Today: Partial derivatives

Next: Strang 4.4

Week 6:

- homework 5 (due Friday, November 4, 11:59 PM)
Limit of a function of two variables

Def. Consider a point \((a, b) \in \mathbb{R}^2\). A \(\delta\)-disk centered at point \((a,b)\) is the open disk of radius \(\delta\) centered at \((a,b)\)

\[
\{ (x,y) \mid (x-a)^2 + (y-b)^2 < \delta^2 \}
\]

Def. The limit of \(f(x,y)\) as \((x,y)\) approaches \((x_0,y_0)\) is \(L\)

\[
\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L
\]

if for each \(\varepsilon > 0\) there exists a small enough \(\delta > 0\) such that all points in a \(\delta\)-disk around \((x_0,y_0)\), except possible \((x_0,y_0)\) itself, \(f(x,y)\) is no more than \(\varepsilon\) away from \(L\). (For any \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(|f(x,y) - L| < \varepsilon\) whenever \(\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\).)
Theorem 4.1  Let \( \lim_{(x,y) \to (a,b)} f(x,y) = L \), \( \lim_{(x,y) \to (a,b)} g(x,y) = M \), \( c \) - constant

- \( \lim_{(x,y) \to (a,b)} c = c \)
- \( \lim_{(x,y) \to (a,b)} x = a \)
- \( \lim_{(x,y) \to (a,b)} y = b \)

- \( \lim_{(x,y) \to (a,b)} [f(x,y) \pm g(x,y)] = L \pm M \)
- \( \lim_{(x,y) \to (a,b)} [f(x,y)g(x,y)] = LM \)

- If \( M \neq 0 \), \( \lim_{(x,y) \to (a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \)

- \( \lim_{(x,y) \to (a,b)} [c \cdot f(x,y)] = cL \)

- \( \lim_{(x,y) \to (a,b)} [f(x,y)]^n = L^n \)
- \( \lim_{(x,y) \to (a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} \)
Examples

\[
\lim_{(x,y) \to (0,0)} \frac{xy + 1}{x^2 + y^2 + 1} =
\]

\[
\lim_{(x,y) \to (2,1)} \frac{x-y-1}{\sqrt{x-y} - 1} =
\]
Partial derivatives of functions of two variables

Functions of one variable $y = f(x)$: the derivative gives the instantaneous rate of change of $y$ as a function of $x$.

Functions of two variables $z = f(x, y)$ have 2 independent variables, we need two (partial) derivatives.

**Def** The partial derivative of $f(x, y)$ with respect to $x$ is

$$f_x = \frac{\partial f}{\partial x} =$$

The partial derivative of $f(x, y)$ with respect to $y$ is

$$f_y = \frac{\partial f}{\partial y} =$$
Partial derivatives of functions of two variables

Partial derivatives measure the instantaneous rate of change of \( f \) if we change only the \( x \) variable \( \frac{\partial f}{\partial x} \) or only the \( y \) variable \( \frac{\partial f}{\partial y} \).
Calculating partial derivatives

Rule To differentiate $f(x,y)$ with respect to $x$, treat the variable $y$ as a constant, and differentiate $f$ as a function of one variable $x$:

$$\frac{\partial}{\partial x} (x^3 - 12xy^2 - x^2y + 4x - y - 3) =$$

To differentiate $f(x,y)$ with respect to $y$, treat the variable $x$ as a constant, and differentiate $f$ as a function of one variable $y$:

$$\frac{\partial}{\partial y} (x^3 - 12xy^2 - x^2y + 4x - y - 3) =$$
Calculating partial derivatives

Example \[ f(x, y) = e^{-\frac{x^2 + y^2}{2}} \]

Compute \[ \frac{\partial f}{\partial x} = \]

\[ \frac{\partial f}{\partial y} = \]
Higher-order partial derivatives

Each partial derivative is itself a function of two variables, so we can compute their partial derivatives, which we call higher-order partial derivatives. For example, there are 4 second-order partial derivatives

\[
\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y^2}.
\]

\(f_{xy}\) and \(f_{yx}\) are called.
\(f_{xy}\) and \(f_{yx}\) are not necessarily equal.

Thm If \(f_{xy}\) and \(f_{yx}\) are continuous on an open disk \(D\), then \(f_{xy} = f_{yx}\) on \(D\).
Higher-order partial derivatives

Example

Let \( f(x,y) = x^2 \cos(2x-y) + xe^{-y^2} \)

\[
\frac{\partial f}{\partial x} = \quad \frac{\partial^2 f}{\partial y \partial x} = \quad \frac{\partial f}{\partial y} = \quad \frac{\partial^2 f}{\partial x \partial y} =
\]

It is not true in general that \( f_{xy} = f_{yx} \).
Tangent planes

Recall, if $f$ is a function of one real variable, then its graph determines a curve $C$ in $\mathbb{R}^2$, and the tangent line to the graph of $f$ at point $x_0$ is the line that "touches" the curve $C$ at point $(x, f(x_0))$.

If $f$ is a function of two variables, then its graph determines a surface $S$, and the tangent plane to $S$ at $(x_0, y_0, f(x_0, y_0))$ is a plane that "touches" $S$ at this point.
Tangent plane

Def. Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface $S$, and let $C$ be any curve passing through $P_0$ and lying entirely in $S$. If the tangent lines to all such curves $C$ at $P_0$ lie in the same plane, then this plane is called the tangent plane to $S$ at $P_0$.

Def. Let $S$ be a surface defined by a differentiable function $z = f(x, y)$. Let $P_0 = (x_0, y_0)$ be in the domain of $f$. Then the equation of the tangent plane to $S$ at $P_0$ is
Tangent plane

To see that this formula is correct, we can find two curves in $S$ that pass through $(x_0, y_0, f(x_0, y_0))$ and determine the equations of the tangent lines.

Take $\vec{p}(t) =$ and $\vec{q}(s) =$

Then for any $t$ (such that $(t, y_0)$ is in the domain of $f$)

$\vec{p}(t)$

Similarly, for any $s$ $\vec{q}(s)$

Moreover,

Tangent line to $\vec{p}(t)$ at $t = x_0$: $\vec{e}_p(t) =$

with $\vec{p}'(t) =$

Similarly, tangent line to $\vec{q}(s)$ at $s = y_0$: $\vec{e}_q(s) =$

$\vec{q}'(s) =$
Tangent plane

Vectors $\vec{p}'(a_0) = \langle 1,0, \frac{\partial f}{\partial x}(x_0,y_0) \rangle$ and $\vec{q}'(y_0) = \langle 0,1, \frac{\partial f}{\partial y}(x_0,y_0) \rangle$ are not parallel, therefore, together with the point $(x_0,y_0,f(x_0,y_0))$ they determine a plane with normal vector

\[ \vec{n} = \]

The equation of a plane passing through $(x_0,y_0,f(x_0,y_0))$ with normal vector $\vec{n}$ is
**Example** Find the equation of the tangent plane to the surface defined by the function $f(x, y) = e^{xy}$ at point $(1, -1)$

- **Step 1:** Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$
  
  $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} =$

- **Step 2:** Evaluate $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$

  $\frac{\partial f}{\partial x}(1, -1) = \frac{\partial f}{\partial y}(1, -1) =$

- **Step 3:** Evaluate $f(x_0, y_0)$: $f(1, -1) =$

- **Step 4:** Plug everything into the equation: