z_0$ such that $x_0 + 2y_0 + 3z_0 - 3 = 0$. We can take $x_0 = 0, y_0 = 0$, which requires that $z_0 = 1$. $P = (0, 0, 1)$ $\vec{PX} = <0, 0, -1>$

Then the distance from $X$ to the plane is $d = \frac{|<1, 2, 3> \cdot <0, 0, -1>|}{||<1, 2, 3>||} = \frac{1-31}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{3}{\sqrt{14}}$
Parallel and intersecting planes

Let $P_1$ and $P_2$ be two planes in $\mathbb{R}^3$. Then the following possibilities exist:

<table>
<thead>
<tr>
<th>$P_1$ and $P_2$ share a common point</th>
<th>YES</th>
<th>NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal vectors of $P_1$ and $P_2$</td>
<td>YES</td>
<td>Equal</td>
</tr>
<tr>
<td>are parallel</td>
<td>NO</td>
<td>Intersecting</td>
</tr>
</tbody>
</table>

If two planes intersect, the intersection is a line!