Today: Partial derivatives

Next: Strang 4.4

Week 6:

- homework 5 (due Friday, November 4, 11:59 PM)
Tangent planes

Recall, if $f$ is a function of one real variable, then its graph determines a curve $C$ in $\mathbb{R}^2$, and the tangent line to the graph of $f$ at point $x_0$ is the line that "touches" the curve $C$ at point $(x, f(x_0))$.

If $f$ is a function of two variables, then its graph determines a surface $S$, and the tangent plane to $S$ at $(x_0, y_0, f(x_0, y_0))$ is a plane that "touches" $S$ at this point.
**Tangent plane**

**Def.** Let \( P_0 = (x_0, y_0, z_0) \) be a point on a surface \( S \), and let \( C \) be any curve passing through \( P_0 \) and lying entirely in \( S \). If the tangent lines to all such curves \( C \) at \( P_0 \) lie in the same plane, then this plane is called the

**Def.** Let \( S \) be a surface defined by a differentiable function \( z = f(x, y) \). Let \( P_0 = (x_0, y_0) \) be in the domain of \( f \). Then the equation of the tangent plane to \( S \) at \( P_0 \) is
**Tangent plane**

To see that this formula is correct, we can find two curves in $S$ that pass through $(x_0, y_0, f(x_0, y_0))$ and determine the equations of the tangent lines.

Take $\vec{p}(t) =$ and $\vec{q}(s) =$

Then for any $t$ (such that $(t, y_0)$ is in the domain of $f$) $\vec{p}(t)$.

Similarly, for any $s$ $\vec{q}(s)$.

Moreover,

Tangent line to $\vec{p}(t)$ at $t = x_0$: $\vec{e}_p(t) =$

with $\vec{p}'(t) =$

Similarly, tangent line to $\vec{q}(s)$ at $s = y_0$: $\vec{e}_q(s) =$

$\vec{q}'(s) =$
Tangent plane

Vectors $\vec{p}'(x_0) = \langle 1, 0, \frac{\partial f}{\partial x}(x_0, y_0) \rangle$ and $\vec{q}'(y_0) = \langle 0, 1, \frac{\partial f}{\partial y}(x_0, y_0) \rangle$ are not parallel, therefore, together with the point $(x_0, y_0, f(x_0, y_0))$ they determine a plane with normal vector $\vec{n}$.

The equation of a plane passing through $(x_0, y_0, f(x_0, y_0))$ with normal vector $\vec{n}$ is
**Tangent plane**

**Example** Find the equation of the tangent plane to the surface defined by the function \( f(x,y) = e^{xy} \) at point \((1, -1)\)

- **Step 1:** Compute \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \)
  \[
  \frac{\partial f}{\partial x} = \quad \frac{\partial f}{\partial y} =
  \]

- **Step 2:** Evaluate \( \frac{\partial f}{\partial x} (x_0, y_0) \) and \( \frac{\partial f}{\partial y} (x_0, y_0) \)
  \[
  \frac{\partial f}{\partial x} (1, -1) = \quad \frac{\partial f}{\partial y} (1, -1) =
  \]

- **Step 3:** Evaluate \( f(x_0, y_0) \):
  \[
  f(1, -1) =
  \]

- **Step 4:** Plug everything into the equation:
Tangent plane does not always exist at every point

Example (tangent plane does not exist at (0,0))

Let \( f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & , \quad (x,y) \neq 0 \\ 0 & , \quad (x,y) = 0 \end{cases} \) (\( f(x,y) \) is continuous)

\( S \) - surface defined by \( f(x,y) \)

Consider the curves:

Consider the curve \( \mathbf{p}(t) = \)

Then \( f(t,t) = \)

For a tangent plane to a surface to exist, it is sufficient that the function that defines the surface is differentiable.
**Linear approximation**

Functions of one variable:
the tangent line at \( x_0 \) can be used as the linear approximation of a function \( f(x) \) at points \( x \) close to \( x_0 \):

\[
f(x) \approx f(x_0) + f'(x_0)(x - x_0)
\]
for \( x \) close to \( x_0 \).

Functions of two variables: the tangent plane at \((x_0, y_0)\) can be used as the linear approximation of \( f(x, y) \) at points close to \((x_0, y_0)\).

