Today: Local minima/maxima

Next: Strang 4.7

Week 7:

- homework 6 (due Friday, November 11)
- Midterm 2: Wednesday, November 16 (lectures 10-19)
Maxima and minima of functions of one variable

Let $f: \mathbb{R} \to \mathbb{R}$ be a function of one variable.

The point $x_0 \in \mathbb{R}$ is called a critical point of $f$ if either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Any local maximum or local minimum of $f$ is a critical point.
Critical points of functions of two variables

Finding local minima/maxima in one dimension:

(i) identify critical points; (ii) determine which critical points are local minima/maxima.

We will extend this to functions of two variables. First, introduce the notion of a critical point for functions of two variables.

**Def.** Let $z = f(x,y)$ be a function of two variables defined at $(x_0,y_0)$. Then $(x_0,y_0)$ is called a

if either

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Critical points. Example

Find the critical points of the function

\[ f(x,y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36} \]

Start by computing \( f_x \) and \( f_y \) and finding \((x,y)\) s.t.

\( f_x(x,y) = 0 \) and \( f_y(x,y) = 0 \) simultaneously

\[ f_x(x,y) = \]

\[ f_y(x,y) = \]

Next, find all \((x,y)\) for which \( f_x \) or \( f_y \) does not exist:

all \((x,y)\) s.t.
Critical points. Example (cont.)

Therefore, and are possible critical points. We have to check that these points are in the domain of definition of $f$. The domain of definition of $f$ consists of all $(x,y)$ s.t.

Clearly, all points satisfying $(x)$

Also, point $(2,-3)$ Therefore, the set of the critical points of $f$ consists of and all points of the hyperbola
Critical points. Example (cont.)

Here is the plot of the domain of $f$ and the critical points of $f$
Local minimum/maximum

Def Let \( z = f(x, y) \) be a function of two variables. Then \( f \) has

if

for all points \( (x, y) \) within some disk centered at \( (x_0, y_0) \). The number \( f(x_0, y_0) \) is called the

if \( (*) \) holds for all \( (x, y) \) in the domain of \( f \), we say that \( f \) has

Function \( f \) has a

if

for all points \( (x, y) \) within some disk centered at \( (x_0, y_0) \). The number \( f(x_0, y_0) \) is called the

if \( (**) \) holds for all \( (x, y) \) in the domain of \( f \), we say that \( f \) has

Local minima and local maxima are called
Local extrema and critical points

Thm 4.16 Let \( z = f(x, y) \) be a function of two variables.

Suppose

Example At the very top of a mountain the ground is flat. If there was slope in some direction, then you could go higher. Similarly, at the lowest point of a crater the ground is also flat (\( \nabla f = 0 \)).

But the fact that the ground is flat (\( \nabla f(x_0, y_0) = 0 \)) that \( f \) has a local extremum at \( (x_0, y_0) \).
**Saddle points**

**Def.** Let \( z = f(x, y) \) be a function of two variables. We say that \((x_0, y_0)\) is a saddle point if \( f \)

Level curves around the saddle point have this shape.
The second derivative test

Thm 4.17 (Second derivative test)
Suppose that \( f(x,y) \) is a function of two variables for which the first- and second-order partial derivatives are continuous around \((x_0,y_0)\). Suppose \( \) and \( \) . Define

(i) If \( \) and \( \), then \( f \) has a
(ii) If \( \) and \( \), then \( f \) has a
(iii) If \( \), then \( f \) has a
(iv) If \( \), then
Problem solving strategy

Problem:
Let \( z = f(x, y) \) be a function of two variables for which the first- and second-ordered partial derivatives are continuous. Find local extrema.

Solution:
1. Determine critical points \((x_0, y_0)\) where \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \)
   Discard any points where \( f_x \) or \( f_y \) does not exist.
2. Calculate \( D \) for each critical point
3. Apply the Second derivative test to determine if \((x_0, y_0)\)
is a local minimum, local maximum or a saddle point.