Today: The dot product

Next: Strang 2.4

Week 2:

- homework 1 (due Monday, October 3)
- survey on Canvas Quizzes (due Friday, October 7)
Dot product (scalar product) of vectors

**Def:** If \( \vec{v} = \langle v_1, v_2, v_3 \rangle \) and \( \vec{w} = \langle w_1, w_2, w_3 \rangle \) are two vectors in \( \mathbb{R}^3 \), then the dot product or the scalar product of \( \vec{v} \) and \( \vec{w} \) is given by the sum of products of vector components

\[
\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3
\]

(in \( \mathbb{R}^2 \) \( \vec{v} = \langle v_1, v_2 \rangle \), \( \vec{u} = \langle u_1, u_2 \rangle \))

\[
\vec{v} \cdot \vec{u} = v_1u_1 + v_2u_2
\]

**Theorem 2.4**

If

\[
\theta = \angle \vec{u}, \vec{v}, \quad 0 \leq \theta \leq \pi,
\]

then

\[
\vec{u} \cdot \vec{v} = ||\vec{u}|| \cdot ||\vec{v}|| \cdot \cos \theta
\]
Dot product and angles between vectors

From Theorem 2.4 we have

\[ \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}, \quad \theta = \arccos \left( \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) \]

Examples

Find the angle between \( \mathbf{u} \) and \( \mathbf{v} \)

(a) \( \mathbf{u} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 2\mathbf{j} \)

\[ \|\mathbf{u}\| = \|\mathbf{v}\| = \]

\[ \mathbf{u} \cdot \mathbf{v} = \]

\[ \Rightarrow \cos \theta = \quad , \quad \theta = \]

(b) \( \mathbf{u} = \langle 1, 2, 3 \rangle, \quad \mathbf{v} = \langle -7, 2, 1 \rangle \)

\[ \mathbf{u} \cdot \mathbf{v} = \]

\[ \Rightarrow \cos \theta = \quad \Rightarrow \theta = \]
Orthogonal vectors

If \( \cos \theta = 0 \), then \( \theta = \frac{\pi}{2} \), which means that the vectors form a right angle.

We call such vectors orthogonal.

Theorem 2.5

The nonzero vectors \( \vec{u} \) and \( \vec{v} \) are orthogonal.

Example: Determine whether \( \vec{p} = (1, 3, 0) \) and \( \vec{q} = (-6, 2, 5) \) are orthogonal. Since \( \vec{p} \cdot \vec{q} = \), we conclude that \( \vec{p} \) and \( \vec{q} \) are orthogonal.
Orthogonality of standard unit vectors

Recall: \( \vec{i} = <1, 0, 0> \), \( \vec{j} = <0, 1, 0> \), \( \vec{k} = <0, 0, 1> \)

Then \( \vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \)

\( \vec{i} \cdot \vec{j} = 0 \)

\( \vec{i} \cdot \vec{k} = 0 \)

\( \vec{j} \cdot \vec{k} = 0 \)

We say that \( \vec{i} \), \( \vec{j} \), \( \vec{k} \) are orthogonal.

Example: \( (10 \vec{i} - \vec{j}) \cdot (-\vec{i} + 2\vec{k}) \)

\( = \)

\( = \)

\( <10, -1, 0> \cdot <-1, 0, 2> = \)
Using vectors to represent data

Fruit vendor sells apples, bananas and oranges. On a given day he sells 30 apples, 12 bananas and 18 oranges. Define the vector
\[ \mathbf{q} = \]

Suppose that the vendor sets the following prices
- 0.5 per apple
- 0.25 per banana
- 1 per orange

Define the vector of prices
\[ \mathbf{p} = \]

Then \[ \mathbf{q} \cdot \mathbf{p} = \]
is vendor's
Projections

Let \( \vec{u} \) and \( \vec{v} \) be two vectors. Sometimes we want to decompose \( \vec{v} \) into two components \( \vec{v} = \vec{a} + \vec{b} \) such that \( \vec{a} \) is parallel to \( \vec{u} \) and \( \vec{b} \) is orthogonal to \( \vec{u} \)

Why?

1. Find the area of the triangle. Area of this triangle is

2. Child pulls a wagon. How much force is actually moving the wagon forward?
**Projections**

The vector projection of $\vec{v}$ onto $\vec{u}$ is the vector labeled $\text{proj}_{\vec{u}} \vec{v}$ given by

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}$$

The length of $\text{proj}_{\vec{u}} \vec{v}$, $\|\text{proj}_{\vec{u}} \vec{v}\|$ is called the scalar projection of $\vec{v}$ onto $\vec{u}$.
Projections

Let \( \vec{v} \) and \( \vec{u} \) be nonzero vectors. Then \( \vec{u} \) and \( \vec{v} - \text{proj}_\vec{u} \vec{v} \) are

\[
\vec{u} \cdot (\vec{v} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}) =
\]

Example Find the projection of \(-2,2\) onto \(4,1\)

resolution of \(\vec{v}\)

\[
\text{proj}_\vec{u} \vec{v} =
\]

\[
\vec{v} - \text{proj}_\vec{u} \vec{v} =
\]
Example: Find the projection of \((-2, 2, 3)\) onto \((10, -1, 0)\).

\[
\text{proj}_u \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}
\]
Example

Ship travels 15° north of east with engine generating a speed of 20 knots in this direction. The ocean current moves the ship NW at a speed of 2 knots. How much does the current slow the movement of the ship in the direction of \( \vec{u} \).

\[
\text{proj}_u \vec{V} =
\]
The cross product

**Def** Let \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and \( \vec{v} = \langle v_1, v_2, v_3 \rangle \).

Then the cross product of \( \vec{u} \) and \( \vec{v} \) is vector

\[
\vec{u} \times \vec{v} =
\]

**Example**

\[
\vec{p} = \langle 1, 2, 3 \rangle, \quad \vec{q} = \langle -1, 2, 0 \rangle
\]

\[
\vec{p} \times \vec{q} =
\]

\[
\vec{p} \cdot (\vec{p} \times \vec{q}) =
\]

\[
\vec{q} \cdot (\vec{p} \times \vec{q}) =
\]
The cross product

Fact: Vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$! and the direction is determined by the right-hand rule.

Indeed,

$$\vec{p} = \langle 1, 2, 3 \rangle, \quad \vec{q} = \langle -1, 2, 0 \rangle, \quad \vec{p} \times \vec{q} = \langle -6, -3, 4 \rangle$$

$$\vec{q} \times \vec{p} =$$
Properties of the cross product

Exercise \( \vec{i} \times \vec{j} = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = \)

\[
\vec{i} \times \vec{j} = \quad \vec{j} \times \vec{k} = \quad \vec{k} \times \vec{i} =
\]

Theorem 2.6 Let \( \vec{u}, \vec{v}, \vec{w} \) be vectors in \( \mathbb{R}^3 \). Then

(i) \( \vec{u} \times \vec{v} = \)

(ii) \( \vec{u} \times (\vec{v} + \vec{w}) = \)

(iii) \( c(\vec{u} \times \vec{v}) = \)

(iv) \( \vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \)

(v) \( \vec{v} \times \vec{v} = \)

(vi) \( \vec{u} \cdot (\vec{v} \times \vec{w}) = \) For proof expand both sides in terms of components of \( \vec{u}, \vec{v}, \vec{w} \)