Def. Given a function \( z = f(x, y) \) with continuous partial derivatives that exist at \((x_0, y_0)\), the linear approximation of \( f \) at point \((x_0, y_0)\) is given by

\[
y = f(x_0) + f'(x_0,y_0)(x - x_0) + R
\]
Linear approximation

Example

Given function \( f(x, y) = e^{xy} \) approximate \( f(1.01, 0.99) \) using points \((1, 1)\) as \((x_0, y_0)\).

- Compute the derivatives
  \[
  f_x(x, y) = , \quad f_y(x, y) = 
  \]

- Evaluate \( f, f_x \) and \( f_y \) at \((x_0, y_0)\)
  \[
  f(1,1) = , \quad f_x(1,1) = , \quad f_y(1,1) = 
  \]

- Write the linear approximation
  \[
  L(x, y) = 
  \]

- Compute the approximation: \( L(1.01, 0.99) = \)
Differentiability

Functions of one variable: if a function is differentiable at $x_0$, the graph at $x_0$ is smooth (no corners), tangent line is well defined and approximates well the function around $x_0$.

Functions of two variables: differentiability gives the condition when the surface at $(x_0, y_0)$ is smooth, by which we mean that the tangent plane at $(x_0, y_0)$ exists. Notice, that whenever $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, we can always write the equation

$$Z = f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0). \quad (*)$$

But this does not mean that the tangent plane exists (if it exists, it is given by $(*)$).
Differentiability

Def. $f$ is differentiable at $(x_0, y_0)$ if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and the error term

$$E(x, y) = f(x, y) - [f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)]$$

satisfies

This means that

$$f(x, y) =$$

and $E(x, y)$ goes to zero faster than the distance between $(x, y)$ and $(x_0, y_0)$.

Remark: If $f(x, y)$ is differentiable at $(x_0, y_0)$, then $f(x, y)$ is continuous at $(x_0, y_0)$. 
Differentiability

The existence of partial derivatives is not sufficient to have differentiability.

Example

\[ f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \]

Then

\[ f_x(x, y) = \begin{cases} -\frac{y^2}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}, \quad f_y(x, y) = \begin{cases} \frac{2xy}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

For \( (x_0, y_0) = (0, 0) \), \( f(0, 0) = \), \( f_x(0, 0) = \), \( f_y(0, 0) = \), so

\[ E(x, y) = \] and
Differentiability

But, if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist AND are continuous in a neighborhood of $(x_0,y_0)$, then $f$ is differentiable at $(x_0,y_0)$.

**Theorem**

If $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$ all exist in a neighborhood of $(x_0,y_0)$ and
The chain rule

Recall that for functions of one variable
\[
\frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x)
\]

Thm (Chain rule for one independent variable)

Let \( x(t) \) and \( y(t) \) be differentiable functions, let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) be a differentiable function. Then

\[
\frac{d}{dt} [f(x(t), y(t))] = \]

Example

Compute \( \frac{d}{dt} [f(\sin t, \cos t)] \) with \( f(x, y) = 4x^2 + 3y^2 \)

\[
\frac{\partial f}{\partial x} = \quad \frac{\partial f}{\partial y} = \quad \frac{d}{dt} \sin t = \quad \frac{d}{dt} \cos t = \]

\[
\frac{d}{dt} [f(\sin t, \cos t)] =
\]
The chain rule

Thm (Chain rule for two independent variables)
Suppose \( x(u,v) \) and \( y(u,v) \) are differentiable, and suppose \( f(x,y) \) is differentiable. Then
\[
z = f(x(u,v), y(u,v))
\]
is differentiable (function from \( \mathbb{R}^2 \) to \( \mathbb{R} \)) and
\[
\frac{\partial z}{\partial u} =
\]
\[
\frac{\partial z}{\partial v} =
\]

Example
\[
z = f(x,y) = e^{x^2+3y}, \quad x(u,v) = u + 2v, \quad y(u,v) = u - v
\]
\[
\frac{\partial f}{\partial x} = e^{x^2+3y}, \quad \frac{\partial f}{\partial y} = 3e^{x^2+3y}, \quad \frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 2, \quad \frac{\partial y}{\partial u} = 1, \quad \frac{\partial y}{\partial v} = -1
\]
\[
\frac{\partial z}{\partial u} =
\]
\[
\frac{\partial z}{\partial v} =
\]